Locally uniformly rotund (F)-norms

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The coauthors

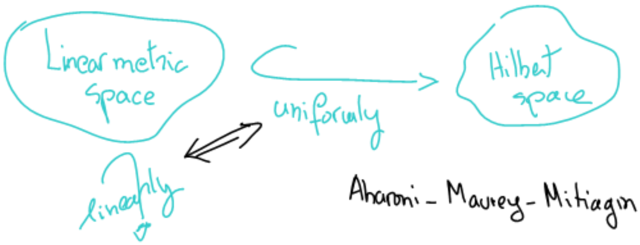
- S. Ferrari, L. Oncina, M. Raja y J.O. *Metrization theory and the Kadec property* Prepint 2013.
- J.O. *On $\mathcal{T}_p$ locally uniformly rotund norms* Set Valued and Variational Analysis, 2013

Contents

- Metrizable topological vector spaces and (F)-norms.
- Kadec norm implies LUR renorming with (F)-norm
- Descriptive Bananch spaces
- Bing-Nagata-Smirnov-Stone meets Kadec
- Construction of LUR (F)-norms in descriptive Bananch spaces

Metrizable topological vector spaces

S. Banach "Theory of linear operations"
V. L. Klee
G. Köthe
H. Jarchow
N. Kalton



$$\mathbb{L}^0(\Omega, \mathfrak{F}, \mathbb{P}) \longrightarrow \mathbb{R}^+$$

$$X \rightsquigarrow \int_\Omega \frac{|X(\omega)|}{1+|X(\omega)|} \, d\mathbb{P}(\omega)$$

Is there an equivalent translation invariant metric on $\mathbb{L}^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that $|X| := d(X, 0)$ verifies

$$\lim_{n\to\infty} \left[2|X_n|^2 + 2|X|^2 - |X + X_n|^2 \right] = 0 \Rightarrow (X_n) \xrightarrow{\mathbb{P}} X \, ?$$

Definition

A function

$$\|\cdot\| : X \longrightarrow [0, +\infty)$$

is called (F)*-norm on the vector space* X *if the following properties are satisfied:*

- $x = 0$ *if, and only if,* $\|x\| = 0$*;*
- $\|\lambda x\| \leq \|x\|$*, if* $|\lambda| \leq 1$ *and* $x \in X$*;*
- $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$*;*
- $\lim_n \|\lambda x_n\| = 0$*, if* $\lim_n \|x_n\| = 0$ *for every* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *and* $\lambda \in \mathbb{R}$*;*
- $\lim_n \|\lambda_n x\| = 0$*, if* $\lim_n \lambda_n = 0$ *for every* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $x \in X$.

Kadec $\Rightarrow$ LUR renormings

Theorem

If a normed space $(X, \|\cdot\|)$ has a Kadec norm there is an equivalent Kadec and locally uniformly rotund (F)-norm $\|\cdot\|_1$ on X, i.e. an (F)-norm $\|\cdot\|_1$ such that the topology determined by the (F)-norm $\|\cdot\|_1$ on X coincides with the norm topology and moreover we have:

1. *the weak and norm topologies coincide on every sphere $\{x \in X : \|x\|_= \rho\}$ for $\rho > 0$.*
2. *For every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ we have $\lim_{n\to\infty} \|x_n - x\| = 0$ whenever*

$$\lim_{n\to\infty} (2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2) = 0$$

Examples

R. Haydon
J. Jayne
I. Namioka
C.A. Rogers

The lexicographic product $K = [0,1]^{\omega_1}$ gives

$C(K)$ with a Kadec renorming

but $C(K)$ does not have equivalent LUR norm

"Continuous functions on totally ordered spaces that are compact in their order topologies" J. Funct. Anal. 178, 23-63 (2000)

A. Moltó
S. Troyanski
M. Valdivia
J.O

If X is a Banach space with a Kadec norm and the Krein-Milman property then X has an equivalent LUR norm

"Kadec and Krein-Milman properties" C.R. Acad. Sci. Paris 131, Série I, 459-464 (2000)

Kadec (F)-renorming $\Leftrightarrow$ descriptiveness

Theorem

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. TFAE:

1. *There is a norm-equivalent and $\sigma(X,Z)$-lower semicontinuous (F)-norm $\|\cdot\|_0$ on X such that $\sigma(X,Z)$ and norm topologies coincide on the unit sphere*

$$\{x \in X : \|x\|_0 = 1\}$$

2. *There are isolated families $\mathcal{B}_n$ for the $\sigma(X,Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with the property that $x \in B$ and*

$$\|\cdot\| - \mathrm{diam}(B) < \epsilon$$

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Descriptive Banach spaces

Fragmentability:

ε-fragmented if $\forall F \subset A$ $\exists W$ w-open $W \cap F \neq \emptyset$ and $\|\cdot\|$-diam$(W \cap F) < \varepsilon$

ε-σ-fragmented $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t. $A_{n,\varepsilon}$ is ε-fragmented $n = 1, 2, \ldots$

ε-DESCRIPTIVE: $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t. $\forall x \in A_{n,\varepsilon}$ $\exists W$ w-open, $x \in W$ and $\|\cdot\|$-diam$(W \cap A_{n,\varepsilon}) < \varepsilon$

isolated families

X DESCRIPTIVE $\iff$ There are families $\mathcal{B}_n$, relatively discrete for the weak topology, s.t. $\forall x \in X$, $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$, $x \in B \in \mathcal{B}_{n_0}$, $\|\cdot\|$-diam$(B_{n_0}) < \varepsilon$

Kadec norm $\Rightarrow$ descriptive $\Rightarrow$ σ-fragmented

w^*-descriptive $\iff$ dual LUR (M. Raja)

Theorem (Kadec metrization)

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. Then the following conditions are equivalent:

1. *The normed space X is $\sigma(X,Z)-$descriptive; i.e there are isolated families $\mathcal{B}_n$ for the $\sigma(X,Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with the property that $x \in B$ and $\|\cdot\| - \mathrm{diam}(B) < \epsilon$*
2. *The norm topology admits a basis $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ such that each one of the families $\mathcal{B}_n$ is norm discrete and $\sigma(X,Z)$-isolated*

Kadec meets Bing-Nagata-Smirnov-Stone

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2. *There is a norm-equivalent, $\sigma(X,Z)$-lower semicontinuous and LUR (F)-norm $\|\cdot\|_0$ on X; i.e. such that for every $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ we have $\lim_{n\to+\infty} \|x_n - x\| = 0$ whenever*

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Method of proof

Definition (p-convex set and hull)

Let A be a subset of a vector space X and $p \in (0,1]$. A is said to be p-convex if for every $x, y \in A$ and $\tau, \mu \in [0,1]$ such that $\tau^p + \mu^p = 1$ we have $\tau x + \mu y \in A$.

If A is p-convex and absorbent, its p-Minkowski functional is

$$p_A(x) := \inf\{\lambda^p : \lambda > 0, x \in \lambda A\}$$

p_A is a p-seminorm, i.e we have

- $p_A(\lambda x) = |\lambda|^p p_A(x)$
- $p_A(x+y) \leq p_A(x) + p_A(y)$.

The Minkowski functional is defined as usual:

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we have $q_A(x)^p = p_A(x)$ for every $x \in X$ and q_A is a quasinorm : $q_A(x+y) \leq 2^{(1/p)-1}(q_A(x) + q_A(y))$.

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Definition

A real function $\phi : X \longrightarrow \mathbb{R}$ is said to be p-convex for $p \in (0,1]$ if

$$\phi(\tau x + \mu y) \leq \tau\phi(x) + \mu\phi(y)$$

whenever $\tau \geq 0$, $\mu \geq 0$ and $\tau^p + \mu^p = 1$.

- the epigraph of ϕ is p-convex if and only if ϕ is p-convex;
- if ϕ is convex and $\phi(0) = 0$, then ϕ is p-convex for every $p \in (0,1]$;
- if ϕ_p is p-convex, ϕ_q is q-convex, with $0 < p \leq q < 1$ and both of them are non-negative, then $\phi_p + \phi_q$ is p-convex;
- if $\phi : X \to \mathbb{R}$ is p-convex for some $0 < p \leq 1$ and bounded from above in a neighbourhood of $x \in X$, then ϕ is locally Lipschitz at x
- $\tau^p\mu^p(\phi(x) - \phi(y))^2 \leq \tau^p\phi(x)^2 + \mu^p\phi(y)^2 - \phi(\tau x + \mu y)^2$ whenever $\tau^p + \mu^p = 1$ and $\tau \geq 0$, $\mu \geq 0$.

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p-convex functions

Definition (p-distance)

Let C be a w^**-compact and p-convex subset of X^{**}, $0 < p \leq 1$,*

$$\varphi(x) := \inf_{c^{**} \in C} \{\sup\{|\langle x - c^{**}, z^* \rangle| : z^* \in B_{X^*} \cap Z\}\}$$

φ is a p-convex, $\sigma(X, Z)$-lower semicontinuous and 1-Lipschitz map from X to $[0, +\infty)$.

Definition

A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets in the normed space X is said to be p-isolated for the $\sigma(X, Z)$-topology if for every $i \in I$

$$B_i \cap \overline{\mathrm{co}_p\{B_j : j \neq i, \ j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

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Theorem

Let $\mathcal{B} := \{B_i\ i \in I\}$ be an uniformly bounded family of subsets of X. The following are equivalent:

1. *The family $\mathcal{B}$ is p-isolated for the $\sigma(X,Z)$-topology; i.e.*

$$B_i \cap \overline{\mathrm{co}_p\{B_j : j \neq i,\ j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

   for every $i \in I$

2. *There exists a family $\mathcal{L} := \{\varphi_i : X \to [0,+\infty) : i \in I\}$ of p-convex and $\sigma(X,Z)$-lower semicontinuous functions such that for every $i \in I$*

$$\{x \in X : \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.$$

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Isolatedness $\Rightarrow$ p-convex isolatedly decomposable

Lemma (Decomposition lemma)

Let $\mathcal{B}$ be a uniformly bounded and isolated family of sets for the $\sigma(X,Z)$ topology. Then for every $B \in \mathcal{B}$ we can write

$$B = \bigcup_{n=1}^{\infty} B_n$$

in such a way that, for every $n \in \mathbb{N}$ fixed, the family

$$\{B_n : B \in \mathcal{B}\}$$

is $\sigma(X,Z)$-q-isolated whenever $q < \frac{\log 2}{\log 4n}$.

p-Localization Theorem

Theorem

Let $\mathcal{B} = \{B_i : i \in I\}$ be a uniformly bounded and p-isolated family of subsets of X for the $\sigma(X, Z)$ topology. Then there is a norm-equivalent $\sigma(X, Z)$-lower semicontinuous p-norm $q_{\mathcal{B}}(\cdot)$ on X such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n\in\mathbb{N}}$ in X the condition

$$\lim_{n\to+\infty} [2q^2_{\mathcal{B}}(x_n) + 2q^2_{\mathcal{B}}(x) - q^2_{\mathcal{B}}(x + x_n)] = 0,$$

implies that:

1. *there exists $n_0 \in \mathbb{N}$ such that* $x_n, \frac{x_n+x}{2^{1/p}} \notin \overline{\mathrm{co}_p\{B_i \ : i \neq i_0,\ i \in I\}}^{\sigma(X,Z)}$ *for every $n \geq n_0$;*
2. *for every positive δ there is $n_\delta \in \mathbb{N}$ such that* $x_n \in \overline{\mathrm{co}(B_{i_0} \cup \{0\}) + B(0, \delta)}^{\sigma(X,Z)}$ *whenever $n \geq n_\delta$.*

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Descriptive $\Rightarrow$ LUR (F)-renorming

- Fix isolated families $\mathcal{B}_n$ for the $\sigma(X, Z)$-topology such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with $x \in B$ and $\|\cdot\| - \operatorname{diam}(B) < \epsilon$.
- $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ are assumed to be p_n-isolated for some sequence $p_n \in (0, 1]$ by decomposition lemma.
- Consider the p_n-norms $q_{\mathcal{B}_n}(\cdot)$ constructed using the p-Localization Theorem
- $F^2_{\mathcal{B}}(x) := \|x\|^2_Z + \sum_{n=1}^{+\infty} \frac{1}{\zeta_n^{2p_n} 2^n} q^2_{\mathcal{B}_n}(x)$ where $q_{\mathcal{B}_n}(x) \leq \zeta_n^{p_n} \|x\|^{p_n} \leq \zeta_n^{p_n} \max\{1, \|x\|\}$.
- If $\lim_{n\to+\infty}[2F^2_{\mathcal{B}}(x_n) + 2F^2_{\mathcal{B}}(x) - F^2_{\mathcal{B}}(x + x_n)] = 0$ then $\lim_{n\to+\infty}[2q^2_{\mathcal{B}_m}(x_n) + 2q^2_{\mathcal{B}_m}(x) - q^2_{\mathcal{B}_m}(x + x_n)] = 0$ for all m.
- If $\epsilon > 0$, $m \in \mathbb{N}$ and $B_0 \in \mathcal{B}_m$ with $x \in B_0 \subseteq x + \frac{\epsilon}{2} B_X$ there exists $n_{\frac{\epsilon}{2}}$ such that $x_n \in \overline{\operatorname{co}(B_0 \cup \{0\}) + B(0, \frac{\epsilon}{2})}^{\sigma(X,Z)}$ whenever $n \geq n_{\frac{\epsilon}{2}}$.

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- By induction we select integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\|x_{n_k} - r_{(n_k,1/k)}x\| \leq \frac{1}{k}$.
- By compactness there is a sequence of integers $k_1 < k_2 < \cdots < k_j < \cdots$ such that $\lim_{j\to+\infty} r_{(n_{k_j},1/k_j)} = r \in [0,1]$ and $\|\cdot\| - \lim_{j\to+\infty} x_{n_{k_j}} = rx$
- If $\|x\|_Z = 1$ we also have $\lim_n \|x_n\|_Z = \|x\|_Z = 1$ and it follows that $r = 1$, so we have found a subsequence (x_{n_j}) of the given sequence (x_n) which norm converges to x
- Since the reasoning is valid for every subsequence too, the proof is over

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Lemma

Let X be a topological space, S be a set and $\varphi_s, \psi_s : X \to [0, +\infty)$ lower semicontinuous functions such that $\sup_{s\in S}(\varphi_s(x) + \psi_s(x)) < +\infty$ for every $x \in X$. Define

$$\varphi(x) = \sup_{s\in S} \varphi_s(x), \quad \theta_m(x) = \sup_{s\in S}(\varphi_s(x) + 2^{-m}\psi_s(x)),$$

and $\theta(x) = \sum_{m\in\mathbb{N}} 2^{-m}\theta_m(x)$. Assume further that $\{x_\sigma : \sigma \in \Sigma\}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma\in\Gamma}$ and a net $\{i_\gamma\}_{\gamma\in\Gamma} \subseteq S$ such that

$$\lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma\in\Gamma} \varphi(x_\gamma) = \sup_{s\in S} \varphi_s(x)$$

and

$$\lim_{\gamma\in\Gamma}(\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0.$$

Theorem

Let $\mathcal{B} := \{B_i : i \in I\}$ be an uniformly bounded and p-isolated family of subsets of X for the $\sigma(X, Z)$-topology and some $p \in (0, 1]$. Then there is an equivalent $\sigma(X, Z)$-lower semicontinuous quasinorm, with p-power a p-norm, $\|\cdot\|_{\mathcal{B}}$ on X such that: for every net $\{x_\alpha : \alpha \in A\}$ and x in X with $x \in B_{i_0}$ for $i_0 \in I$, the conditions $\sigma(X, Z) - \lim_\alpha x_\alpha = x$ and $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ imply that

1. *there exists $\alpha_0 \in A$ such that x_α is not in $\overline{\mathrm{co}_p\{B_i\ i \neq i_0,\ i \in I\}}^{\sigma(X,Z)}$ for $\alpha \geq \alpha_0$;*
2. *for every positive δ there exists $\alpha_\delta \in A$ such that*

$$x, x_\alpha \in \overline{\mathrm{co}(B_{i_0} \cup \{0\}) + B(0, \delta)}^{\sigma(X,Z)}$$

whenever $\alpha \geq \alpha_\delta$.

- We can construct norm-equivalent and $\sigma(X, Z)$-lower semicontinuous F-norms F_1 and F_2 such that F_1 has the LUR property and F_2 the Kadec property.
- Then we define

$$\|\cdot\|_1(x)^2 := F_1(\cdot)^2 + F_2(\cdot)^2$$

which is an equivalent $\sigma(X, Z)$-lower semicontinuous F-norm which has both Kadec and the LUR property.

- $\lim_{n\to\infty}[2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2] = 0$ is equivalent to $\lim_{n\to\infty}[2F_i(x)^2 + 2F_i(x_n)^2 - F_i(x + x_n)^2] = 0$ for $i = 1, 2$, and LUR property of F_1 is translated to $\|\cdot\|_1$.
- If $\{x_\alpha : \alpha \in (A, \succ)\}$ is a net in X which converges to x in the topology $\sigma(X, Z)$ and $\lim_{\alpha\in A}\|x_\alpha\|_1 = \|x\|_1$ it follows that $\lim_{\alpha\in A} F_i^2(x_\alpha) = F_i^2(x)$ for $i = 1, 2$. Thus Kadec property of F_2 is translated to $\|\cdot\|_1$.

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THANK YOU VERY MUCH !!!!!