Lebesgue Risk Measures

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The coauthors

- M. Ruiz Galán and J.O. *A coercive and nonlinear James's weak compactness theorem* Preprint
- M. Ruiz Galán and J.O. *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces* Work in progress.

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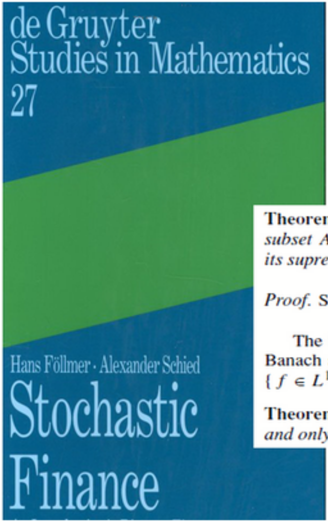

Theorem A.66 (James). *In a Banach space E, a bounded and weakly closed convex subset A is weakly compact if and only if every continuous linear functional attains its supremum on A.*

Proof. See, for instance, [86]. □

The following result characterizes the weakly relatively compact subsets of the Banach space $L^1 := L^1(\Omega, \mathcal{F}, P)$. It implies, in particular, that a set of the form $\{ f \in L^1 \mid |f| \leq g \}$ with given $g \in L^1$ is weakly compact in L^1.

Theorem A.67 (Dunford–Pettis). *A subset A of L^1 is weakly relatively compact if and only if it is bounded and uniformly integrable.*

$\sup_{f \in \mathcal{K}} \|f\|_1 < \infty$, ***and given*** $\varepsilon > 0$ ***there is a*** $\delta > 0$ ***such that if*** $\lambda(A) \leq \delta$, ***then*** $\int_A |f| \, d\lambda \leq \varepsilon$ ***for all*** $f \in \mathcal{K}$.

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The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

  on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact

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Convex Analysis

- We fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- We are going to work in a duality $\langle \mathcal{X}, \mathcal{X}^* \rangle$ where $\mathbb{L}^\infty(\Omega, \mathcal{F}) \subseteq \mathcal{X} \subseteq \mathbb{L}^0(\Omega, \mathcal{F})$
- Examples: $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$, $\langle \mathbb{L}^p, \mathbb{L}^q \rangle$, $\langle \mathbb{L}^\infty, \mathbf{ba}(\Omega, \mathcal{F}) \rangle$
- $f : \mathcal{X} \to (-\infty, +\infty]$, $f^* : \mathcal{X}^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in \mathcal{X}\}$$

Theorem

If $f : \mathcal{X} \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous, then

- $f^{**} \upharpoonright_{\mathcal{X}} = f$
- *for all $x \in \mathcal{X}, x^* \in \mathcal{X}$ we have $\langle x, x^* \rangle \leq f(x) + f^*(x^*)$*

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Risk measures

Definition

A monetary utility function is a concave non-decreasing map

$$U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, +\infty)$$

with $\operatorname{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset$ and

$$U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^\infty, c \in \mathbb{R}$$

Defining $\rho(X) = -U(X)$ the above definition of monetary utility function yields the definition of a convex risk measure.Both U, ρ are called coherent if $U(0) = 0$, $U(\lambda X) = \lambda U(X)$ for all $\lambda > 0, X \in \mathbb{L}^\infty$

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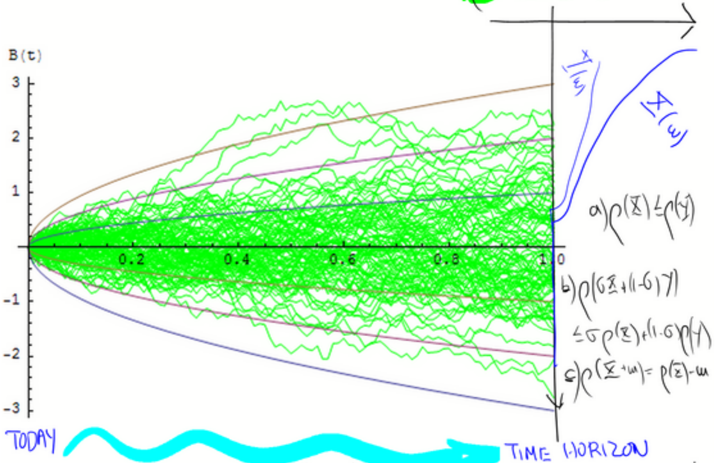
CONVEX MONETARY RISK MEASURE: $\rho : \mathcal{X} \to \mathbb{R}$



has Fatou if $X_n \searrow X \Rightarrow \rho(X_n) \nearrow \rho(X) \Leftrightarrow \sigma(L^\infty, L^1)$ lower semicont.

is order sequentially continuous $\Leftrightarrow |X_n| \leq Z \ \forall n \ X_n \xrightarrow{a.s.} X$

$\Updownarrow$

$\lim_{n \to \infty} \rho(X_n) = \rho(X)$

ρ has Lebesgue property

Representing risk measures

Theorem

A convex (resp. coherent) risk measure $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *admits a representation*

$$\rho(X) = \sup\{\mu(-X) - \alpha(\mu) : \mu \in \mathbf{ba}, \mu \geq 0 \mu(\Omega) = 1\}$$

(resp.
$\rho(X) = \sup\{\mu(-X) : \mu \in \mathcal{S} \subseteq \{\mu \in \mathbf{ba}, \mu \geq 0, \mu(\Omega) = 1\}\}$*) If in addition* ρ *is* $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$*-lower semicontinuous we have:*

$$\rho(X) = \sup\{\mathbb{E}_\mathbb{Q}(-X) - \alpha(\mathbb{Q}) : \mathbb{Q} << \mathbb{P} \text{ and } \mathbb{E}_\mathbb{P}(d\mathbb{Q}/d\mathbb{P}) = 1\}\}$$

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Minimizing $\{\alpha(Y) + \mathbb{E}(X \cdot Y) : Y \in \mathbb{L}^1\}$

Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. *$\{U^* \leq c\}$ is $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$-compact subset for all $c \in \mathbb{R}$*
2. *For every $X \in \mathbb{L}^\infty$ the infimum in the equality*
$$U(X) = \inf_{Y \in \mathbb{L}^1} \{U^*(Y) + \mathbb{E}[XY]\},$$
is attained
3. *For every uniformly bounded sequence (X_n) tending a.s. to X we have*
$$\lim_{n\to\infty} U(X_n) = U(X).$$

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Tools for the proof

- The proof in [JST] is for separable $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. The separability is needed to show 2) $\Rightarrow$ 1) with a variant of the separable James' compactness Theorem.
- Delbaen has given a proof for general non separable spaces using an homogenisation trick. He shows how to apply directly the non separable James' compactness Theorem in the duality $\langle \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rangle$.

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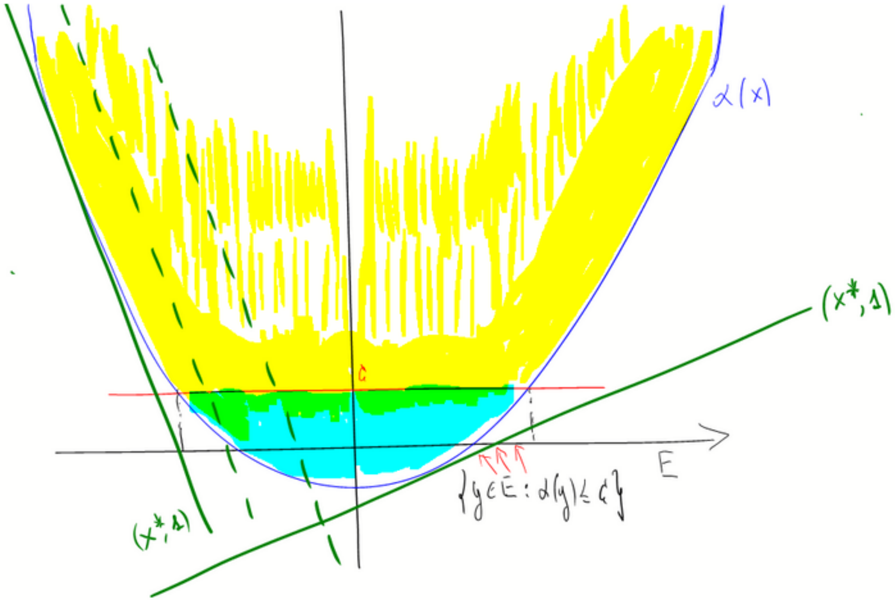

α(x)
(x*, 1)
c
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Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be weakly compact. Then there is $x^* \in E^*$ such that,the infimum

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Minimizing $\{\alpha(x) + x^*(x) : x \in E\}$

Theorem (M. Ruiz and J. Orihuela)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, lower semicontinuous function with

$$\lim_{\|x\|\to\infty} \frac{\alpha(x)}{\|x\|} = +\infty$$

$\partial\alpha(E^*) = E$ dom

⇓ E reflexive

Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be weakly compact. Then there is $x^* \in E^*$ such that, the infimum

$$\inf_{x\in E}\{\langle x, x^*\rangle + \alpha(x)\}$$

is not attained.

Sup-limsup Theorem

Theorem (Simons)

Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then we have:

$$\sup_{\lambda \in \Lambda} \limsup_{k \to \infty} x_k(\lambda) = \sup_{\gamma \in \Gamma} \limsup_{k \to \infty} x_k(\gamma)$$

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Weak Compactness through Sup–limsup Theorem

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k \in K}\{\limsup_{n \to \infty} x_n^*(k)\} = \sup_{\kappa \in \overline{K}^{w^*}} \{\limsup_{n \to \infty} x_n^*(\kappa)\}$$

De la Vallée Poussin's Theorem

Definition

A family $\mathcal{H} \in \mathbb{L}^1$ is uniformly integrable if it is bounded and $\lim_{\mathbb{P}(A)\searrow 0} \int_A |X| d\mathbb{P} = 0$ uniformly in $X \in \mathcal{H}$

Theorem

A family $\mathcal{H} \subset L^0$ is uniformly integrable if, and only if there is a convex function $\Phi : \mathbb{R} \to [0, +\infty)$ s.t $\Phi(0) = 0, \Phi(x) = \Phi(-x), \lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$ which

$$\sup\{\int \Phi(X) d\mathbb{P} : X \in \mathcal{H}\} < \infty$$

Orlicz spaces

An even, convex function $\Psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

1. $\Psi(0) = 0$
2. $\lim_{x\to\infty} \Psi(x) = +\infty$
3. $\Psi < +\infty$ in a neighbourhood of 0

is called a Young function

1. $\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathbb{L}^0 : \exists\alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\}$
2. $N_{\Psi}(X) := \inf\{c > 0 : \mathbb{E}_{\mathbb{P}}[\Psi(\frac{1}{c}X)] \leq 1\}$
3. $\mathbb{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^{1}(\Omega, \mathcal{F}, \mathbb{P})$
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Lebesgue measures go to Orlicz spaces

- A Lebesgue risk measure $\rho : \mathbb{L}^\infty \to \mathbb{R}$ can be extended to a risk measure on some Orlicz space $\overline{\rho} : \mathbb{L}^\Psi \to \mathbb{R}$

Theorem (F. Delbaen)

Every risk measure $\rho : \mathbb{L}^\Psi \to \mathbb{R}$ defined on an Orlicz space $\mathbb{L}^\Psi$ with $\mathbb{L}^\Psi \setminus \mathbb{L}^\infty \neq \emptyset$ has the Lebesgue property restricted to $\mathbb{L}^\infty$

- (Biagini-Fritelli) In general financial markets, the indifference price is a (except for the sign) a convex risk measure on an Orlicz space $\mathbb{L}^{\hat{u}}$ naturally induced by the utility function *u* of the agent.

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Theorem

Any linear and positive functional $\varphi : \mathcal{X} \to \mathbb{R}$ on a Fréchet lattice $\mathcal{X}$ is continuous

Theorem (S. Biagini and M. Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ be an order continuous Frechet lattice. Any convex monotone increasing functional $U : \mathcal{X} \to \mathbb{R}$ is order continuous and it admits a dual representation as

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Komlos C-Properties

- A linear topology $\mathcal{T}$ on $\mathcal{X}$ has the *C*-property if for every $A \subset X$ and every $x \in \overline{A}^{\mathcal{T}}$ there is a sequence $(x_n) \in A$ together with $z_n \in \mathrm{co}\{x_p : p \geq n\}$ such that (z_n) is order convergent to x.
- If $\{v_n\}_{n\geq 1} \in A \subset \mathcal{X}$, another one $\{u_n\}_{n\geq 1}$ is a *convex block sequence of* $\{v_n\}_{n\geq 1}$ if there are finite subsets of $\mathbb{N}$ $\max F_1 < \min F_2 \leq \cdots < \max F_n < \min F_{n+1} < \cdots$ and $\{\lambda_i^n : i \in F_n\} \subset (0,1], \sum_{i\in F_n} \lambda_i^n = 1$ with $u_n = \sum_{i\in F_n} \lambda_i^n v_i$.
- When each sequence $\{x_n\}_{n\geq 1}$ in A has a convex block $\mathcal{T}$-convergent sequence we say that A is $\mathcal{T}$-convex block compact.

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When is it possible to turn $\sup$ to $\max$ on $\mathcal{X}_n^\sim$?

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Is it true Jouni-Schachermayer-Tuzi on $(\mathcal{X}, \mathcal{T})$?

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A Nonlinear James Theorem

Theorem

Let E be a Banach space with B_{E^*} convex-block compact for $\sigma(E^*, E)$. If

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

is a proper map such that for every $x^* \in E^*$ the minimization problem

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is attained at some point of E, then the level sets

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Order Continuity of Risk Measures

Theorem (Lebesgue Risk Measures)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$ be a finite convex risk measure on L^Ψ with $\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to (-\infty, +\infty]$ a penalty function w^*-lower semicontinuos. T.F.A.E.:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

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THANK YOU!!!!