James boundaries in Banach spaces

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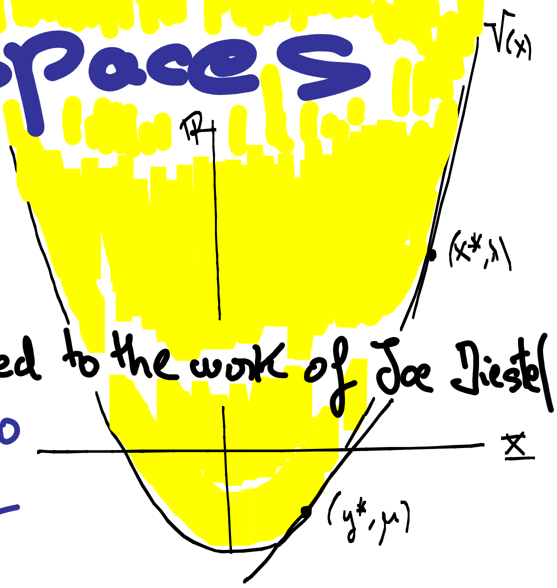

B. Cascales., I. Namioka, and J.O. The Lindelof property in Banach spaces, Studia Math. 154 (2003), no. 2, 165 {192. MR MR1949928 (2003m:54028)

B. Cascales, M. Muñoz and J.O., James boundaries and σ-fragmented selectors, Studia Math. 2008. 188 - 2, 97-122

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M. Ruiz Galan and J.O. A convex non linear James Theorem. Preprint

Simons' inequality

Lemma (Simons)

Let K be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(K)$. If B is a subset of K such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b^* \in B$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in K\} = \sum_{n=1}^{\infty} \lambda_n z_n(b^*),$$

then

$$\sup_{b^* \in B} \{\limsup_{n\to\infty} z_n(b^*)\} \geq \inf\left\{\sup_K w : w \in co\{z_n : n \in \mathbb{N}\}\right\}$$

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A central problem

Problem: When is a boundary strong?

Let $K \subset X^*$ be a w^*-compact convex subset of a dual Banach space and $B \subset K$ be a boundary of K, i.e. for any $x \in X$ there is $f \in B$ with

$$sup\{g(x) : g \in K\} = f(x).$$

Under which conditions a boundary verifies $\overline{coB}^{\|\cdot\|} = K$?

- A subset $B \subset B_{X^*} = \{x^* \in X^* \; ; \; \| x^* \| \leq 1\}$ is a boundary for B_{X^*} if for any $x \in X$, there is $x^* \in B$ such that $x^*(x) = \| x \|$.
- A simple example of boundary is provided by Ext (B_{X^*}) the set of extreme points of B_{X^*}.

$B \subset B_{X^*}$ boundary, $\tau_p(B)$ topology of pointwise convergence on B.

When is a boundary strong?

Let X be Banach space, $B \subset B_{X^*}$ boundary.

When do we have $B_{X^*} = \overline{\mathrm{co} B}^{\| \|}$?

The boundary problem (Godefroy)

Let H be a norm bounded and $\tau_p(B)$-compact subset of X.

Is H weakly compact?

$B_{X^*} = \overline{\mathrm{co} B}^{w^*}$ = replace w^* by $\| \|$.

Extremal test = lift compactness.

$B_{X^*} = \overline{\mathrm{co}\, B}^{\| \|}$?

Is H weakly compact?

1. 1952, Grothendieck: $X = C(K)$ and $B = \mathrm{Ext}(B_{C(K)^*})$;
2. 1963, Rainwater: $B = \mathrm{Ext}(B_{X^*})$, H $\tau_p(B)$-seq.compact;
3. 1972, James: $B_X \subset B_{X^{**}}$ boundary;
4. 1972, Simons: H $\tau_p(B)$-seq.compact and B arbitrary;
5. 1974, de Wilde: H convex and B arbitrary;
6. 1982, Bourgain-Talagrand: $B = \mathrm{Ext}(B_{X^*})$, arbitrary H.

The question?

K conv. w^*-comp. $B \subset K$ boundary, conditions (X, B or K?) $\Rightarrow K = \overline{\operatorname{co} B}^{\|\ \|}$.

What are the techniques that have been used?

1. 1976, Haydon [Hay76]: if $\ell^1 \not\subset X$ then $K = \overline{\operatorname{co} \operatorname{Ext} K}^{\|\ \|}$ using independent sequences and Bishop-Phelps theorem.
2. 1987, Namioka [Nam87]: $K \subset X^*$ is norm fragmented, then $\overline{\operatorname{co} K}^{w^*} = \overline{\operatorname{co} K}^{\|\ \|}$ using the existence of barycenters.
3. 1987, Godefroy [God87]: if $B \subset K$ is norm separable then $K = \overline{\operatorname{co} B}^{\|\ \|}$ using Simons' inequality.
4. 1987, Godefroy [God87] using Simons' inequality proves that if X is separable and $\ell^1 \not\subset X$ then $K = \overline{\operatorname{co} B}^{\|\ \|}$.
5. 2003, Fonf-Lindenstrauss used the so-called (I)-formula.

Godefroy, [God87]

X Banach space, $K \subset X^*$ w^*-compact convex set, $B \subset K$ a James boundary for K. We have the formula

$$K = \overline{\operatorname{co}(B)}^{\|\ \|}.$$

provided that for every convex bounded subset $C \subset X$ and every $x^{**} \in \overline{C}^{\sigma(X^{**},B)}$ there is a sequence $(x_n)_n$ in C such that $x^{**} = \sigma(X^{**}, B) - \lim_n x_n$.

G. Rodé, [Rod81]
V. P. Fonf, J. Lindenstrauss and R. R. Phelps, [FLP01] -new proof

X Banach space, $K \subset X^*$ w^*-compact convex set, $B \subset K$ a James boundary for K, i.e., for every $x \in X$, there is $b \in B$ such that

$$b(x) = \sup\{g(x) : g \in K\}.$$

If B is norm separable, then we have the formula

$$K = \overline{\text{co}(B)}^{\|\ \|}.$$

We show next how the Bishop-Phelps theorem may be used in the study of boundaries.
Theorem 5.7 ([147]). Let X be a real Banach space, ȿ ʉ X* be convex and w*-compact and ɺ ʉ ȿ be a norm separable subset such that for each ɢ e X there exists f e ɺ with f(x) = suplg(x): g e K]. Then ȿ is the closure in the norm topology of the convex hull of ɺ .
Rode obtained this result from his minimax theorem of "superconvex analysis" (cf. [146]); the proof we present here seems to be new.

MR1998108 (2004g:46021) 46B20 (46B50 52A07)
Fonf, Vladimir P. (IL-BGUN); **Lindenstrauss, Joram** (IL-HEBR-IM)
Boundaries and generation of convex sets. (English summary)
Israel J. Math. **136** (2003), 157–172.

Let X be a Banach space, and K be a weak* compact convex subset of the dual space X^*. A boundary of K is a subset B of K such that every $x \in X$ attains its supremum on K at some point of B.

It is shown in this article that in the above notation, the boundary B "(I)-generates" K in full generality, in the following sense: for every representation of B as a countable union $B = \bigcup_{n\geq 1} C_n$, the set K is equal to the norm-closed convex hull of the union of the weak* closed convex hulls of the sets C_n. This "(I)-generation" is an intermediate notion between being the norm closed convex hull of B and the weak* closed convex hull of B. Its interest lies in the fact that it opens the way to a remarkably simple proof of James' characterization of weakly compact sets in the separable case, as well as of several known extensions of this theorem, usually shown through Simons' inequality. Interesting new results are also proven, such as this one: if X is separable and nonreflexive, the set of linear functionals which do not attain their norm is not a subset of a proper operator range (although it can be meager, e.g. when X has the Radon-Nikodým property).

Our results

1. We prove that when B is "descriptive" then $K = \overline{\text{co}\, B}^{\|\ \|}$: this extends results by Godefroy, Contreras-Payá and *solve* problems asked by Plichko and Talagrand.
2. We apply the techniques developed to give new characterizations of Asplund spaces.
3. We prove that Fonf-Lindenstrauss techniques can be reduced to the *old techniques* coming from Simons inequality: there are no new techniques nor can be stronger applications derived from Fonf-Lindenstrauss.
4. We characterize Banach spaces X without copies of ℓ^1 via *boundaries* extending the results by Godefroy for the separable case.
5. For Asplund spaces we characterize boundaries for which $K = \overline{\text{co}\, B}^{\|\ \|}$. We extend in several different ways results by Namioka and Fonf.
6. We obtain non linear convex versions of James weak compactness theorem.

Our first result: answer to a question by Talagrand

Theorem [Cascales, Namioka and J.O., 2003]

Let X be a Banach space, $K \subset X^*$ a w^*-compact and weakly Lindelof subset, then

$$\overline{co(K)}^{w^*} = \overline{co(B)}^{\|\cdot\|}$$

for any James boundary $B \subset K$ and $\overline{spanK}^{\|\cdot\|}$ is weakly Lindelof determined.

This answers an old question by Talagrand (1979) on the Lindelof property of a Banach space generated by a Lindelof and w^*-compact subset.

Answering a question by Plichko

Theorem, [Cascales, Muñoz and J.O.]

Let X be a Banach space, B a boundary for B_{X^*}, $1 > \varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t \in T} B(t, \varepsilon)$

- If $T = f(X)$ for a σ-fragmented map $f : X \longrightarrow X^*$ with $\|\cdot\|$-dist$(f(x), J(x)) < \varepsilon$ for every $x \in X$,where we are denoting wit J the duality mapping

$$J(x) = \{f \in B_{X^*} : f(x) = \|x\|\},$$

  then $X^* = \overline{spanT}^{\|\cdot\|}$ and X is an Asplund space.
- If (T, w) is countably K-determined (resp. K-analytic) then:
  - (i) $X^* = \overline{\text{span}\, T}^{\|\ \|}$ and X^* is weakly countably K-determined (resp. weakly K-analytic).
  - (ii) Every boundary for B_{X^*} is strong. In particular $B_{X^*} = \overline{\text{co}(B)}^{\|\ \|}$.

Descriptive boundaries $\ell^1 \not\subset X$

Theorem

Let X be a Banach space. The following statement are equivalent:

(i) $\ell^1 \not\subset X$;

(ii) for every w*-compact convex subset K of X^* and any boundary B of K we have $K = \overline{\mathrm{co}(B)}^{\gamma}$.

(iii) for every w*-compact convex subset K of X^* any w^*-K-analytic boundary is strong.

Proof.-

1. $(B,\gamma)^n$ is Lindelöf for every $n \in \mathbb{N}$ if X is Asplund, Namioka-B.C.
2. $\mathrm{co}(B)$ is γ-Lindelöf.
3. If $Z \subset X^*$ is γ-Lindelöf, then $\overline{Z}^{\gamma} = \overline{Z}^{\|\ \|}$.
4. $K \overset{\ell^1 \not\subset X}{=} \overline{\mathrm{co}(B)}^{\gamma} = \overline{\mathrm{co}(B)}^{\|\ \|}$. ■

Boundaries in Asplund spaces

Definition

Let X be a Banach space and $C \subset X^*$. We say that C is finitely-self-predictable (FSP in short) if there is a map

$$\xi : \mathscr{F}_X \to \mathscr{F}_{\mathrm{co}C}$$

from the family of all finite subsets of X into the family of all finite subsets of co$C$ such that for any increasing sequence $\sigma_n \subset \mathscr{F}_X$, $n = 1, 2, \ldots$, with

$$E = [\sigma_n]_{n=1}^{\infty}, \quad D = \cup_{n=1}^{\infty} \xi(\sigma_n),$$

we have

$$C|_E \subset \overline{\mathrm{co}(D|_E)}^{\|\ \|}.$$

Let X be an Asplund space and B be a boundary for X. TFAE:

1. B is strong.
2. B is FSP.

Let X be a Banach space. TFAE:

1. X is an Asplund space.
2. X admits an FSP boundary.
3. Any strong boundary $B \subset B_{X^*}$ is FSP.

Simons versus Fonf-Lindenstrauss

Theorem [Cascales, Fonf, Troyanski and J.O.]

Let X be a Banach space, $K \subset X^*$ be w^*-compact convex, $B \subset K$, TFAE:

1. For any covering $B \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of w^*-closed convex subsets $D_n \subset K$, we have
$$\overline{\cup_n^{\infty} \overline{D_n}^{w^*}}^{\|\cdot\|} = K.$$
2. $\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in K} (\limsup_k g(x_k))$ for every sequence $\{x_k\} \subset B_X$.
3. $\sup_{f \in B} (\limsup_k f(x_k)) \geq \inf_{\sum \lambda_i = 1, \lambda_i \geq 0} (\sup_{g \in K} g(\sum \lambda_i x_i))$ for every sequence $\{x_k\} \subset B_X$.

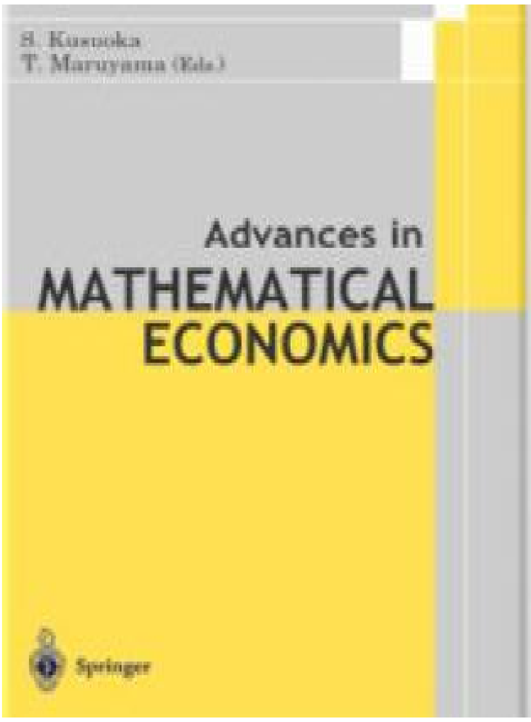
Simons versus Fonf-Lindenstrauss

Theorem [Cascales, Fonf, Troyanski and J.O.]

Let X be a Banach space not containing ℓ^1 and K be a w^*-compact convex subset of X^* with a James boundary B. Let $(z_n)_n$ be a bounded sequence in $(X^*, \gamma)'$, then

$$\sup_{f \in B} \left(\limsup_{k -> \infty} z_k(f)\right) = \sup_{g \in K} \left(\limsup_{k -> \infty} z_k(g)\right)$$

CONVEX AND RISK MEASURES



2 The Jouini-Schachermayer-Touzi Theorem

This section is devoted to the characterisation of a weak compactness theorem. The theorem is a generalisation of the beautiful result of James' on weakly compact sets, see [7]. The original proof of [9] followed the rather complicated proof of James. The proof below uses the homogenisation trick and allows to apply the original version of James' theorem. Let us recall this theorem

Theorem 2 (Jouini-Schachermayer-Touzi [9]) *If u is a concave monetary utility function satisfying the Fatou property then are equivalent:*

(1) *For each $\xi \in L^\infty$ there is a $\mathbb{Q} \in \mathscr{P}$ such that $u(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})$.*

(2) *If $(\xi, t) \in L^\infty(\Omega_1)$ there is a $\mathbb{Q}_1 \in \mathscr{S}_1$ such that $u_1(\xi, t) = \int_{\Omega_1} (\xi, t)\, d\mathbb{Q}_1$.*

(3) *The set $\mathscr{S}_1$ is weakly compact in $L^1(\Omega_1)$.*

(4) *The homogenisation u_1 satisfies the Lebesgue property. This means that for uniformly bounded sequences $(\theta_n)_n$ in $L^\infty(\Omega_1)$, converging in probability to say θ, we have $u_1(\theta_n) \to u_1(\theta)$.*

(5) *If ξ_n is a uniformly bounded sequence in L^∞ converging in probability to a function ξ, then $u(\xi_n) \to u(\xi)$, i.e. u has the Lebesgue property.*

(6) *For each $k \in \mathbb{R}$ the set $\{\mathbb{Q} \mid c(\mathbb{Q}) \le k\}$ is weakly compact (or uniformly integrable and closed) in L^1, in particular $c(\mu) = +\infty$ for non countably additive elements of $\mathscr{P}^{\mathbf{ba}}$.*

A non linear convex James theorem

Theorem [M. Ruiz and J.O.]

Let E be a Banach space with B_{E^*} w^*-sequentially compact, and $V : E \longrightarrow \mathbb{R} \cup \{\infty\}$ a proper convex l.s.c. function such that

$$\partial V(E) = E^*$$

and

$$\lim_{\|x\| -> \infty} \frac{V(x)}{\|x\|} = \infty.$$

Then the level sets $L_c := \{x \in E : V(x) \leq c\}$ are weakly compact for every $c \in \mathbb{R}$.

This answers a question by **S. Simons** after the erratum in the result of **B. Calvert and S. Fitzpatrick:** *In a nonreflexive space the subdifferential is not onto,* Math. Z. 189 (1985), 555-560 and 235 (2000), 627.

MAIN TOOL: A bounded subset of $(\mathfrak{X}, \|\cdot\|)$ B-space

If there is $(a_n) \subset A$ without w^* cluster point in $\mathfrak{X}$, ie $\overline{\{a_n : n = 1, 2 \dots\}}^{w^*} \setminus \{a_n : n = 1, 2 \dots\} \subset \mathfrak{X}^{**} \setminus \mathfrak{X}$

Then there exists a sequence $(x_n^*) \in B_{\mathfrak{X}^*}$ and $g_0 \in co_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell^\infty(A)$ with $\underline{\lim}_n x_n^*(a) \leq h(a) \leq \overline{\lim}_n x_n^*(a)$ $\forall a \in A$ we have that $g_0 - h$ doest not attains its supremum on A