One side James' Compactness Theorem

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Topological Methods in Analysis and Optimization. On the occasion of the 70th birthday of Prof. Petar Kenderov

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A birthday Theorem

- P. Kenderov 2003
- J. Lindenstrauss 2006
- M. Valdivia 2010
- W. Schachermayer 2010
- J. Borwein 2011
- I. Karatzas 2012
- F. Delbaen 2012
- A. Defant 2013
- P. Kenderov 2013
- N. Ghoussoub 2013

The coauthors

- M. Ruiz Galán and J.O. *A coercive and nonlinear James's weak compactness theorem* Nonlinear Analysis 75 (2012) 598-611.
- M. Ruiz Galán and J.O. *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces* Math. Finan. Econ. 6(1) (2012) 15–35.
- B. Cascales, M. Ruiz Gal'an and J.O. *Compactness, Optimality and Risk* Computational and Analytical Mathematics. Conference in honour of J.M Borwein 60'th birthday. Chapter 10, Springer Verlag 2013, 153–208.
- B. Cascales and J. O. *One side James' Theorem* Preprint 2013.

Contents

- Compactness and Optimization.
- Variational problems and reflexivity.
- One-side James' Theorem.
- Conic Godefroy's Theorem.
- Dual variational problems.

One-Perturbation Variational Principle

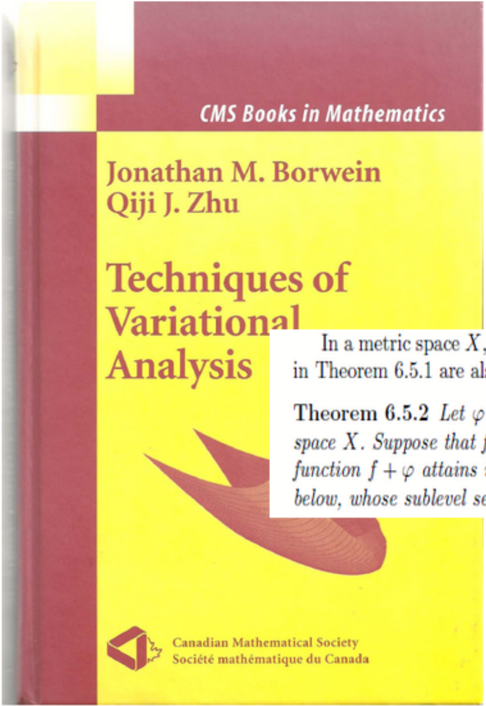
Compact domain $\Rightarrow$ lsc functions attain their minimum

Theorem (Borwein-Fabian-Revalski)

Let X be a Hausdorff topological space and $\alpha : X \to (-\infty, +\infty]$ proper, lsc map s.t. $\{\alpha \leq c\}$ is compact for all $c \in \mathbb{R}$. Then for any proper lsc map $f : X \to (-\infty, +\infty]$ bounded from below, the function $\alpha + f$ attains its minimum.

Theorem (Borwein-Fabian-Revalski)

If X is metrizable and $\alpha : X \to (-\infty, +\infty]$ is a proper function such that for all bounded continuous function $f : X \to (-\infty, +\infty]$, the function $\alpha + f$ attains its minimum, then α is a lsc map, bounded form below, whose sublevel sets $\{\alpha \leq c\}$ are all compact



In a metric space X, the conditions imposed on the unique perturbation φ in Theorem 6.5.1 are also necessary.

Theorem 6.5.2 *Let* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function on a metric space* X*. Suppose that for every bounded continuous function* $f\colon X \to \mathbb{R}$*, the function* $f + \varphi$ *attains its minimum. Then* φ *is a lsc function, bounded from below, whose sublevel sets are all compact.*

Theorem

A Banach space is reflexive if, and only if, each continuous linear functional attains its supremum on the unit ball

Theorem

A bounded and weakly closed subset K of a Banach space is weakly compact if, and only if, each continuous linear functional attains its supremum on K

R.C. James 1964, 1972, J.D. Pryce 1964, S. Simons 1972, G. Rodé 1981, G. Godefroy 1987, V. Fonf, J. Lindenstrauss, B. Phelps 2000-03, M. Ruiz, S. Simons 2002, B. Cascales, I. Namioka, J.O. 2003, O. Kalenda 2007, the boundary problem ...

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The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact

- When the minimization problem

$$\min\{\alpha(\cdot) + x^*(\cdot)\}$$

on E has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?

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Theorem (M. Ruiz and J. O.)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, (lower semicontinuous) function with

$$\lim_{\|x\|\to\infty} \frac{\alpha(x)}{\|x\|} = +\infty$$

Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be (relatively) weakly compact. Then there is $x^* \in E^*$ such that,the infimum

$$\inf_{x\in E} \{\langle x, x^*\rangle + \alpha(x)\}$$

is not attained.

Minimizing $\{\alpha(x) + x^*(x) : x \in E\}$

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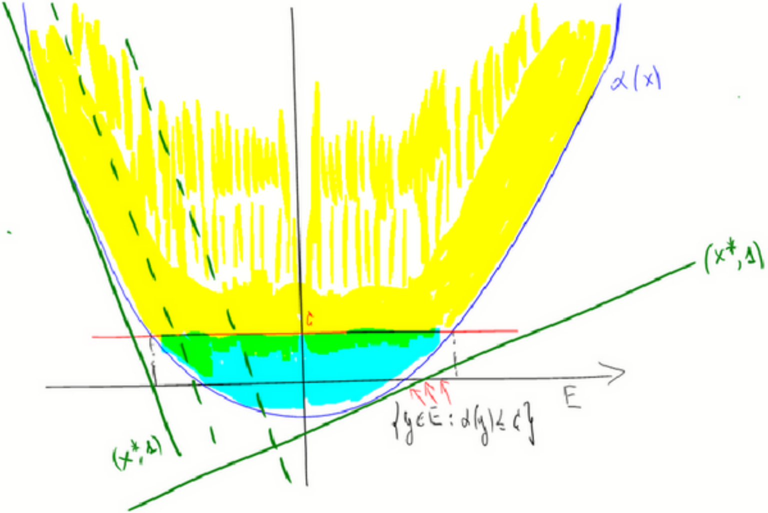

$$\partial\alpha(x_0) = \{x^* \in E^* : x^*(x - x_0) \le \alpha(x) - \alpha(x_0) \ \forall x\}$$

$$\alpha(x_0) - x^*(x_0) \le \alpha(x) - x^*(x) \quad \forall x \in E$$

Lemma

Let A be a bounded but not relatively weakly compact subset of the Banach space E. If $(a_n) \subset A$ is a sequence without weak cluster point in E, then there is $(x_n^*) \subset B_{E^*}$, $g_0 = \sum_{n=1}^{\infty} \lambda_n x_n^*$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ such that: for every $h \in l^\infty(A)$, with

$$\liminf_n x_n^*(a) \leq h(a) \leq \limsup_n x_n^*(a)$$

for all $a \in A$, we will have that $g_0 + h$ doest not attain its minimum on A

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Theorem (M. Ruiz, J. O. and J. Saint Raymond)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, lower semicontinuous function, then we have:

- *If $\partial\alpha(E) = E^*$ then the level sets $\{\alpha \leq c\}$ are weakly compact for all $c \in \mathbb{R}$.*
- *If α has weakly compact level sets and the Fenchel-Legendre conjugate α^* is finite, i.e. $\sup\{x^*(x) - \alpha(x) : x \in E\} < +\infty$ for all $x^* \in E^*$, then $\partial\alpha(E) = E^*$*

Risk meausures

Definition

A monetary utility function is a concave non-decreasing map

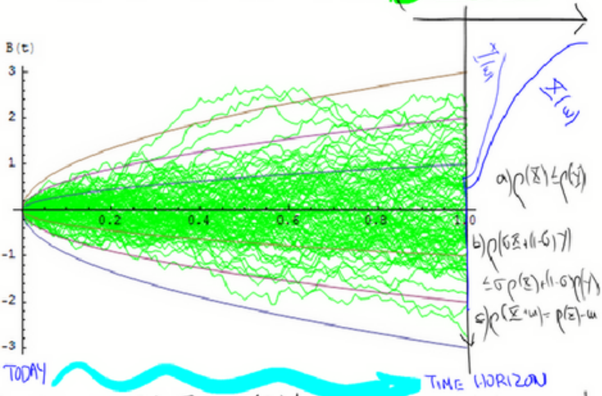
$$U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, +\infty)$$

with $\mathrm{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset$ and

$$U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^\infty, c \in \mathbb{R}$$

Defining $\rho(X) = -U(X)$ the above definition of monetary utility function yields the definition of a convex risk measure.Both U, ρ are called coherent if $U(0) = 0$, $U(\lambda X) = \lambda U(X)$ for all $\lambda > 0, X \in \mathbb{L}^\infty$

CONVEX MONETARY RISK MEASURE: $\rho: \mathcal{X} \to \mathbb{R}$



a) $\rho(X) \le \rho(Y)$

b) $\rho(\sigma X + (1-\sigma)Y) \le \sigma\rho(X) + (1-\sigma)\rho(Y)$

c) $\rho(X + m) = \rho(X) - m$

has Fatou if $X_n \searrow X \Rightarrow \rho(X_n) \uparrow \rho(X) \iff \sigma(L^\infty, L^1)$ lower semicont.

is order sequentially continuous $\iff |X_n| \le Z \ \forall n \quad X_n \xrightarrow{a.s.} X$

$\Downarrow$

$\lim_{n \to \infty} \rho(X_n) = \rho(X)$

ρ has Lebesgue property

Representing risk measures

Theorem

A convex (resp. coherent) risk measure $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *admits a representation*

$$\rho(X) = \sup\{\mu(-X) - \alpha(\mu) : \mu \in \mathbf{ba}, \mu \geq 0 \mu(\Omega) = 1\}$$

(resp.
$\rho(X) = \sup\{\mu(-X) : \mu \in \mathcal{S} \subseteq \{\mu \in \mathbf{ba}, \mu \geq 0, \mu(\Omega) = 1\}\}$*) If in addition* ρ *is* $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$*-lower semicontinuous we have:*

$$\rho(X) = \sup\{\mathbb{E}_\mathbb{Q}(-X) - \alpha(\mathbb{Q}) : \mathbb{Q} << \mathbb{P} \textit{ and } \mathbb{E}_\mathbb{P}(d\mathbb{Q}/d\mathbb{P}) = 1\}\}$$

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$\rho(X) = \sup\{\mathbb{E}_\mathbb{Q}(-X)) : \mathbb{Q} \in \{\mathbb{Q} << \mathbb{P} \textit{ and } \mathbb{E}_\mathbb{P}(d\mathbb{Q}/d\mathbb{P}) = 1\}\}$*)*

Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. $\{U^* \leq c\}$ *is* $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$*-compact subset for all* $c \in \mathbb{R}$
2. *For every* $X \in \mathbb{L}^\infty$ *the infimum in the equality*

$$U(X) = \inf_{Y \in \mathbb{L}^1} \{U^*(Y) + \mathbb{E}[XY]\},$$

   is attained

3. *For every uniformly bounded sequence* (X_n) *tending a.s. to* X *we have*

$$\lim_{n \to \infty} U(X_n) = U(X).$$

Order Continuity of Risk Measures

Theorem (Lebesgue Risk Measures on Orlicz spaces)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$ be a finite convex risk measure on L^Ψ with $\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to (-\infty, +\infty]$ a penalty function w^*-lower semicontinuos. T.F.A.E.:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$$

is attained.

(iii) *ρ is sequentially order continuous*

Theorem (Reflexivity frame)

Let E be a real Banach space and

$$\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$$

a a function such that $\mathrm{dom}(\alpha)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

$$\alpha(x_0) + x^*(x_0) = \inf_{x \in E} \{\alpha(x) + x^*(x)\}$$

Then E is reflexive.

$[\partial\alpha(E) = E^*] \Rightarrow E = E^{**}$

- Fix an open ball $B \subseteq \mathrm{dom}(\alpha)$
- $B = \bigcup_{p=1}^{+\infty} B \cap \overline{\alpha^{-1}((-\infty, p])}^{\sigma(E,E^*)}$
- Baire Category Theorem $\Rightarrow$ there is $q \in \mathbb{N}$:

$$B \cap \overline{\alpha^{-1}((-\infty, q])}^{\sigma(E,E^*)}$$

  has non void interior relative to B
- There is G open in E such that $\emptyset \neq B \cap G \subset B \cap \overline{\alpha^{-1}((-\infty, q])}^{\sigma(E,E^*)}$
- $\overline{\alpha^{-1}((-\inf, q])}^{\sigma(E,E^*)}$ weakly compact $\Rightarrow$ G contains an open relatively weakly compact ball
- B_E is weakly compact

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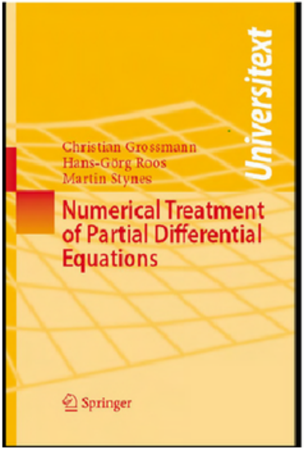
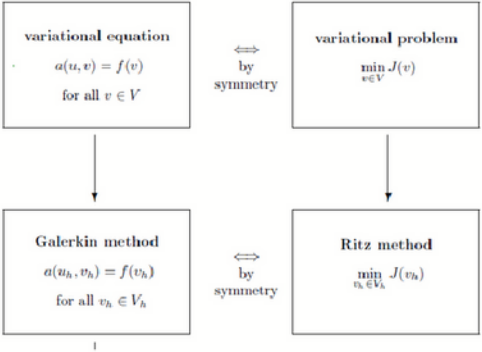
Corollary 2.101 (Main Theorem on Monotone Operators). *Let X be a real, reflexive Banach space, and let $A : X \to X^*$ be a monotone, hemicontinuous, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.*





Applications to nonlinear variational problems

Given an operator $\Phi : E \longrightarrow E^*$ it is said to be *monotone* provided that

$$\text{for all } x, y \in E, \quad (\Phi x - \Phi y)(x - y) \geq 0,$$

and *symmetric* if for all $x, y \in E$, $< \Phi(x), y >=< \Phi(y), x >$

Corollary

A real Banach space E is reflexive whenever there exists a monotone, symmetric and surjective operator $\Phi : E \longrightarrow E^*$

Question

Let E be a real Banach space and $\Phi : E \to 2^{E^*}$ a monotone multivalued map with non void interior domain.

$$[\Phi(E) = E^*] \Rightarrow E = E^{**}?$$

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Theorem (Simons)

Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then we have:

$$\sup_{\lambda \in \Lambda} \limsup_{k \to \infty} x_k(\lambda) = \sup_{\gamma \in \Gamma} \limsup_{k \to \infty} x_k(\gamma)$$

Sup-limsup Theorem

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Weak Compactness through Sup–limsup Theorem

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k \in K}\{\limsup_{n \to \infty} x_n^*(k)\} = \sup_{\kappa \in \overline{K}^{w^*}}\{\limsup_{n \to \infty} x_n^*(\kappa)\}$$

Sup–limsup Theorem ⇒ Compactness

- If K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} \geq$$

$$= \sup_{v^{**}\in\overline{K}^{w^*}}\{\limsup_{n\to\infty} x_n^*(v^{**})\} = \limsup_{n\to\infty} x_n^*(x_0^{**}) \geq \alpha/2 > 0$$

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$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} \geq$$

$$= \sup_{v^{**}\in\overline{K}^{w^*}}\{\limsup_{n\to\infty} x_n^*(v^{**})\} = \limsup_{n\to\infty} x_n^*(x_0^{**}) \geq \alpha/2 > 0$$

Sup–limsup Theorem ⇒ Compactness

- If K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

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Weak Compactness through I-generation

Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For any covering $K \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of closed convex subsets $D_n \subset K$, we have*

$$\overline{\cup_n^{\infty} \overline{D_n}^{w^*}}^{\|\cdot\|} = \overline{K}^{w^*}.$$

- The proof uses Krein Milman and Bishop Phelps theorems

Fonf-Lindenstrauss = Simons

Theorem (Cascales, Fonf, Troyanski and J.O., J.F.A.-2010)

Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

1. *For any covering $B \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of convex subsets $D_n \subset K$, we have*

$$\overline{\bigcup_{n=1}^{\infty} \overline{D_n}^{w^*}}^{\|\cdot\|} = K.$$

2. $\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in K} (\limsup_k g(x_k))$ *for every sequence $\{x_k\} \subset B_X$.*
3. $\sup_{f \in B} (\limsup_k f(x_k)) \geq \inf_{\sum \lambda_i = 1, \lambda_i \geq 0} (\sup_{g \in K} g(\sum \lambda_i x_i))$ *for every sequence $\{x_k\} \subset B_X$.*

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F. Delbaen problem

Let C be a convex, bounded and closed, but not weakly compact subset of the Banach space E with $0 \notin C$. The following problem has been posed by F. Delbaen motivated by risk measures theory:

Question

Is it possible to find a linear functional not attaining its minimum on C and that stays strictly positive on C?

Example (R. Haydon)

In every non reflexive Banach space there is a closed, convex and bounded subset C with non void interior and $0 \notin C$ such that every linear form $x^* \in E^*$ such that $x^*(C) > 0$ attains its minimum on C.

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$0 < \varepsilon < 1/4$

$\varepsilon \le \delta < 1$

$\frac{1+\delta}{M} < \varepsilon \le \frac{2}{M} < 1$

$g(z) = \varepsilon\delta + (1-\delta)\, g(y')$

$\geq \varepsilon\delta + (1-\delta) > \varepsilon > 1/M$

$\Downarrow$

$g(z') > 1/M$

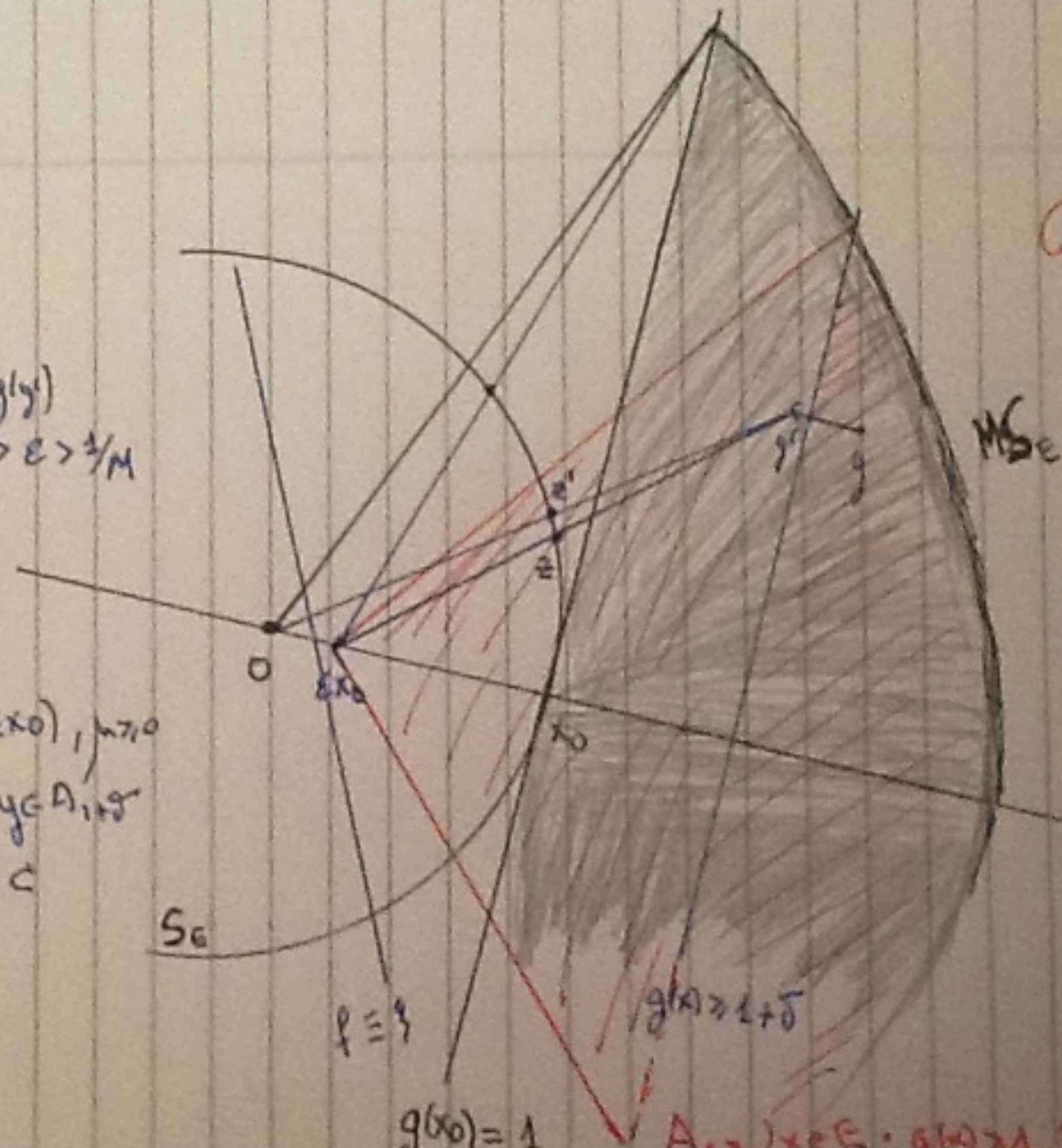
$\Downarrow$

Si $f(C) > 0 \Rightarrow f(A_1) > 0$

$\forall c \in C$, $c = \varepsilon x_0 + \mu(y - \varepsilon x_0)$, $\mu \geq 0$ and $y \in A_{1+\delta}$

and f attains its inf on C at εx_0

$C = \overline{co}\{\varepsilon x_0, A_{1+\delta}\}$



$A_1 = \{x \in E : g(x) \geq 1, \|x\| \leq M\}$

$f \in E^*$ $f(A_1) > 0 \underset{\text{wlog}}{\Rightarrow} \|f - g\| \leq 3/M$

$A_{1+\delta} = \{x \in E : g(x) \geq 1+\delta, \|x\| \leq M\}$

Positive results

Theorem (Birthday's Theorem)

Let E be a separable Banach space. Let C be a closed, convex and bounded subset of $E \setminus \{0\}$, $D \subset C$ a relatively weakly compact set of directions such that, for every $x^* \in E^*$, we have that

$$\inf\{x^*(c) : c \in C\}$$

is attained at some point of C whenever

$$x^*(d) > 0 \text{ for every } d \in D.$$

Then C is weakly compact.

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Then C is weakly compact.

Unbounded Simon's inequality

Theorem (Simons's Theorem in $\mathbb{R}^{\mathbf{X}}$)

Let X be a nonempty set, let (f_n) be a pointwise bounded sequence in $\mathbb{R}^X$ and let Y be a subset of X such that for every $g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}$ there exists $y \in Y$ with

$$g(y) = \sup\{g(x) : x \in X\}.$$

Then the following statements hold true:

$$\inf\{\sup_{x\in X} g(x) : g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}\} \leq \sup_{y\in Y}(\limsup_n f_n(y)) \qquad (1)$$

and

$$\sup\{\limsup_n f_n(x) : x \in X\} = \sup\{\limsup_n f_n(y) : y \in Y\}. \qquad (2)$$

Unbounded Rainwater's Theorem

Theorem (Unbounded Rainwater-Simons's theorem)

If E is a Banach space, $B \subset C$ are nonempty subsets of E^* and (x_n) is a bounded sequence in E such that for every

$$x \in \mathrm{co}_\sigma\{x_n : n \geq 1\}$$

there exists $b^* \in B$ with $\langle x, b^* \rangle = \sup\{\langle x, c^* \rangle : c^* \in C\}$, then

$$\sup_{b^* \in B} \left(\limsup_n \langle x_n, b^* \rangle \right) = \sup_{c^* \in C} \left(\limsup_n \langle x_n, c^* \rangle \right).$$

As a consequence

$$\sigma(E, B) - \lim_n x_n = 0 \;\Rightarrow\; \sigma(E, C) - \lim_n x_n = 0.$$

Unbounded Godefroy's Theorem

Theorem (Unbounded Godefroy's Theorem)

Let E a Banach space and B a nonempty subset of E^*. Let us assume there is a relatively weakly compact subset $D \subset E^*$ such that:

1. $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$
2. *For every $x \in E$ with $x(d^*) < 0$ for all $d^* \in D$ we have $\sup\{x(c^*) : c^* \in B\} = x(b^*)$ for some $b^* \in B$.*
3. *For every convex bounded subset $L \subset E$ and every $x^{**} \in \overline{L}^{\sigma(E^{**}, B \cup \overline{D}^{w})}$ there is a sequence (x_n) in L such that $\langle x^{**}, z^* \rangle = \lim_n \langle x_n, z^* \rangle$ for every $z^* \in B \cup \overline{D}^{w}$*

Then

$$\overline{\mathrm{co}(B)}^{w^*} \subset \bigcup \{\overline{\mathrm{co}(B)}^{\|\cdot\|} + \lambda \overline{\mathrm{co}(D)}^{\|\cdot\|} : \lambda \in [0, +\infty)\}.$$

Theorem (Conic Godefroy's Theorem)

Let E a Banach space and B a nonempty subset of E^*. Let us assume $0 \notin \overline{\mathrm{co}(B)}^{\|\cdot\|}$ and fix $D \subset B$, a relatively weakly compact set so that:

1. *For every $x \in E$ with $x(d^*) > 0$ for every $d^* \in D$, we have $\inf\{x(c^*) : c^* \in B\} = x(b^*) > 0$ for some $b^* \in B$.*
2. *For every convex bounded subset $L \subset E$, and every $x^{**} \in \overline{L}^{\sigma(E^{**}, B \cup \overline{D}^{w})}$, there is a sequence (x_n) in L such that $\langle x^{**}, z^* \rangle = \lim_n \langle x_n, z^* \rangle$, for every $z^* \in B \cup \overline{D}^{w}$.*

Then the norm closed convex truncated cone C generated by B, i.e. $C := \overline{\bigcup\{\lambda \mathrm{co}(B) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}$, is w^*-closed.

Theorem

Let E be a separable Banach space without copies of $\ell^1(\mathbb{N})$,

$$f : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$

norm lower semicontinuous, convex and proper map, such that

$$\text{for all } x \in E, \quad x - f \text{ attains its supremum on } E^*.$$

Then the map f is w^*-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, \mu])$ is w^*-compact.

Dual variational problems

Theorem

Let E be a Banach space,

$$f : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$

convex, proper and lower semicontinuous map with either a weakly Lindelöf-Σ or a strongly pseudocompact epigraph, such that

$$\text{for all } x \in E, \quad x - f \text{ attains its supremum on } E^*.$$

Then f is w^*-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, \mu])$ is w^*-compact.

Dual variational problems

Theorem

Let E be a Banach space without copies of $\ell^1(\mathbb{N})$,

$$f : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$

convex, proper and norm lower semicontinuous map with w^*-K-analytic epigraph, such that

$$\text{for all } x \in E, \quad x - f \text{ attains its supremum on } E^*.$$

Then f is w^*-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, \mu])$ is w^*-compact.

THANK YOU VERY MUCH FOR YOUR ATTENTION AND MY FAVORITE POEM FOR PETAR:

- Cultivo la rosa blanca I grow the white rose
- tanto en julio como en enero, as much in July as January,
- para el amigo sincero for the real friend
- que me da su mano franca. who gives me his frank hand.
- Y para el cruel que arranca And for the cruel who drags
- el corazón con que vivo, the heart with I am living,
- cardo ni oruga cultivo neither thistle or larve I am growing
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