An interplay between Topology, Functional Analysis and Risk

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The coauthors

- M. Ruiz Galán and J.O. *A coercive and nonlinear James's weak compactness theorem* Preprint.
- M. Ruiz Galán and J.O. *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces* Preprint.

Contents

- Compactness and Optimization
- Compactness, Convex Analysis and Risk
- Risk measures on Orlicz spaces
- Variational problems and reflexivity.

One-Perturbation Variational Principle

Compact domain $\Rightarrow$ lsc functions attain their minimum

Theorem (Borwein-Fabian-Revalski)

Let X be a Hausdorff topological space and $\alpha : X \to (-\infty, +\infty]$ proper, lsc map s.t. $\{\alpha \leq c\}$ is compact for all $c \in \mathbb{R}$. Then for any proper lsc map $f : X \to (-\infty, +\infty]$ bounded from below, the function $\alpha + f$ attains its minimum.

Theorem (Borwein-Fabian-Revalski)

If X is metrizable and $\alpha : X \to (-\infty, +\infty]$ is a proper function such that for all bounded continuous function $f : X \to (-\infty, +\infty]$, the function $\alpha + f$ attains its minimum, then α is a lsc map, bounded form below, whose sublevel sets $\{\alpha \leq c\}$ are all compact

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Let X be a Hausdorff topological space and $\alpha : X \to (-\infty, +\infty]$ *proper, lsc map s.t.* $\{\alpha \leq c\}$ *is compact for all* $c \in \mathbb{R}$*. Then for any proper lsc map* $f : X \to (-\infty, +\infty]$ *bounded from below, the function* $\alpha + f$ *attains its minimum.*

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1. One Theorem

Theorem 1 *Let X be a Hausdorff topological space which admits a proper lsc function*

$$\varphi : X \to \mathbb{R} \cup \{+\infty\}$$

whose level sets are all compact. Then for any proper lsc and bounded from below function $f : X \to \mathbb{R} \cup \{+\infty\}$ the function $f + \varphi$ attains its minimum. In particular, if dom φ *is relatively compact, the conclusion is true for any proper lsc function f.*

J. Borwein's talk
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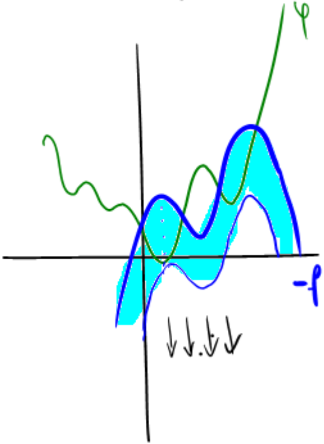

Key application. In separable Banach space, a *nice* convex choice is:

$$\varphi(x) := \begin{cases} \tan\left(\|S^{-1}x\|_H^2\right), & \text{if } \|S^{-1}x\|_H^2 < \frac{\pi}{2}, \\ +\infty, & \text{otherwise.} \end{cases}$$

for an appropriate compact, linear and injective mapping $S\colon H \to X$ $(H := \ell_2)$. Also φ is *almost Hadamard* smooth:

$$\lim_{t \searrow 0} \sup_{h \in \operatorname{dom} \varphi} \frac{\varphi(x+th) + \varphi(x-th) - 2\varphi(x)}{t} = 0,$$

Remark 2 If $(X, \|\cdot\|)$ is *normed* and φ is *convex*, the result above holds for every proper lsc convex f, provided only that the level sets of φ are *weakly* compact, or that dom φ is.

Remark 3 In a normed space $(X, \|\cdot\|)$, by allowing translations of φ, we get a *localization* of the minimum of the perturbation (as in Bishop-Phelps, Ekeland, Borwein-Preiss [B-P], etc.).

With the same proof:

> *Suppose X admits a function φ as above. For any proper lsc (bounded below) function $f : X \to \mathbb{R} \cup \{+\infty\}$, for any $\bar{x} \in$ dom f and each $\lambda > 0$, the function*
>
> $$f + \varphi((\cdot - \bar{x})/\mu)$$
>
> *(for some $\mu > 0$), attains its minimum at a u with $\|u - \bar{x}\| \leq \lambda$.*

- Observe that in this case, formally, the perturbation function is *now* varying.

- The core requirement of Theorem 1 is also necessary.

Namely, we have:

Theorem 4 *Let* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function on a metric space* X *with the property that* for every bounded continuous function $f : X \to \mathbb{R}$, the function $f + \varphi$ attains its minimum.

Then φ *is (i) a lower semicontinuous function, (ii) bounded from below, (iii) whose level sets are all compact.*

- This proof is significantly more subtle.

$\alpha + f \in C_b(X)$ attain minimum $\Rightarrow \{\alpha \leq c\}$ compact

If not there are open sets:

(1) $U_{x_m} \cap U_{x_n} = \emptyset \quad n \neq m$

(2) $\alpha(x) > \alpha(x_n) - 1/2n \quad \forall x \in U_{x_n}$

$\{\alpha \leq c\} \supset \{x_n : n \in \mathbb{N}\}' = \emptyset$

$W_x \cap \{x_n : n \in \mathbb{N}\} = \emptyset$

$\{W_x : x \in \{\alpha \leq c\} \setminus \{x_n\}\} \cup \{U_{x_n} : n \in \mathbb{N}\}$ open cover

$\{V_\alpha : \alpha \in A\}$ open refinement loc. finite

$h_n : \{\alpha \leq c\} \to [-\alpha(x_n) - 1 + 1/n, 0]$

1) $h_n(x_n) = -\alpha(x_n) - 1 + 1/n$

2) $h_n(x) = 0 \quad \forall x \notin V_{\alpha_n}$

$h = \sum_{n=1}^{\infty} h_n : \{\alpha \leq c\} \to [-c-1, 0]$

X — f continuous extension

$f(x) + \alpha(x) \geq -1 \quad \forall x \in X$

$f(x_n) + \alpha(x_n) = -1 + 1/n \to -1$

Weak Compactness Theorem of R.C. James

Theorem

A Banach space is reflexive if and only if each continuous linear functional attains its supremum on the unit ball

Theorem

A bounded and weakly closed subset K of a Banach space is weakly compact if and only if each continuous linear functional attains its supremum on K

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Compactness, Functional Analysis and Risk

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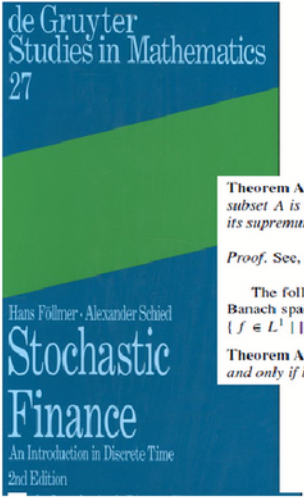

Theorem A.66 (James). *In a Banach space E, a bounded and weakly closed convex subset A is weakly compact if and only if every continuous linear functional attains its supremum on A.*

Proof. See, for instance, [86]. □

The following result characterizes the weakly relatively compact subsets of the Banach space $L^1 := L^1(\Omega, \mathcal{F}, P)$. It implies, in particular, that a set of the form $\{ f \in L^1 \mid |f| \le g \}$ with given $g \in L^1$ is weakly compact in L^1.

Theorem A.67 (Dunford–Pettis). *A subset A of L^1 is weakly relatively compact if and only if it is bounded and uniformly integrable.*

$\sup_{f \in \mathcal{X}} \|f\|_1 < \infty$, *and given* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $\lambda(A) \le \delta$, *then* $\int_A |f| \, d\lambda \le \varepsilon$ *for all* $f \in \mathcal{X}$.

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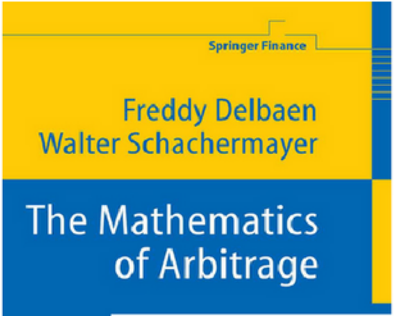
Theorem 15.1.3. *Given a bounded sequence* $(f_n)_{n\geq 1} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ *then there are convex combinations*

$$g_n \in \operatorname{conv}\{f_n, f_{n+1}, \dots)\}$$

such that $(g_n)_{n\geq 1}$ *converges in measure to some* $g_0 \in L^1(\Omega, \mathcal{F}, \mathbf{P})$.



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A Compactness Principle

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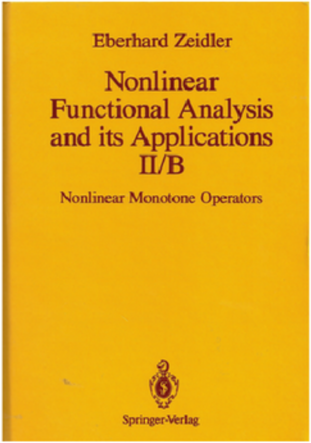

Definition 25.16. Let $f: M \subseteq X \to \mathbb{R}$ be a functional on the subset M of the B-space X.

(i) f is called *coercive* iff $f(u)/\|u\| \to +\infty$ as $\|u\| \to \infty$ on M.
(ii) f is called *weakly coercive* iff $f(u) \to +\infty$ as $\|u\| \to \infty$ on M.

EXAMPLE 25.17. Let $a: X \times X \to \mathbb{R}$ be a strongly positive bilinear functional on the B-space X. Then a is coercive.

PROOF. For all $u \in X$ and fixed $c > 0$, $a(u, u) \geq c\|u\|^2$. Hence $a(u, u)/\|u\| \to +\infty$ as $\|u\| \to \infty$. □

In contrast to Theorem 25.C, the set M can be unbounded in the following theorem.

Theorem 25.D (Main Theorem on Weakly Coercive Functionals). *Suppose that the functional* $f: M \subseteq X \to \mathbb{R}$ *has the following three properties:*

(i) *M is a nonempty closed convex set in the reflexive B-space X (e.g., $M = X$).*
(ii) *f is weakly sequentially lower semicontinuous on M.*
(iii) *f is weakly coercive.*

Then f has a minimum on M.

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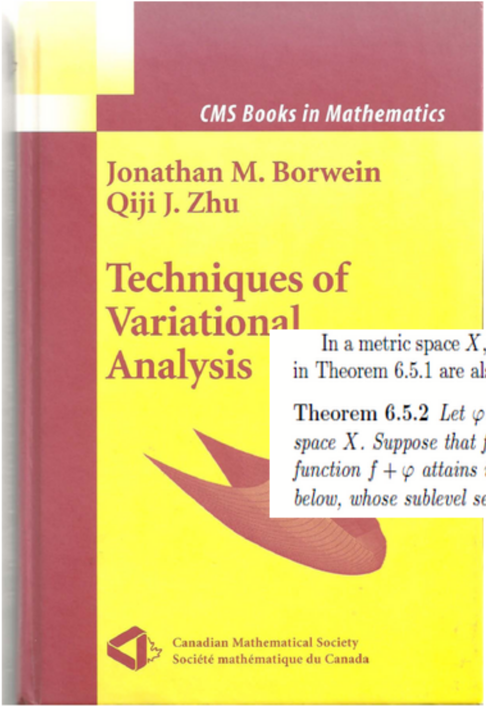

In a metric space X, the conditions imposed on the unique perturbation φ in Theorem 6.5.1 are also necessary.

Theorem 6.5.2 *Let* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function on a metric space* X*. Suppose that for every bounded continuous function* $f\colon X \to \mathbb{R}$*, the function* $f + \varphi$ *attains its minimum. Then* φ *is a lsc function, bounded from below, whose sublevel sets are all compact.*

The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

  on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact
- When the minimization problem

$$\min\{\alpha(\cdot) + x^*(\cdot)\}$$

  on E has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?

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Minimizing $\{\alpha(x) + x^*(x) : x \in E\}$

Theorem (M. Ruiz and J. Orihuela)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, (lower semicontinuous) function with

$$\lim_{\|x\|\to\infty} \frac{\alpha(x)}{\|x\|} = +\infty$$

Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be (relatively) weakly compact. Then there is $x^* \in E^*$ such that,the infimum

$$\inf_{x\in E}\{\langle x, x^*\rangle + \alpha(x)\}$$

is not attained.

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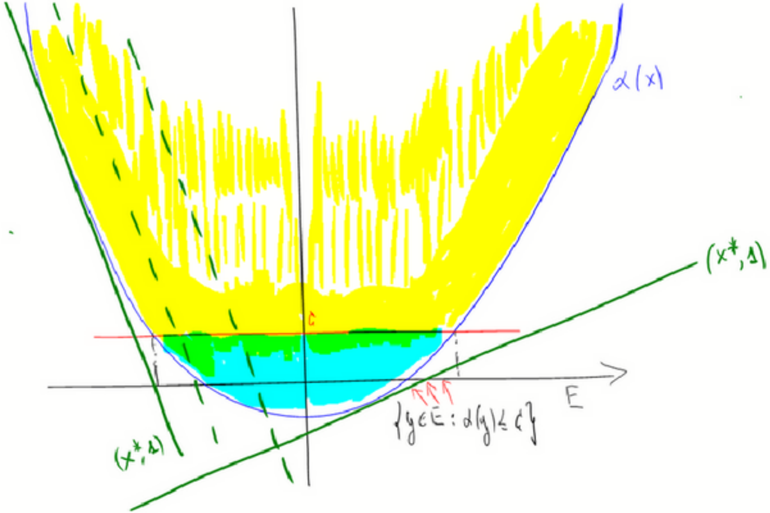

$$\partial\alpha(x_0) = \{x^* \in E^* : x^*(x - x_0) \leq \alpha(x) - \alpha(x_0) \ \forall x\}$$

$$\alpha(x_0) - x^*(x_0) \leq \alpha(x) - x^*(x) \quad \forall x \in E$$

$\{\alpha \leq c\}$ not w.c.$\Rightarrow \exists x^*$: $\inf_E\{x^*(\cdot) + \alpha(\cdot)\}$ not attained

Lemma

Let A be a bounded but not relatively weakly compact subset of the Banach space E. If $(a_n) \subset A$ is a sequence without weak cluster point in E, then there is $(x_n^*) \subset B_{E^*}$, $g_0 = \sum_{n=1}^{\infty} \lambda_n x_n^*$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ such that: for every $h \in l^\infty(A)$, with

$$\liminf_n x_n^*(a) \leq h(a) \leq \limsup_n x_n^*(a)$$

for all $a \in A$, we will have that $g_0 + h$ doest not attain its minimum on A

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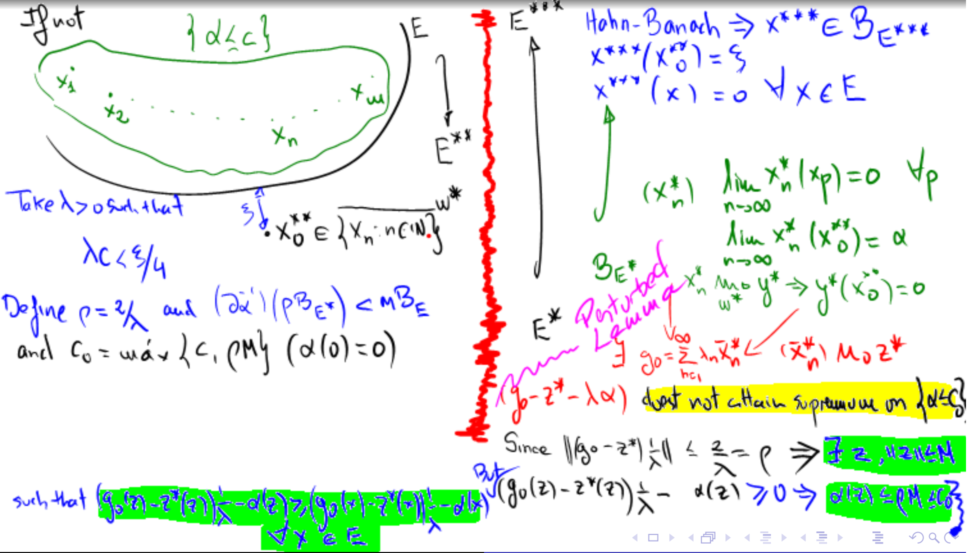

Maximizing $\{x^*(x) - \alpha(x) : x \in E\}$

Theorem (M. Ruiz and J. Orihuela)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, lower semicontinuous function, then we have:

- *If $\partial\alpha(E) = E^*$ then the level sets $\{\alpha \leq c\}$ are weakly compact for all $c \in \mathbb{R}$ whenever α is a coercive map, i.e. $\lim_{\|x\|\to\infty} \frac{\alpha(x)}{\|x\|} = +\infty$. ,*
- *It α has weakly compact level sets and the Fenchel-Legendre conjugate α^* is finite, i.e. $\sup\{x^*(x) - \alpha(x) : x \in E\} < +\infty$ for all $x^* \in E^*$, then $\partial\alpha(E) = E^*$*

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Convex Analysis

- We fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- We are going to work in a duality $\langle \mathcal{X}, \mathcal{X}^* \rangle$ where $\mathbb{L}^\infty(\Omega, \mathcal{F}) \subseteq \mathcal{X} \subseteq \mathbb{L}^0(\Omega, \mathcal{F})$
- Examples: $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$, $\langle \mathbb{L}^p, \mathbb{L}^q \rangle$, $\langle \mathbb{L}^\infty, \mathbf{ba}(\Omega, \mathcal{F}) \rangle$
- $f : \mathcal{X} \to (-\infty, +\infty]$, $f^* : \mathcal{X}^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in \mathcal{X}\}$$

Theorem

If $f : \mathcal{X} \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous, then

- $f^{**} \restriction_{\mathcal{X}} = f$
- *for all* $x \in \mathcal{X}, x^* \in \mathcal{X}$ *we have* $\langle x, x^* \rangle \leq f(x) + f^*(x^*)$

Convex Analysis

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If $f : \mathcal{X} \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous, then

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Risk meausures

Definition

A monetary utility function is a concave non-decreasing map

$$U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, +\infty)$$

with $\mathrm{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset$ and

$$U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^\infty, c \in \mathbb{R}$$

Defining $\rho(X) = -U(X)$ the above definition of monetary utility function yields the definition of a convex risk measure.Both U, ρ are called coherent if $U(0) = 0$, $U(\lambda X) = \lambda U(X)$ for all $\lambda > 0, X \in \mathbb{L}^\infty$

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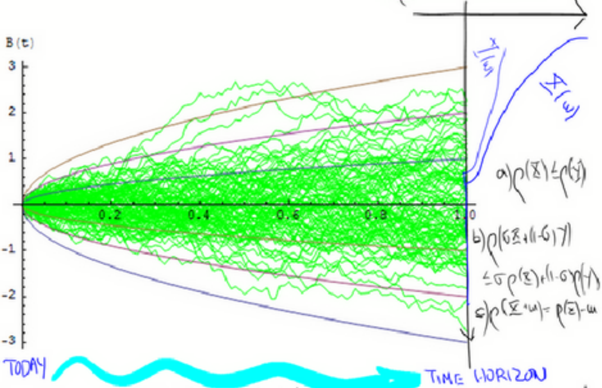
CONVEX MONETARY RISK MEASURE: $\rho : \mathcal{X} \to \mathbb{R}$



a) $\rho(X) \le \rho(Y)$

b) $\rho(\sigma X + (1-\sigma) Y) \le \sigma \rho(X) + (1-\sigma)\rho(Y)$

c) $\rho(X + m) = \rho(X) - m$

has Fatou if $X_n \downarrow X \Rightarrow \rho(X_n) \uparrow \rho(X) \Leftrightarrow \sigma(L^\infty, L^1)$ lower semicont.

is order sequentially continuous $\Leftrightarrow |X_n| \le Z \ \forall n \quad X_n \xrightarrow{a.s.} X$

$\Downarrow$

$\lim_{n \to \infty} \rho(X_n) = \rho(X)$

ρ has Lebesgue property

Representing risk measures

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A convex (resp. coherent) risk measure $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *admits a representation*

$$\rho(X) = \sup\{\mu(-X) - \alpha(\mu) : \mu \in \mathbf{ba}, \mu \geq 0 \mu(\Omega) = 1\}$$

(resp.
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Minimizing $\{\alpha(Y) + \mathbb{E}(X \cdot Y) : Y \in \mathbb{L}^1\}$

Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. $\{U^* \leq c\}$ *is* $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$*-compact subset for all* $c \in \mathbb{R}$
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- The proof in [JST] is for separable $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. The separability is needed to show 2) $\Rightarrow$ 1) with a variant of the separable James' compactness Theorem we provided to authors.
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De la Vallée Poussin's Theorem

Definition

A family $\mathcal{H} \in \mathbb{L}^1$ is uniformly integrable if it is bounded and $\lim_{\mathbb{P}(A)\searrow 0} \int_A |X| d\mathbb{P} = 0$ uniformly in $X \in \mathcal{H}$

Theorem

A family $\mathcal{H} \subset L^0$ is uniformly integrable if, and only if there is a convex function $\Phi : \mathbb{R} \to [0, +\infty)$ s.t $\Phi(0) = 0, \Phi(x) = \Phi(-x), \lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$ which

$$\sup\{\int \Phi(X) d\mathbb{P} : X \in \mathcal{H}\} < \infty$$

Orlicz spaces

An even, convex function $\Psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

1. $\Psi(0) = 0$
2. $\lim_{x \to \infty} \Psi(x) = +\infty$
3. $\Psi < +\infty$ in a neighbourhood of 0

is called a Young function

1. $\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathbb{L}^0 : \exists \alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\}$
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3. $\mathbb{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^{1}(\Omega, \mathcal{F}, \mathbb{P})$
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Lebesgue measures go to Orlicz spaces

- A Lebesgue risk measure $\rho : \mathbb{L}^\infty \to \mathbb{R}$ can be extended to a risk measure on some Orlicz space $\overline{\rho} : \mathbb{L}^\Psi \to \mathbb{R}$

Theorem (F. Delbaen)

Every risk measure $\rho : \mathbb{L}^\Psi \to \mathbb{R}$ defined on an Orlicz space $\mathbb{L}^\Psi$ with $\mathbb{L}^\Psi \setminus \mathbb{L}^\infty \neq \emptyset$ has the Lebesgue property restricted to $\mathbb{L}^\infty$

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Namioka-Klee Theorem

Theorem

Any linear and positive functional $\varphi : \mathcal{X} \to \mathbb{R}$ on a Fréchet lattice $\mathcal{X}$ is continuous

Theorem (S. Biagini and M. Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ be an order continuous Frechet lattice. Any convex monotone increasing functional $U : \mathcal{X} \to \mathbb{R}$ is order continuous and it admits a dual representation as

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Komlos C-Properties

- A linear topology $\mathcal{T}$ on $\mathcal{X}$ has the C-property if for every $A \subset X$ and every $x \in \overline{A}^{\mathcal{T}}$ there is a sequence $(x_n) \in A$ together with $z_n \in \mathrm{co}\{x_p : p \geq n\}$ such that (z_n) is order convergent to x.
- If $\{v_n\}_{n\geq 1} \in A \subset \mathcal{X}$, another one $\{u_n\}_{n\geq 1}$ is a *convex block sequence of* $\{v_n\}_{n\geq 1}$ if there are finite subsets of $\mathbb{N}$ $\max F_1 < \min F_2 \leq \cdots < \max F_n < \min F_{n+1} < \cdots$ and $\{\lambda_i^n : i \in F_n\} \subset (0,1], \sum_{i\in F_n} \lambda_i^n = 1$ with $u_n = \sum_{i\in F_n} \lambda_i^n v_i$.
- When each sequence $\{x_n\}_{n\geq 1}$ in A has a convex block $\mathcal{T}$-convergent sequence we say that A is $\mathcal{T}$-convex block compact.

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- If $\{v_n\}_{n\geq 1} \in A \subset \mathcal{X}$, another one $\{u_n\}_{n\geq 1}$ is a *convex block sequence of* $\{v_n\}_{n\geq 1}$ if there are finite subsets of $\mathbb{N}$ $\max F_1 < \min F_2 \leq \cdots < \max F_n < \min F_{n+1} < \cdots$ and $\{\lambda_i^n : i \in F_n\} \subset (0,1], \sum_{i\in F_n} \lambda_i^n = 1$ with $u_n = \sum_{i\in F_n} \lambda_i^n v_i$.
- When each sequence $\{x_n\}_{n\geq 1}$ in A has a convex block $\mathcal{T}$-convergent sequence we say that A is $\mathcal{T}$-convex block compact.

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Theorem (S.Biagini and M.Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ a locally convex Frechet lattice and $U : \mathcal{X} \to (-\infty, +\infty]$ proper and convex. If $\sigma(\mathcal{X}, \mathcal{X}_n^{\sim})$ has the C-property then U is order lower semicontinuous if, and only if

$$U(x) = \sup_{y' \in (\mathcal{X}_n^{\sim})} \{y'(x) - U^*(y')\}$$

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Sup-limsup Theorem

Theorem (Simons)

Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then we have:

$$\sup_{\lambda \in \Lambda} \limsup_{k \to \infty} x_k(\lambda) = \sup_{\gamma \in \Gamma} \limsup_{k \to \infty} x_k(\gamma)$$

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Weak Compactness through Sup–limsup Theorem

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k \in K}\{\limsup_{n\to\infty} x_n^*(k)\} = \sup_{\kappa \in \overline{K}^{w^*}}\{\limsup_{n\to\infty} x_n^*(\kappa)\}$$

Sup–limsup Theorem ⇒ Compactness

- If K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} \geq$$

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Weak Compactness through I-generation

Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

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Theorem (Cascales, Fonf, Troyanski and Orihuela, J.F.A.-2010)

Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

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Inf-liminf Theorem in $\mathbb{R}^\Gamma$

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Let $\{\Phi_k\}_{k\geq 1}$ *be a pointwise bounded sequence in* $\mathbb{R}^\Gamma$. *We set* $\Lambda \subseteq \Gamma$ *satisfying the following boundary condition:*
For all $\Phi = \sum_{i=1}^\infty \lambda_i \Phi_i, \sum_{i=1}^\infty \lambda_i = 1, 0 \leq \lambda_i \leq 1$, *there exists*

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Then

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A Nonlinear James Theorem

Theorem (M.Ruiz and J. Orihuela)

Let E be a Banach space with B_{E^*} convex-block compact for $\sigma(E^*, E)$. If

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

is a proper map such that for every $x^* \in E^*$ the minimization problem

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is attained at some point of E, then the level sets

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Order Continuity of Risk Measures

Theorem (Lebesgue Risk Measures)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$ be a finite convex risk measure on L^Ψ with $\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to (-\infty, +\infty]$ a penalty function w^*-lower semicontinuos. T.F.A.E.:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

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Nonlinear Variational Problems

Theorem (Reflexivity frame)

Let E be a real Banach space and

$$\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$$

a coercive function such that $\mathrm{dom}(\alpha)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

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Moreover, if the dual ball B_{E^*} is a w^*- convex-block compact no coercive assumption is needed for α

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- Fix an open ball $B \subseteq \mathrm{dom}(\alpha)$
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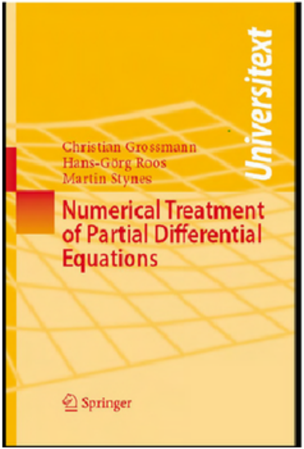
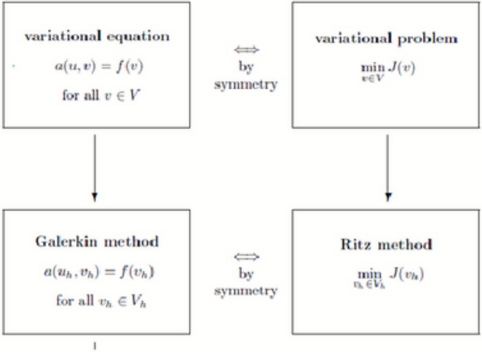
Corollary 2.101 (Main Theorem on Monotone Operators). *Let X be a real, reflexive Banach space, and let $A : X \to X^*$ be a monotone, hemicontinuous, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.*





Nonlinear variational problems

Corollary

- *A real Banach space E is reflexive, provided there exists a monotone, coercive, symmetric and surjective operator $\Phi : E \longrightarrow E^*$*
- *A real Banach space with dual ball w^*-convex-block compact is reflexive whenever there exists a monotone, symmetric and surjective operator $\Phi : E \longrightarrow E^*$*

Question

Let E be a real Banach space and $\Phi : E \to 2^{E^*}$ a monotone multivalued map with non void interior domain.

$$[\Phi(E) = E^*] \Rightarrow E = E^{**}?$$

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THANK YOU!!!!