Compacidad, Optimización y Riesgo

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On Compactness in Locally Convex Spaces

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1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is as good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

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Network characterization of Gul'ko compact spaces and their relatives ☆

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Received 4 February 2004

Submitted by R.M. Aron

Dedicated to Professor John Horváth on the occasion of his 80th birthday

Abstract

In this paper we characterize the classes of Gul'ko and Talagrand compact spaces through a network condition leading us to answer two questions posed by G. Gruenhage [Proc. Amer. Math. Soc. 100 (1987) 371–376] on covering properties.

Keywords: Gul'ko compact; Network; Covering properties

Nonlinear Analysis 74 (2011) 1937–1944

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Average locally uniform rotundity and a class of nonlinear maps

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ABSTRACT

We consider some topological characterizations of dual Banach spaces that admit an equivalent dual average locally uniformly rotund norm and provide a criterion for such renorming which involves the class of σ-slicely continuous maps.

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ORIGINAL PAPER

The number of *K*-determination of topological spaces

B. Cascales · M. Muñoz · J. Orihuela

Received: 9 October 2011 / Accepted: 3 January 2012

Abstract We introduce a cardinal function that assigns to each topological space Y a cardinal number $\ell\Sigma(Y)$ that *measures* how the space is determined by its compact subsets via upper semicontinuous compact valued maps defined on metric spaces. By doing so we extend and take to a different dimension the study of the so-called countably K-determined spaces (or Lindelöf Σ-spaces) and their associates Gul'ko compacta. We study the behaviour of $\ell\Sigma(\cdot)$ with respect to the usual operations for topological spaces as well as some of the standard operations within the category of Banach spaces. We study the rela-

A CONTINUOUS IMAGE OF A RADON-NIKODÝM COMPACT SPACE WHICH IS NOT RADON-NIKODÝM

ANTONIO AVILÉS AND PIOTR KOSZMIDER

Abstract. We construct a continuous image of a Radon-Nikodým compact space which is not Radon-Nikodým compact, solving the problem posed in the 80ties by Isaac Namioka.

J. Orihuela, W. Schachermayer and M. Valdivia, Every Radon-Nikodym Corson Compact Space Is Eberlein Compact, Studia Math. 98 (2) (1991), 157-174.

INTEGRATION IN HILBERT GENERATED BANACH SPACES

ROBERT DEVILLE AND JOSÉ RODRÍGUEZ

Abstract. We prove that McShane and Pettis integrability are equivalent for functions taking values in a subspace of a Hilbert generated Banach space. This generalizes simultaneously all previous results on such equivalence. On the other hand, for any super-reflexive generated Banach space having density character greater than or equal to the continuum, we show that Birkhoff integrability lies strictly between Bochner and McShane integrability. Finally, we give a ZFC example of a scalarly null Banach space-valued function (defined on a Radon probability space) which is not McShane integrable.

Israel J. Math. 177 (2010), no. 1, 285-306.

Dentability indices with respect to measures of non compactness

M. Raja*

April, 2007

Dedicated to the memory of my father

Abstract

We study the relationships between the ordinal indices of set derivations associated to several measures of non compactness. We obtain applications to the Szlenk index, improving a result of Lancien, and LUR renorming, providing a non probabilistic proof of a result of Troyanski.

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A new look at compactness via distances to function spaces

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Many classical results about compactness in functional analysis can be derived from suitable inequalities involving distances to spaces of continuous or Baire one functions: this approach gives an extra insight to the classical results as well as triggers a number of open questions in different exciting research branches. We exhibit here, for instance, *quantitative* versions of Grothendieck's characterization of weak compactness in spaces $C(K)$ and also of the Eberlein-Šmulyan and Krein-Šmulyan theorems. The above results specialized in Banach spaces lead to several equivalent measures of *non-weak compactness*. In a different direction we envisage a method to measure the distance from a function $f \in \mathbb{R}^X$ to $B_1(X)$ –space of Baire one functions on X– which allows us to obtain, when X is Polish, a quantitative version of the well known Rosenthal's result stating that in $B_1(X)$ the pointwise relatively countably compact sets are pointwise compact. Other results and applications are commented too.

Keywords: Eberlein-Grothendieck theorem, Krein-Smulyan theorem, oscillations, iterated limits, compactness, measures of non compactness, distances to function spaces, Rosenthal theorem, Baire one functions

DISTANCES TO SPACES OF MEASURABLE AND INTEGRABLE FUNCTIONS

C. ANGOSTO, B. CASCALES, AND J. RODRÍGUEZ

ABSTRACT. Given a complete probability space (Ω, Σ, μ) and a Banach space X we establish formulas to compute the distance from a function $f \in X^\Omega$ to the spaces of strongly measurable functions and Bochner integrable functions. We study the relationship between these distances and use them to prove some quantitative counterparts of Pettis' measurability theorem. We also give several examples showing that some of our estimates are sharp.

A QUANTITATIVE VERSION OF JAMES' COMPACTNESS THEOREM

BERNARDO CASCALES, ONDŘEJ F.K. KALENDA AND JIŘÍ SPURNÝ

ABSTRACT. We introduce two measures of weak non-compactness Ja_E and Ja that quantify, via distances, the idea of boundary behind James' compactness theorem. These measures tell us, for a bounded subset C of a Banach space E and for given $x^* \in E^*$, how far from E or C one needs to go to find $x^{**} \in \overline{C}^{w^*} \subset E^{**}$ with $x^{**}(x^*) = \sup x^*(C)$. A quantitative version of James' compactness theorem is proved using Ja_E and Ja, and in particular it yields the following result: *Let C be a closed convex bounded subset of a Banach space E and $r > 0$. If there is an element x_0^{**} in $\overline{C}^{w^*}$ whose distance to C is greater than r, then there is $x^* \in E^*$ such that each $x^{**} \in \overline{C}^{w^*}$ at which $\sup x^*(C)$ is attained has distance to E greater than $r/2$.* We indeed establish that Ja_E and Ja are equivalent to other measures of weak non-compactness studied in the literature. We also collect particular cases and examples showing when the inequalities between the different measures of weak non-compactness can be equalities and when the inequalities are sharp.

To appear Proc. Edinburgh Math. Soc. (2012)

The coauthors

- M. Ruiz Galán and J.O. *A coercive and nonlinear James's weak compactness theorem* Nonlinear Analysis 75 (2012) 598-611
- M. Ruiz Galán and J.O. *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces* Math. Finan. Econ. (2012)
- B. Cascales, M. Ruiz Gal'an and J.O. *Compactness, Optimality and Risk* Computational and Analytical Mathematics Conference, in honour of J.M Borwein 60'th birthday. Springer 2012.

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- Compactness, Convex Analysis and Risk
- Risk measures on Orlicz spaces
- Variational problems and reflexivity.

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One-Perturbation Variational Principle

Compact domain ⇒ lsc functions attain their minimum

Theorem (Borwein-Fabian-Revalski)

Let X be a Hausdorff topological space and $\alpha : X \to (-\infty, +\infty]$ proper, lsc map s.t. $\{\alpha \leq c\}$ is compact for all $c \in \mathbb{R}$. Then for any proper lsc map $f : X \to (-\infty, +\infty]$ bounded from below, the function $\alpha + f$ attains its minimum.

Theorem (Borwein-Fabian-Revalski)

If X is metrizable and $\alpha : X \to (-\infty, +\infty]$ is a proper function such that for all bounded continuous function $f : X \to (-\infty, +\infty]$, the function $\alpha + f$ attains its minimum, then α is a lsc map, bounded form below, whose sublevel sets $\{\alpha \leq c\}$ are all compact

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1. One Theorem

J. Borwein's talk
2.003

Theorem 1 *Let X be a Hausdorff topological space which admits a proper lsc function*

$$\varphi : X \to \mathbb{R} \cup \{+\infty\}$$

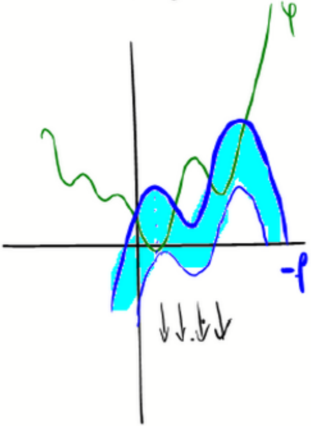
whose level sets are all compact. Then for any proper lsc and bounded from below function $f : X \to \mathbb{R} \cup \{+\infty\}$ the function $f + \varphi$ attains its minimum. In particular, if dom φ *is relatively compact, the conclusion is true for any proper lsc function f.*

Key application. In separable Banach space, a *nice* convex choice is:

$$\varphi(x) := \begin{cases} \tan\left(\|S^{-1}x\|_H^2\right), & \text{if } \|S^{-1}x\|_H^2 < \frac{\pi}{2}, \\ +\infty, & \text{otherwise.} \end{cases}$$

for an appropriate compact, linear and injective mapping $S\colon H \to X$ $(H := \ell_2)$. Also φ is *almost Hadamard smooth*:

$$\lim_{t \searrow 0} \sup_{h \in \operatorname{dom}\varphi} \frac{\varphi(x+th) + \varphi(x-th) - 2\varphi(x)}{t} = 0,$$



- The core requirement of Theorem 1 is also necessary.

Namely, we have:

Theorem 4 *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function on a metric space X with the property that* for every bounded continuous function $f : X \to \mathbb{R}$, the function $f + \varphi$ attains its minimum.

Then φ is (i) a lower semicontinuous function, (ii) bounded from below, (iii) whose level sets are all compact.

- This proof is significantly more subtle.

$\alpha + f \in C_b(X)$ attain minimum $\Rightarrow \{\alpha \leq c\}$ compact

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If not there are open sets:

(1) $U_{x_m} \cap U_{x_n} = \emptyset \quad n \neq m$

(2) $\alpha(x) > \alpha(x_n) - 1/2n \quad \forall x \in U_{x_n}$

$\{\alpha \le c\} \supset \{x_n : n \in \mathbb{N}\}' = \emptyset$

$W_x \cap \{x_n : n \in \mathbb{N}\} = \emptyset$

$\{W_x : x \in \{\alpha \le c\} \setminus \{x_n\}\} \cup \{U_{x_n} : n \in \mathbb{N}\}$ open cover

$\{V_\alpha : \alpha \in A\}$ open refinement loc. finite

$h_n : \{\alpha \le c\} \to [-\alpha(x_n) - 1 + 1/n, 0]$

1) $h_n(x_n) = -\alpha(x_n) - 1 + 1/n$

2) $h_n(x) = 0 \quad \forall x \notin V_{\alpha_n}$

$h = \sum_{n=1}^{\infty} h_n : \{\alpha \le c\} \to [-c-1, 0]$ ⟶ $\bar h$ ⟶ f continuous extension

$f(x) + \alpha(x) > -1 \quad \forall x \in X$

$f(x_n) + \alpha(x_n) = -1 + 1/n \to -1$ ↯

Weak Compactness Theorem of R.C. James

Theorem

A Banach space is reflexive if and only if each continuous linear functional attains its supremum on the unit ball

Theorem

A bounded and weakly closed subset K of a Banach space is weakly compact if and only if each continuous linear functional attains its supremum on K

- R.C. James 1964, 1972, J.D. Pryce 1964, S. Simons 1972, G. RodÃ© 1981, G. Godefroy 1987, V. Fonf, J. Lindenstrauss, B. Phelps 2000-03, M. Ruiz, S. Simons 2002, B. Cascales, I. Namioka, J.O. 2003, O. Kalenda 2007, the boundary problem ...

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Compactness, Functional Analysis and Risk

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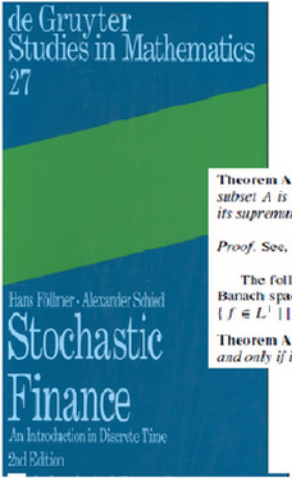

Theorem A.66 (James). *In a Banach space E, a bounded and weakly closed convex subset A is weakly compact if and only if every continuous linear functional attains its supremum on A.*

Proof. See, for instance, [86]. □

The following result characterizes the weakly relatively compact subsets of the Banach space $L^1 := L^1(\Omega, \mathcal{F}, P)$. It implies, in particular, that a set of the form $\{ f \in L^1 \mid |f| \leq g \}$ with given $g \in L^1$ is weakly compact in L^1.

Theorem A.67 (Dunford–Pettis). *A subset A of L^1 is weakly relatively compact if and only if it is bounded and uniformly integrable.*

$\sup_{f \in \mathcal{K}} \|f\|_1 < \infty$, *and given* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $\lambda(A) \leq \delta$, *then* $\int_A |f|\, d\lambda \leq \varepsilon$ *for all* $f \in \mathcal{K}$.

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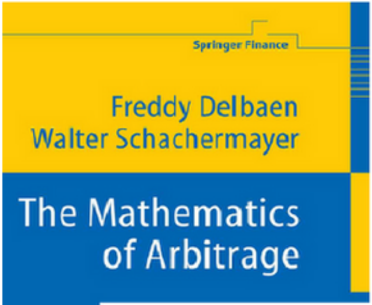
Theorem 15.1.3. *Given a bounded sequence* $(f_n)_{n\geq 1} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ *then there are convex combinations*

$$g_n \in \operatorname{conv}\{f_n, f_{n+1}, \ldots)\}$$

such that $(g_n)_{n\geq 1}$ *converges in measure to some* $g_0 \in L^1(\Omega, \mathcal{F}, \mathbf{P})$.



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A Compactness Principle

Compactness, Functional Analysis and Risk

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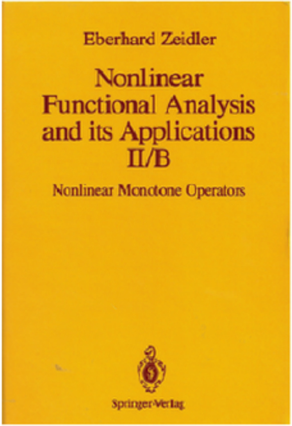

Definition 25.16. Let $f: M \subseteq X \to \mathbb{R}$ be a functional on the subset M of the B-space X.

(i) f is called *coercive* iff $f(u)/\|u\| \to +\infty$ as $\|u\| \to \infty$ on M.
(ii) f is called *weakly coercive* iff $f(u) \to +\infty$ as $\|u\| \to \infty$ on M.

EXAMPLE 25.17. Let $a: X \times X \to \mathbb{R}$ be a strongly positive bilinear functional on the B-space X. Then a is coercive.

PROOF. For all $u \in X$ and fixed $c > 0$, $a(u,u) \geq c\|u\|^2$. Hence $a(u,u)/\|u\| \to +\infty$ as $\|u\| \to \infty$. $\square$

In contrast to Theorem 25.C, the set M can be unbounded in the following theorem.

Theorem 25.D (Main Theorem on Weakly Coercive Functionals). *Suppose that the functional $f: M \subseteq X \to \mathbb{R}$ has the following three properties:*

(i) *M is a nonempty closed convex set in the reflexive B-space X (e.g., $M = X$).*
(ii) *f is weakly sequentially lower semicontinuous on M.*
(iii) *f is weakly coercive.*

Then f has a minimum on M.

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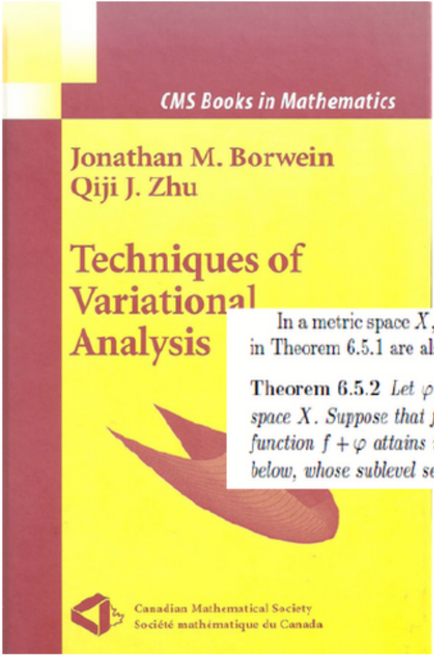

In a metric space X, the conditions imposed on the unique perturbation φ in Theorem 6.5.1 are also necessary.

Theorem 6.5.2 *Let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper function on a metric space X. Suppose that for every bounded continuous function $f\colon X \to \mathbb{R}$, the function $f + \varphi$ attains its minimum. Then φ is a lsc function, bounded from below, whose sublevel sets are all compact.*

The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

  on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact
- When the minimization problem

$$\min\{\alpha(\cdot) + x^*(\cdot)\}$$

  on E has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?

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Minimizing $\{\alpha(x) + x^*(x) : x \in E\}$

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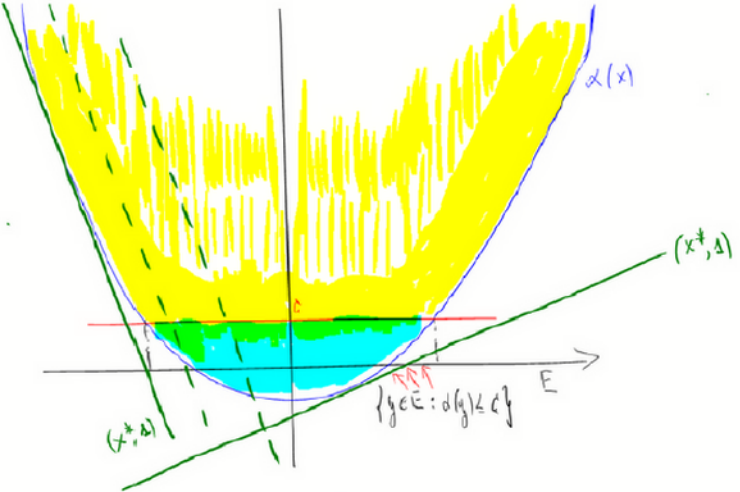

$$\partial\alpha(x_0) = \{x^* \in E^* : x^*(x - x_0) \le \alpha(x) - \alpha(x_0)\ \forall x\}$$

$$\alpha(x_0) - x^*(x_0) \le \alpha(x) - x^*(x) \quad \forall x \in E$$

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Let A be a bounded but not relatively weakly compact subset of the Banach space E. If $(a_n) \subset A$ is a sequence without weak cluster point in E, then there is $(x_n^*) \subset B_{E^*}$, $g_0 = \sum_{n=1}^{\infty} \lambda_n x_n^*$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ such that: for every $h \in l^\infty(A)$, with

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Convex Analysis

- We fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
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Risk meausures

Definition

A monetary utility function is a concave non-decreasing map

$$U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, +\infty)$$

with $\mathrm{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset$ and

$$U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^\infty, c \in \mathbb{R}$$

Defining $\rho(X) = -U(X)$ the above definition of monetary utility function yields the definition of a convex risk measure.Both U, ρ are called coherent if $U(0) = 0$, $U(\lambda X) = \lambda U(X)$ for all $\lambda > 0, X \in \mathbb{L}^\infty$

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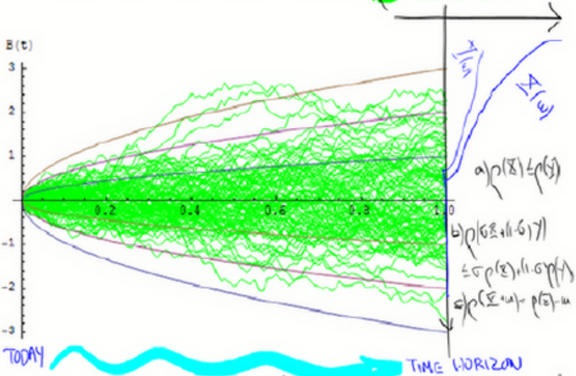
CONVEX MONETARY RISK MEASURE: $\rho : \mathcal{X} \longrightarrow \mathbb{R}$



a) $\rho(X) \le \rho(Y)$

b) $\rho(\sigma X + (1-\sigma) Y) \le \sigma \rho(X) + (1-\sigma)\rho(Y)$

c) $\rho(X + m) = \rho(X) - m$

has Fatou if $X_n \searrow X \Rightarrow \rho(X_n) \nearrow \rho(X) \Leftrightarrow \sigma(L^\infty, L^1)$ lower semicont.

is order sequentially continuous $\Leftrightarrow$ $|X_n| \le Z \ \forall n \ X_n \xrightarrow{a.s.} X$ $\Rightarrow$ $\lim_{n \to \infty} \rho(X_n) = \rho(X)$

has Lebesgue property

Representing risk measures

Theorem

A convex (resp. coherent) risk measure $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *admits a representation*

$$\rho(X) = \sup\{\mu(-X) - \alpha(\mu) : \mu \in \mathbf{ba}, \mu \geq 0 \mu(\Omega) = 1\}$$

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Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

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De la Vallée Poussin's Theorem

Definition

A family $\mathcal{H} \in \mathbb{L}^1$ is uniformly integrable if it is bounded and $\lim_{\mathbb{P}(A)\searrow 0} \int_A |X| d\mathbb{P} = 0$ uniformly in $X \in \mathcal{H}$

Theorem

A family $\mathcal{H} \subset L^0$ is uniformly integrable if, and only if there is a convex function $\Phi : \mathbb{R} \to [0, +\infty)$ s.t $\Phi(0) = 0, \Phi(x) = \Phi(-x), \lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$ *which*

$$\sup\{\int \Phi(X) d\mathbb{P} : X \in \mathcal{H}\} < \infty$$

Orlicz spaces

An even, convex function $\Psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

1. $\Psi(0) = 0$
2. $\lim_{x\to\infty} \Psi(x) = +\infty$
3. $\Psi < +\infty$ in a neighbourhood of 0

is called a Young function

1. $\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathbb{L}^0 : \exists \alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\}$
2. $N_{\Psi}(X) := \inf\{c > 0 : \mathbb{E}_{\mathbb{P}}[\Psi(\frac{1}{c}X)] \leq 1\}$
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An even, convex function $\Psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

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- A Lebesgue risk measure $\rho : \mathbb{L}^\infty \to \mathbb{R}$ can be extended to a risk measure on some Orlicz space $\overline{\rho} : \mathbb{L}^\Psi \to \mathbb{R}$

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Every risk measure $\rho : \mathbb{L}^\Psi \to \mathbb{R}$ defined on an Orlicz space $\mathbb{L}^\Psi$ with $\mathbb{L}^\Psi \setminus \mathbb{L}^\infty \neq \emptyset$ has the Lebesgue property restricted to $\mathbb{L}^\infty$

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Theorem

Any linear and positive functional $\varphi : \mathcal{X} \to \mathbb{R}$ on a FrÃ©chet lattice $\mathcal{X}$ is continuous

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Let $(\mathcal{X}, \mathcal{T})$ be an order continuous Frechet lattice. Any convex monotone increasing functional $U : \mathcal{X} \to \mathbb{R}$ is order continuous and it admits a dual representation as

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- A linear topology $\mathcal{T}$ on $\mathcal{X}$ has the C-property if for every $A \subset X$ and every $x \in \overline{A}^{\mathcal{T}}$ there is a sequence $(x_n) \in A$ together with $z_n \in \mathrm{co}\{x_p : p \geq n\}$ such that (z_n) is order convergent to x.
- If $\{v_n\}_{n\geq 1} \in A \subset \mathcal{X}$, another one $\{u_n\}_{n\geq 1}$ is a *convex block sequence of* $\{v_n\}_{n\geq 1}$ if there are finite subsets of $\mathbb{N}$ $\max F_1 < \min F_2 \leq \cdots < \max F_n < \min F_{n+1} < \cdots$ and $\{\lambda_i^n : i \in F_n\} \subset (0,1], \sum_{i\in F_n} \lambda_i^n = 1$ with $u_n = \sum_{i\in F_n} \lambda_i^n v_i$.
- When each sequence $\{x_n\}_{n\geq 1}$ in A has a convex block $\mathcal{T}$-convergent sequence we say that A is $\mathcal{T}$-convex block compact.

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Let $(\mathcal{X}, \mathcal{T})$ a locally convex Frechet lattice and $U : \mathcal{X} \to (-\infty, +\infty]$ proper and convex. If $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$ has the C-property then U is order lower semicontinuous if, and only if

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Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in \Lambda$ such that

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Weak Compactness through Sup–limsup Theorem

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
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$$\sup_{k \in K}\{\limsup_{n \to \infty} x_n^*(k)\} = \sup_{\kappa \in \overline{K}^{w^*}}\{\limsup_{n \to \infty} x_n^*(\kappa)\}$$

Sup–limsup Theorem $\Rightarrow$ Compactness

- If K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
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Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

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- Since $K \subset \bigcup_{m=1}^{\infty} D_m$, the I-generation says that

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- So $(\bigcup_m^\infty B_m) + 2\delta B_{E^{**}} \supset \overline{K}^{w^*}$.
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Theorem (Cascales, Fonf, Troyanski and Orihuela, J.F.A.-2010)

Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

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Inf-liminf Theorem in $\mathbb{R}^\Gamma$

Theorem (Inf-liminf Theorem in $\mathbb{R}^\Gamma$)

Let $\{\Phi_k\}_{k\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^\Gamma$. We set $\Lambda \subseteq \Gamma$ satisfying the following boundary condition:
For all $\Phi = \sum_{i=1}^\infty \lambda_i \Phi_i, \sum_{i=1}^\infty \lambda_i = 1, 0 \leq \lambda_i \leq 1$, there exists

$$\lambda_0 \in \Lambda \text{ with } \Phi(\lambda_0) = \inf\{\Phi(\gamma) : \gamma \in \Gamma\}$$

Then

$$\inf_{\{\lambda\in\Lambda\}} \left(\liminf_{k\geq 1} \Phi_k(\lambda) \right) = \inf_{\{\gamma\in\Gamma\}} \left(\liminf_{k\geq 1} \Phi_k(\gamma) \right).$$

Inf-liminf Theorem in $\mathbb{R}^\Gamma$

Theorem (Inf-liminf Theorem in $\mathbb{R}^\Gamma$)

Let $\{\Phi_k\}_{k\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^\Gamma$. We set $\Lambda \subseteq \Gamma$ satisfying the following boundary condition:
For all $\Phi = \sum_{i=1}^{\infty} \lambda_i \Phi_i, \sum_{i=1}^{\infty} \lambda_i = 1, 0 \leq \lambda_i \leq 1$, there exists

$$\lambda_0 \in \Lambda \text{ with } \Phi(\lambda_0) = \inf\{\Phi(\gamma) : \gamma \in \Gamma\}$$

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A Nonlinear James Theorem

Theorem (M.Ruiz and J. Orihuela)

Let E be a Banach space with B_{E^*} convex-block compact for $\sigma(E^*, E)$. If

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

is a proper map such that for every $x^* \in E^*$ the minimization problem

$$\inf\{\alpha(y) + x^*(y) : y \in E\}$$

is attained at some point of E, then the level sets

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Order Continuity of Risk Measures

Theorem (Lebesgue Risk Measures)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$ be a finite convex risk measure on L^Ψ with $\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to (-\infty, +\infty]$ a penalty function w^*-lower semicontinuos. T.F.A.E.:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

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Nonlinear Variational Problems

Theorem (Reflexivity frame)

Let E be a real Banach space and

$$\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$$

a coercive function such that $\mathrm{dom}(\alpha)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

$$\alpha(x_0) + x^*(x_0) = \inf_{x \in E} \{\alpha(x) + x^*(x)\}$$

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$[\partial\alpha(E) = E^*] \Rightarrow E = E^{**}$

- Fix an open ball $B \subseteq \mathrm{dom}(\alpha)$
- $B = \bigcup_{p=1}^{+\infty} B \cap \overline{\alpha^{-1}((-\infty, p])}^{\sigma(E,E^*)}$
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- There is G open in E such that $\emptyset \neq B \cap G \subset B \cap \overline{\alpha^{-1}((-\infty, q])}^{\sigma(E,E^*)}$
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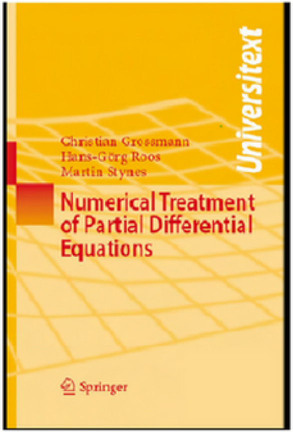
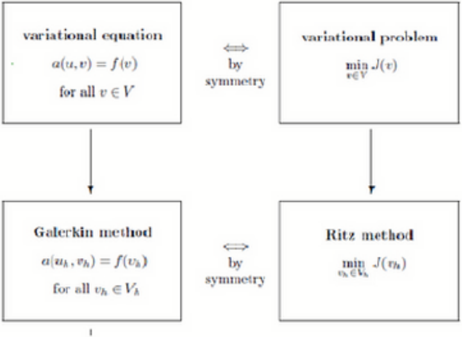
Corollary 2.101 (Main Theorem on Monotone Operators). *Let X be a real, reflexive Banach space, and let $A : X \to X^*$ be a monotone, hemicontinuous, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.*





Nonlinear variational problems

Corollary

- *A real Banach space E is reflexive, provided there exists a monotone, coercive, symmetric and surjective operator $\Phi : E \longrightarrow E^*$*
- *A real Banach space with dual ball w^*-convex-block compact is reflexive whenever there exists a monotone, symmetric and surjective operator $\Phi : E \longrightarrow E^*$*

Question

Let E be a real Banach space and $\Phi : E \to 2^{E^*}$ a monotone multivalued map with non void interior domain.

$$[\Phi(E) = E^*] \Rightarrow E = E^{**}?$$

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Handbook of the geometry of Banach Spaces

CHAPTER 9

Applications to Mathematical Finance

Freddy Delbaen

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Contents

The fundamental theorem of asset pricing in its most general form can now be stated (see [11] and [15]).

THEOREM 3.4. *For a semi-martingale* $S: \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ *the following two properties are equivalent*:

(a) *S satisfies the (NFLVR) property.*

(b) *There is an equivalent probability measure* $\mathbb{Q} \sim P$ *such that under* $\mathbb{Q}$ *the process S is a sigma-martingale.*

If we assume that the semi-martingale S is locally bounded (respectively bounded) the term "sigma-martingale" in (b) *may be replaced by the term "local martingale" (respectively "martingale").*

As indicated above, the central point of the proof is the fact that C is weak*-closed. This is done using the Kreĭn–Šmulian theorem, also called the Banach–Diedonné theorem. This theorem says that a convex set C in the dual X^* of a Banach space X is $\sigma(X^*, X)$ (i.e., weak*-closed) if and only if $C \cap (nB_{X^*})$ is $\sigma(X^*, X)$ closed for each $n \geqslant 1$. If $X = L_1$ and $X^* = L_\infty$ we can, using the characterization of relatively weakly compact sets in L_1 as uniformly integrable subsets of L_1, make this even more precise. A convex set $C \subset L_\infty$ is weak*-closed if and only if, for each sequence $(f_n)_{n \geqslant 1}$ in C that is uniformly bounded and converges in probability to a function f, we have that $f \in C$. Since in our context the set C is a cone we have to show the following fact.

Another application of Banach space theory is given by James' theorem on weakly compact sets. We state the result in its negative form, see [9] for details.

THEOREM 5.4. *Suppose S is continuous, satisfies (NFLVR) and suppose that all martingales with respect to $(\mathcal{F}_t)_{0\leqslant t}$ are continuous (i.e., all stopping times are predictable). Then we have*

(a) *either $\mathbb{M}$ is a singleton,*
(b) *or $\mathbb{M}$ is so big that it has no extreme points.*

It turns out that (a) occurs if and only if $\mathbb{M}$ is weakly compact and the proof uses James' theorem.

In the case when S is only assumed to be locally bounded (and not necessarily continuous), the above theorem is false and the implications of $\mathbb{M}$ being weakly compact are not yet fully understood.

Teorema fundamental de los mercados financieros

Theorem (Harrison, Pliska, Kreps, Delbaen, Schachermayer)

Los siguientes enunciados son equivalentes para un modelo (S_t) *de mercado financiero sobre* $(\Omega, \mathcal{F}, \mathbb{P})$

1. (S_t) *no admite posibilidad de arbitraje*
2. *Existe una medida de probabilidad* $\mathbb{Q}$ *equivalente a* $\mathbb{P}$ *bajo la cual el proceso* (S_t) *se convierte en una martingala*

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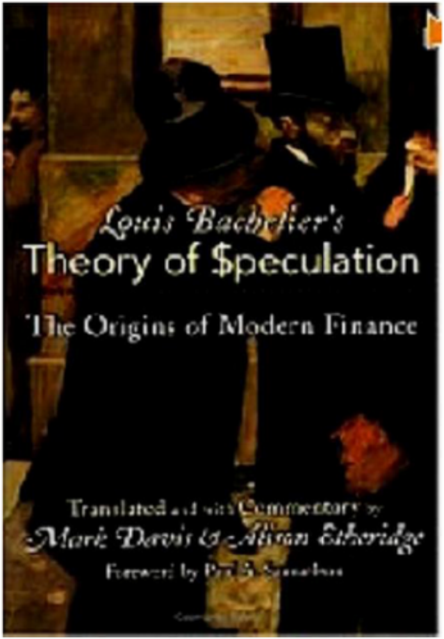

Louis Bachelier's
Theory of $peculation
The Origins of Modern Finance
Mark Davis & Alison Etheridge

Louis Bachelier 1870-1946

- 1989 Mueren los padres de L. Bachelier
- 1900 Defiende su tesis *Teoría de la Especulación* que es informada por Appell, **Poincaré** y Boussinesq.
- 1909-14 Imparte clases en la Sorbona de carácter honorífico
- 1912 Publicación de su *Calcul des Probabilités*
- 1914 Publicación de su *Le Jeu, la Chance et le Hasard*
- 1919-22 Profesor asistente en Besançon
- 1922-25 Profesor asistente en Dijon
- 1926 Enero, es censurado por la Univesidad de Dijon
- 1926-27 Profesor asociado en Rennes
- 1927-37 Profesor en Besançon
- 1941 Su última publicación
- 1996 Se funda la Bachelier Finance Society
- 2006 La tesis de Bachelier es reeditada y traducida al inglés por M. Davis y A. Etheridge con un prólogo de Paul A. Samuelson

Movimiento Browniano

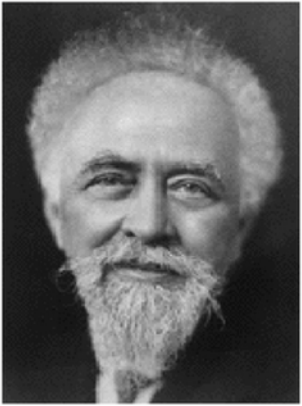
Bachelier-Einstein-Perrin

Jean Perrin experimentally verified Einstein's predictions

In his letter to Einstein:

"I did not believe it was possible to study Brownian motion with such precision"

Accurate calculation of Avogadro number



PREMIO NOBEL DE FISICA EN 1926

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Wolfgang Doeblin
Springer VideoMATH
Agnes Handwerk
Harrie Willems
Wolfgang Doeblin
A mathematician rediscovered
$X_t = x + \beta\left(\int_0^t \sigma^2(s, X_s)\,\mathrm{d}s\right) + \int_0^t a(s, X_s)\,\mathrm{d}s$
Springer
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Martingalas

Sea $\mathcal{F}_t$ la σ-algebra generada por $\{W_s : s \leq t\}$. El movimiento Browniano (W_t) verifica:

1. $W_t - W_s$ es independiente de $\mathcal{F}_s$ si $0 \leq s < t$
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y tendremos operadores proyección ortogonal entre estos espacios de Hilbert.

$\mathbb{E}(\cdot, \mathcal{F}_s)$ coincide con la proyección ortogonal sobre $L^2(\Omega, \mathcal{F}_s, \mathbb{P})$

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$$\mathbb{E}(S_t|\mathcal{F}_s) = S_s$$

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$\mathbb{E}(\cdot, \mathcal{F}_s)$ coincide con la proyección ortogonal sobre $L^2(\Omega, \mathcal{F}_s, \mathbb{P})$

Un proceso (S_t) adaptado a la filtración anterior se dice que es una martingala si

$$\mathbb{E}(S_t | \mathcal{F}_s) = S_s$$

siempre que $0 \le s \le t$, esto es siempre que $S_t - S_s$ sea ortogonal a $L^2(\Omega, \mathcal{F}_s, \mathbb{P})$ cuando $0 \le s < t$

Martingalas

Sea $\mathcal{F}_t$ la σ-algebra generada por $\{W_s : s \leq t\}$. El movimiento Browniano (W_t) verifica:

1. $W_t - W_s$ es independiente de $\mathcal{F}_s$ si $0 \leq s < t$
2. $\mathbb{E}(W_t|\mathcal{F}_s) = W_s$ siempre que $0 \leq s \leq t$

$$L^2(\Omega, \mathcal{F}_s, \mathbb{P}) \subset L^2(\Omega, \mathcal{F}_t, \mathbb{P}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$$

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MUCHAS GRACIAS POR VUESTRA ATENCIÓN!!!!