# The Slice Localization Theorem

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# Contents

# The coauthors

- Antonio José Guirao (Pointwise LUR)
- Matias Raja (Uniform rotundity)
- Richard Smith (Strict convexity)
- Stanimir Troyanski (all contents)


A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia *A* Nonlinear Transfer Technique for Renorming Springer LNM 1951, 2009.

J. Orihuela and S. Troyanski, *Devilles's master lemma and Stone discretness in renorming theory* Journal Convex Analysis 16, 2009

J. Orihuela and S. Troyanski, *LUR renormings through Deville's master Lemma* RACSAM 103 (1), 2009 75–85

# Summary

- The Valencia and Murcia Seminar in Functional Analysis, 25 years of research
- Renorming techniques without coordinates.
- Nonlinear transfer techniques for renormings.
- Interplay between geometry and topology in Banach spaces.

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**Theorem** (Lindenstrauss) $X$ reflexive Banach space
Then there exists $T: X \longrightarrow c_0(\Gamma)$ linear, continuous
one to one operator

Valdivia approach $\{x_\gamma, f_\gamma\}$ M-basis

$$T(x) := (f_\gamma(x))_{\gamma \in \Gamma}$$

↳ $|||x|||^2 := \|x\|^2 + \|Tx\|^2$ is equivalent strictly convex norm.

↳ $X$ admits equivalent LUR norm.

$$S(x,y) := 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2$$

$S(x,y) = 0 \Rightarrow x = y$ (strictly convex)

$S(x_n, x) \to 0 \Rightarrow \lim_{n\to\infty} x_n = x$ (LUR)

$S(x_n, y_n) \to 0 \Rightarrow \|x_n - y_n\| \to 0$ (UR)

$S(x_n, x) \to 0 \Rightarrow T_p - \lim_{n\to\infty} x_n = x$ ($T_p$-LUR) ($X \hookrightarrow \ell^\infty(\Gamma)$)

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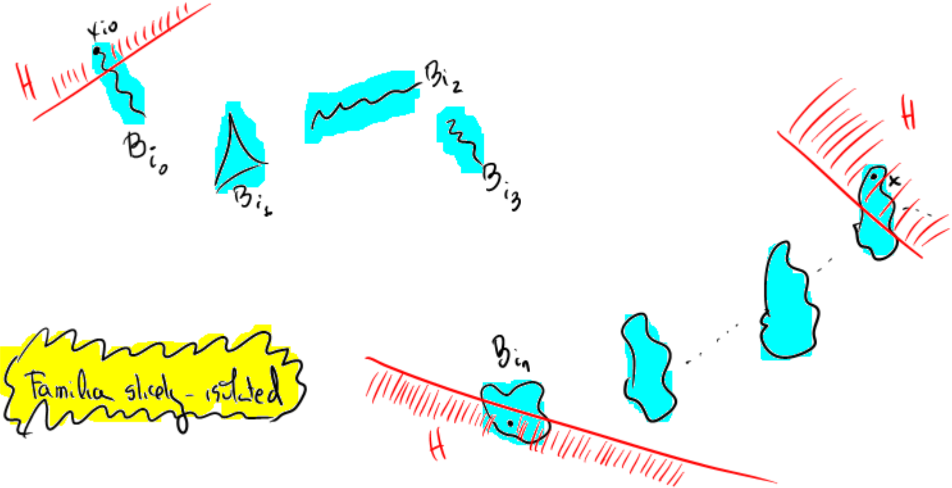
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Familia slicely - isolated

# The Connection Lemma.

## Lemma (Connection lemma)

*If $\mathcal{B}$ is a uniformly bounded and $\sigma(X,F)$-slicely isolated family of subsets of $X$, there is an equivalent $\sigma(X,F)$- lower semicontinuous norm $\|\cdot\|_{\mathcal{B}}$ such that*

$$\lim_n \left(2\|x_n\|_{\mathcal{B}}^2 + 2\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

*with $x \in B_0 \in \mathcal{B}$ implies that:*

1. *There is $n_0 \in \mathbb{N}$ such that $x_n, \frac{1}{2}(x_n + x) \notin \overline{\mathrm{co} \bigcup \{B : B \neq B_0; B \in \mathcal{B}\}}$ for all $n \geq n_0$.*
2. *For every positive $\delta$ there is $n_\delta \in \mathbb{N}$ such that $x_n \in \mathrm{co}(B_0) + \delta B_X$ whenever $n \geq n_\delta$.*

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## Corollary

*In a normed space $X$, with a norming subspace $F$ in $X^*$, we have an equivalent $\sigma(X, F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, there are slicely isolated families for the $\sigma(X, F)$ topology*

$$\{\mathcal{B}_n : n = 1, 2, ...\}$$

*such that for every $x$ in $X$ and every $\epsilon > 0$ there is some positive integer $n$ with the property that*

$$x \in B \in \mathcal{B}_n \text{ and that } \| \cdot \| - diam(B) < \epsilon$$

# Pointwise LUR renorming

## Theorem

*Let $X$ be a subspace of the Banach space $l^\infty(\Gamma)$. They are equivalent:*

1. *$X$ admits an equivalent $T_p$-locally uniformly rotund norm.*
2. *There is a metric $\rho$ on $X$ generating a topology finer then $T_p$ and $Id : X \to (X, \rho)$ $\sigma$-slicely continuous.*
3. *The pointwise convergence topology $T_p$ on $X$ has a $\sigma$-slicely isolated network.*
4. *There is a sequence of sets $(A_n)$ in $X$ and families of $T_p$-open half spaces $(\mathcal{H}_n)$ such that the slicing families of sets $(A_n \cap \mathcal{H}_n)$ verifies:*
    - *For every $T_p$-neighbourhood of the origin $W$ and every $x \in X$ there is $p \in \mathbb{N}$ such that $x \in A_p \cap H_0$ for some $H_0 \in \mathcal{H}_p$ and $A_p \cap H \subset x + W$ whenever $x \in H \in \mathcal{H}_p$*

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# *Weak*$^*$ LUR renorming

**Theorem (Matias Raja)**

*A compact space $K$ has a $\sigma$-discrete network (i.e. it is descriptive) if, and only if, $C(K)^*$ admits an equivalent $w^*$-LUR norm*

# $\sigma$-slicely isolated network implies $T_p$ LUR renorming

- Let $\mathcal{M} = \cup_{n=1}^{\infty} \mathcal{M}_n$ the $\sigma$-slicely isolated network for $T_p$.
- Take for every $n \in \mathbb{N}$ open half spaces $\mathcal{H}_n$ such that:
- $\cup \mathcal{M}_n \subset \cup \mathcal{H}_n$ and $H \in \mathcal{H}_n$ meets just one element in $\mathcal{M}_n$
- Apply the Connection Lemma to get $\| \cdot \|_{n,r} := \| \cdot \|_{\mathcal{H}_n, (\cup \mathcal{M}_n) \cap B(0,r)}$
- Glue the countable information we have: $|||\cdot|||^2 := \sum_{n,r=1}^{\infty} c_{n,r} \| \cdot \|_{n,r}^2$
- Given $\lim_n (2|||x_n|||^2 + 2|||x|||^2 - |||x + x_n|||^2) = 0$, a convex $T_p$ neighbourhood of the origin $W$, $M_0 \in \mathcal{M}_p$ with $x \in M_0 \subset x + W/2$
- If $\|x\| < r$, since $\lim_n (2\|x_n\|_{p,r}^2 + 2\|x\|_{p,r}^2 - \|x + x_n\|_{p,r}^2) = 0$
- we have $x, x_n \in \mathrm{co}(M_0) + B(0, \delta)$ for $n \geq n_0^\delta$
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# Nonlinear transfer for $T_p$ LUR renorming

**Theorem**

*Let $X \subset l^\infty(\Gamma)$ and*

$$\Phi : X \to (Y, \rho)$$

*a $\sigma$-slicely continuous map. If there is a sequence of sets $(D_n)$ in $Y$ such that for every $T_p$-neighbourhood of the origen $W$ and $x \in X$ there is some $\delta > 0$, $p \in \mathbb{N}$ with*

$$\Phi x \in D_p \text{ and } \Phi^{-1}(D_p \cap B_\rho(\Phi x, \delta)) \subset x + W,$$

*then $X$ admits an equivalent $T_p$-LUR norm.*

**Corollary (Moltó, Orihuela, Troyanski, Valdivia)**

*If $X$ has an F-smooth norm $\|\cdot\|$ with dual norm G-smooth on the set of norm attaining functionals, then $X$ admits an equivalent LUR norm*

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**Theorem**

*$X \subset l^\infty(\Gamma)$, $Y \subset l^\infty(\Delta)$ normed spaces, $\Phi : X \to Y$ be a map with sequences of sets $(A_n)$ in $X$, $(D_n)$ in $Y$ such that:*

- *For every $T_p$-open half space $G \subset Y$, $x \in X$ with $\Phi x \in G$ there is $p \in \mathbb{N}$ and a $T_p$-open half space $H \subset X$ with*

$$x \in A_p \cap H \text{ and } \Phi(A_p \cap H) \subset G$$

- *For every $T_p$-open half space $H \subset X$, $y \in Y$ with $\Phi^{-1} y \cap H \neq \emptyset$ there is $q \in \mathbb{N}$, a $T_p$-open half space $G \subset Y$ with*

$$y \in D_q \cap G \text{ and } \Phi^{-1}(D_q \cap G) \subset H$$

*Then $X$ admits an equivalent $T_p$ LUR norm if, and only if $Y$ does it.*

# The Slice Localization Theorem.

**Theorem**

*Given a bounded subset $A \subset X$ and a family $\mathcal{H}$ of open half spaces slicing $A$ there is an equivalent norm $\|\cdot\|_{\mathcal{H},A}$ such that: for $(x_n)$ in $X$ and $x \in A \cap H$, with $H \in \mathcal{H}$, the LUR condition*

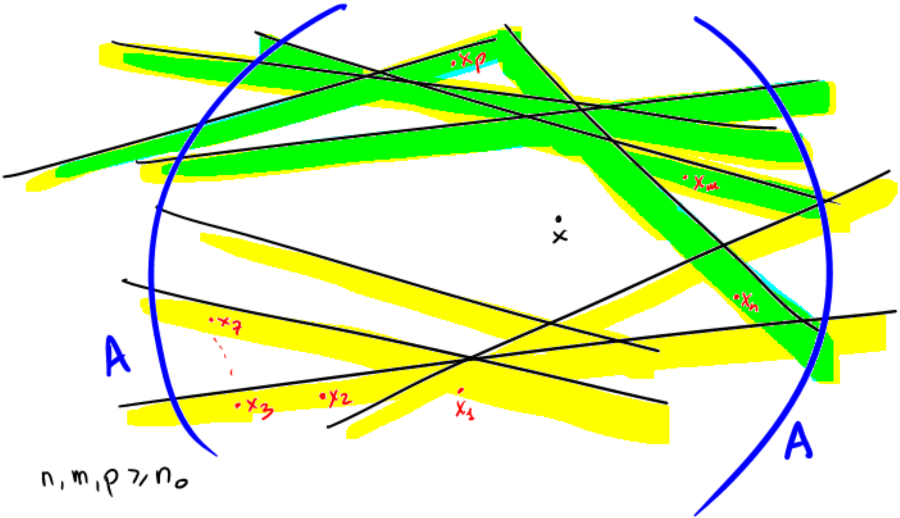
$$\lim_n (2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x + x_n\|^2_{\mathcal{H},A}) = 0$$

*implies the existence of a sequence of half spaces $\{H_n \in \mathcal{H}\}$:*

- *There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.*
- *For every $\delta > 0$ there is some $n_\delta$ such that*

$$x, x_n \in \mathrm{co}(A \cap H_n)) + B(0, \delta)$$

*for all $n \geq n_\delta$*



$\cdot x_p$
$\cdot x_m$
$\cdot x_n$
$x$
A
A
$\cdot x_3$
$\cdot x_2$
$x_1$
$n, m, p \geqslant n_0$

# Application for LUR renorming

**Theorem (Bing-Nagata-Stone meet renorming)**

*Let $X$ be a normed space with a norming subspace $F \subset X^*$. $X$ admits an equivalent $\sigma(X, F)$-lower semicontinuous and LUR norm if, and only if, the norm topology admits a basis where $\mathcal{B}_n$ is norm discrete and $\sigma(X, F)$-slicely isolated family for every $n \in \mathbb{N}$.*

**Theorem (Matias Raja's formulation)**

*Let $X$ be a as above. We assume that there are subsets $(A_n)$ such that for every $x$ and $\varepsilon > 0$ we can find $p \in \mathbb{N}$ and a $\sigma(X, F)$-open half space $H$ such that $x \in A_p \cap H$ and $diam(A_p \cap H) \leq \epsilon$. Then $X$ admits an equivalent $\sigma(X, F)$-lower semicontinuous LUR norm.*

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# Strictly convex renormings

**Theorem (Slice localization for the rotundity condition )**

*Given a bounded subset $A \subset X$ and a family $\mathcal{H}$ of open half spaces slicing $A$, there is an equivalent norm $\|\cdot\|_{\mathcal{H},A}$ such that: for $x, y$ in $X$ and $x \in A \cap H$, with $H \in \mathcal{H}$, the R condition*

$$2\|x\|^2_{\mathcal{H},A} + 2\|y\|^2_{\mathcal{H},A} - \|x+y\|^2_{\mathcal{H},A} = 0$$

*implies the existence of half spaces $(H_n \in \mathcal{H})$:*

- *Both $x, y \in H_n, n = 1, 2, \cdots$ if $y \in A$.*
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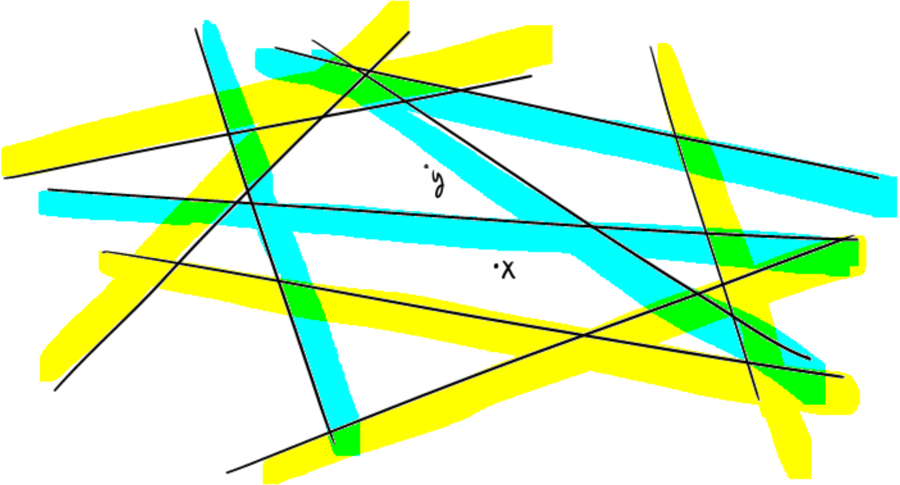

y
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# Strictly Convex Renorming Present.

**Theorem ( Birthday's Theorem)**

*Let $(X, \|\cdot\|)$ be a normed space, $F$ a norming subspace in $X^*$. $X$ admits an equivalent $\sigma(X, F)$-lower semicontinuous rotund norm if, and only if,*
*there are families of $\sigma(X, F)$-open half spaces $\mathcal{H}_n$ in $X$ $(*)$-separating points in $X$, i.e.*

- *For every two different points $x$ and $y$ in $X$ there is some integer $p$ with*
  1. $\{x, y\} \cap H_0 \neq \emptyset$ *for some* $H_0 \in \mathcal{H}_p$
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# Half-spaces $(*)$-separation $\Rightarrow \langle$ strictly convex $\rangle$

- $\mathcal{H}_n$ families of half-spaces $(*)$-separating $X$.
- For every $n, R > 0$ we apply SLT to construct an equivalent norm $\|\cdot\|_{n,R}$ for non empty slices in $\mathcal{H}_n \cap B(0,R)$.
- $\|x\|_{n,R} = \|y\|_{n,R} = \|(x+y)/2\|_{n,R} \Rightarrow x, y \in H' \in \mathcal{H}_n$ for some $H'$ whenever one of them is in $\cup\{H : H \in \mathcal{H}_n\}$ and both are in $B(0,R)$.
- If we define an equivalent norm on $X$ by the expression:

$$|||x|||^2 := \sum_{n=1,R=1}^{\infty} c_{n,R}\|x\|_n^2$$

  for every $x \in X$, where $(c_{n,R})$ has beeen chosen accordingly for the uniform convergence of the series on bounded sets.
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# Nonlinear Transfer for Strictly Convex Renorming

## Theorem

*$X \subset l^\infty(\Gamma)$ and $Y \subset l^\infty(\Delta)$ normed spaces,
$\Phi : X \to Y$ be a one to one map
with an increasing sequences of sets*

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \in X$$

*such that:*

- *For every $T_p$-open half space $G \subset Y$, $x \in X$ with $\Phi x \in G$ there is $p \in \mathbb{N}$ and a $T_p$-open half space $H \subset X$ with*

$$x \in A_p \cap H \text{ and } \Phi(A_p \cap H) \subset G$$

*Then $X$ admits an equivalent $T_p$-lower semicontinuous strictly convex norm whenever $Y$ has it.*

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*$\Phi : X \to Y$ be a one to one map*
*with a sequences of convex sets $(A_n)$ in $X$ such that:*

- *For every $T_p$-open half space $G \subset Y$, $x \in X$ with $\Phi x \in G$ there is $p \in \mathbb{N}$ and a $T_p$-open half space $H \subset X$ with*

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# Nonlinear Transfer for Strictly Convex Renorming

**Theorem**

*$X$ normed space and $\theta_n : X \to l^\infty(\Gamma)$ bounded on bounded sets such that:*

$$|\theta_n(\frac{u+v}{2})| \leq |\frac{\theta_n u + \theta_n v}{2}| \text{ in } l^\infty(\Gamma)$$

*for all $u, v \in X, n \in \mathbb{N}$. If given $x \neq y$ in $X$ there is $p \in \mathbb{N}, A \subset \Gamma$ with $\theta_p x_{\restriction A} \neq \theta_p y_{\restriction A}$ and either*

- $\min\{|\theta_p x(\alpha)| : \alpha \in A\} > |\theta_p x(\gamma)|$ *for all* $\gamma \in \Gamma \setminus A$.
- $\min\{|\theta_p y(\alpha)| : \alpha \in A\} > |\theta_p y(\gamma)|$ *for all* $\gamma \in \Gamma \setminus A$.

*Then $X$ admits an equivalent strictly convex norm*

# Internal characterization for spaces $C(K)^*$

**Theorem**

*Let $K$ be an scattered compact space.*
*$C(K)^*$ admits an equivalent dual and strictly convex norm if, and only if, there are families of open sets*

$$(\mathcal{U}_n)_{n=1}^{\infty}(*) - \text{ separating points of } K,$$

*i.e.*

- *For every two different points $x$ and $y$ in $K$ there is some integer $p$ such that*
  1. $\{x, y\} \cap U_0 \neq \emptyset$ *for some* $U_0 \in \mathcal{U}_p$
  2. $\{x, y\} \cap U$ *has at most one element for every* $U \in \mathcal{U}_p$

## Theorem (UR case)

*Given a bounded subset $A \subset X$ and open half spaces $\mathcal{H}$ slicing $A$,*
*there is an equivalent norm $\|\cdot\|_{\mathcal{H},A}$ such that:*
*for $(x_n) \in A \cap \mathcal{H}$, $(y_n)$ bounded sequence in $X$, the UR condition*

$$\lim_n (2\|x_n\|^2_{\mathcal{H},A} + 2\|y_n\|^2_{\mathcal{H},A} - \|y_n + x_n\|^2_{\mathcal{H},A}) = 0$$

*implies, for every $\varepsilon > 0$, the existence of half spaces $\{H_n^\varepsilon \in \mathcal{H}, n \in \mathbb{N}\}$:*

- $y_n, x_n \in (H_n^\varepsilon + \varepsilon)$ *eventually if $(y_n)$ is eventually in $A$.*
- *In any case, for every $\delta > 0$*

  $$y_n, x_n \in \overline{\mathrm{co}(A \cap (H_n^\varepsilon + \varepsilon))} + B(0, \delta)$$

  *for all $n$ big enough.*

# Uniformly rotund renorming.

## Theorem (UR case)

*A Banach space admits an equivalent uniformly rotund norm if, and only if, for every* $\epsilon > 0$ *we have:*

$$B_X = \bigcup_{n=1}^{\mathbb{N}_\epsilon} B_n^\epsilon$$

*and every set* $B_n^\epsilon$ *is uniformly* $\epsilon$*-denting; i.e there is* $\delta_n^\epsilon > 0$ *such that for every* $x \in B_n^\epsilon$ *there is an open half space* $H$ *with*

- $\mathrm{diam}(H \cap B_n^\epsilon) < \epsilon$
- $B(x, \delta_n^\epsilon) \subset H \cap B_n^\epsilon$