Lebesgue property for convex risk measures on Orlicz spaces

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Abstract

We present a robust representation theorem for monetary convex risk measures $\rho : \mathcal{X} \to \mathbb{R}$ such that

$$\lim_n \rho(X_n) = \rho(X) \text{ whenever } (X_n) \text{ almost surely converges to } X,$$

$|X_n| \le Z \in \mathcal{X}$, for every $n \in \mathbb{N}$ and $\mathcal{X}$ is an arbitrary Orlicz space. The separable $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$ case of Jouini, Schachermayer and Touzi, [Jou-Scha-Tou06], as well as the non-separable version of Delbaen [Del09], are contained as a particular case here. We answer a natural question posed by Biagini and Fritelli in [Bia-Fri09]. Our approach is based on the study for unbounded sets, as the epigraph of a given penalty function associated with ρ, of the celebrated weak compactness Theorem due to R. C. James [Jam72].

1 Introduction

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with $\mathcal{X}$, a linear space of functions in $\mathbb{R}^\Omega$ that contains the constant functions. We assume here that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. The space $\mathcal{X}$ is going to describe all possible financial positions $X : \Omega \to \mathbb{R}$ where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. The problem of quantifying the risk of a financial position $X \in \mathcal{X}$ is modeled with function $\rho : \mathcal{X} \to \mathbb{R}$ that satisfy:

- Monotonicity : If $X \le Y$, then $\rho(X) \ge \rho(Y)$
- Cash invariance: If $m \in \mathbb{R}$ then $\rho(X + m) = \rho(X) - m$

Such a function ρ is called *monetary measure of risk*, see Chapter 4 in [Foll-Schi04]. When it is a convex function too, *i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y) \text{ for } 0 \le \lambda \le 1,$$

then ρ is called *convex measure of risk*. If $\mathcal{X}$ is a Frechet lattice, the convexity and monotonicity of ρ lead to continuity and subdifferentiability at all positions $X \in \mathcal{X}$ by the extended Namioka-Klee Theorem of [Bia-Fri09], as well as to strong representations of the form

$$\rho(X) = \max_{Y \in (\mathcal{X}')^+} \{\langle Y, -X\rangle - \rho^*(-Y)\},$$

for every $X \in \mathcal{X}$ whenever $\mathcal{X}$ is order continuous (see Theorem 1 and Corollary 1 in [Bia-Fri09]). Here $(\mathcal{X}')^+$ denotes the positive cone of continuous linear functionals, ρ^* is the Fenchel conjugate to ρ:

$$\rho^*(Y) = \sup_{X \in \mathcal{X}} \{\langle Y, X\rangle - \rho(X)\}$$

for all $Y \in \mathcal{X}'$, and $\langle \cdot, \cdot \rangle$ denotes the bilinear form for the duality between $\mathcal{X}$ and its topological dual $\mathcal{X}'$.
When the Frechet lattice $\mathcal{X}$ is not order continuous, for instance the Lebesgue space of bounded and measurable functions $\mathcal{X} = \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with its canonical essential supremum norm $\|\cdot\|_\infty$, we have the representation formula with a supremum instead of maximum:

$$\rho(X) = \sup_{Y \in (\mathcal{X}_n^\sim)^+} \{\langle Y, -X\rangle - \rho^*(-Y)\}, \tag{1}$$

where $(\mathcal{X}_n^\sim)^+$ denotes the positive cone of the order continuous linear functionals $\mathcal{X}_n^\sim \subset \mathcal{X}'$, whenever the risk measure ρ is $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$-lower semicontinuous, as seen in Proposition 1 in [Bia-Fri09]. A natural question posed by them is whether the sup in formula (1) is attained.

For risk measures defined on L^∞, the representation formula with supremum is equivalent to the so–called Fatou property, given in Theorem 4.31 in [Foll-Shci04]. The fact that the order continuity of ρ is equivalent to turning the sup into a max in (1) i.e:

$$\rho(X) = \max_{Y \in (L^1)^+} \{-\mathbb{E}[Y \cdot X] - \rho^*(-Y)\},$$

for every $X \in L^\infty$, is the statement of the so called Jouini-Schachermayer-Touzi Theorem in [Del09] (see Theorem 5.2 in [Jou-Sch-Tou06]).
S. Biagini and M. Fritelli show in [Bia-Fri09] that order continuity of the risk measure ρ is sufficient to turn a maximum in (1) for an arbitrary locally convex Frechet lattice $\mathcal{X}$. Our main contribution here shows that sequential order continuity is a necessary and sufficient condition for it when $\mathcal{X}$ is an arbitrary Orlicz space L^Ψ such that the order continuous linear functionals coincides with the Orlicz heart $\mathbb{M}^{\Psi^*}$, i.e., we present the Jouini-Schachermayer-Touzi Theorem for risk measures defined on Orlicz spaces.

A Young function Ψ is an even, convex function $\Psi : E \to [0, +\infty]$ with the properties: $\Psi(0) = 0, \lim_{x\to\infty} \Psi(x) = +\infty$ and $\Psi < +\infty$ in a neighborhood of 0. The Orlicz space L^Ψ is defined as:

$$L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : \exists \alpha > 0 \text{ with } \mathbb{E}_\mathbb{P}[\Psi(\alpha X)] < +\infty\}$$

with the Luxemburg norm on it: $N_\Psi(X) := \inf\{c > 0 : \mathbb{E}_\mathbb{P}[\Psi(\frac{1}{c}X)] \le 1\}$. With the usual pointwise lattice operations, $L^\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is a Banach lattice and we have inclusions: $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, $(L^\Psi)^* = L^{\Psi^*} \oplus G$ where G is the singular band and L^{Ψ^*} is the order continuous band identified with the Orlicz space L^{Ψ^*}, where

$$\Psi^*(y) := \sup_{x \in \mathbb{R}} \{yx - \Psi(x)\}$$

is the Young function conjugate to Ψ.

2 Lebesgue risk measures

Theorem 1 (Main result) *Let Ψ be a Young function with finite conjugate Ψ^* and*

$$\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}))^* \to \mathbb{R} \cup \{+\infty\}$$

be a $\sigma((\mathbb{L}^\Psi)^*, \mathbb{L}^\Psi)$-lower semicontinuous penalty function representing a finite monetary risk measure ρ as

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}.$$

The following are equivalent:

(i) *For each $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}$$

is attained.

(iii) *ρ is order sequentially continuous.*

Let us remark that order sequential continuity for a map ρ in $\mathbb{L}^\Psi$ is equivalent to having

$$\lim_n \rho(X_n) = \rho(X)$$

whenever (X_n) is a sequence in L^Ψ almost surely convergent to X and bounded by some $Z \in L^\Psi$, i.e. $|X_n| \le Z$ for every $n \in \mathbb{N}$. For that reason we say that a map $\rho : L^\Psi \to (-\infty, +\infty]$ verifies the Lebesgue property whenever it is sequentially order continuous.

When Ψ is a Young function that verifies the Δ_2 condition, i.e. if there are $t_0 > 0$ and $K > 0$ such that: $\Phi(2t) \le K\Phi(t)$ for all $t > t_0$, we have a result for the risk measures studied by P. Cheredito and T. Li., see [Che-Li09]. The Orlicz heart M^{Ψ^*} is the Morse subspace of all $X \in L^{\Psi^*}$ such that

$$\mathbb{E}_\mathbb{P}[\Psi^*(\beta X)] < +\infty$$

for all $\beta > 0$.

Corollary 2 *Let Ψ a Young function that verifies the Δ_2 condition and finite Ψ^*. Let $\rho : \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a finite convex risk measure with the Fatou property and*

$$\rho^* : \mathbb{L}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$$

its Fenchel–Legendre conjugate defined on the dual space. The following are equivalent:

(i) *For each $c \in \mathbb{R}$, $(\rho^*)^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in (\mathbb{M}^{\Psi^*})^+} \{\mathbb{E}[-XY] - \rho^*(-Y)\}$$

is attained.

(iii) *ρ is sequentially order continuous.*

(iv) *$\lim_n \rho(X_n) = \rho(X)$ whenever $X_n \nearrow X$ in $\mathbb{L}^\Psi$.*

(v) *Dom$(\rho^*) \subset \mathbb{M}^{\Psi^*}$.*

Corollary 3 *Let Ψ be a Young function with conjugate Ψ^* that verifies the Δ_2 condition. Let $\rho : \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a finite convex risk measure with the Fatou property and*

$$\rho^* : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to \mathbb{R} \cup \{+\infty\}$$

its Fenchel–Legendre conjugate defined on the dual space. The following are equivalent:

(i) *For each $c \in \mathbb{R}$, $(\rho^*)^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{L}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in (\mathbb{L}^{\Psi^*})^+} \{\mathbb{E}[-XY] - \rho^*(-Y)\}$$

is attained.

(iii) *ρ is sequentially order continuous.*

(iv) *$\lim_n \rho(X_n) = \rho(X)$ whenever $X_n \nearrow X$ in $\mathbb{L}^\Psi$.*

(v) *Dom$(\rho^*) \subset \mathbb{L}^{\Psi^*}$.*

F. Delbaen has proved we always are forced to have the compactness-continuity property ((i)-(iii) in Theorem 1) restricted to the duality $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$, whenever the convex risk measure ρ is defined on a rearrangement invariant solid space $\mathcal{X}$ such that $\mathcal{X} \setminus L^\infty \neq \emptyset$, as seen in Section 4.16 in [Del12]. This result is even true for non-continuous risk measures since F. Delbaen only asks for the property

$$\rho(X) \ge 0 = \rho(0) \text{ whenever } X \ge 0$$

instead of monotonicity. Our Theorem 1 complements Delbaen's results when looking for compactness-continuity properties in the duality $\langle \mathbb{M}^{\Psi^*}, \mathbb{L}^\Psi \rangle$.

3 Main tool

Theorem 4 (Perturbed James' Theorem) *Let E be a real Banach space and*

$$\alpha : E \longrightarrow \mathbb{R} \cup \{\infty\}$$

be a proper map such that

for every $x^* \in E^*$, $\quad x^* - \alpha$ attains its supremum on E.

Then, for each $c \in \mathbb{R}$, the corresponding sublevel set $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of E.

Theorem 4 has been recently proved by J. SaintRaymond [Sain12] solving our conjecture in [Ori-Rui02] where it was proved assuming that

$$\lim_{\|x\| \to +\infty} \frac{\alpha(x)}{\|x\|} = 0.$$

Theorem 4 were firstly presented at the meeting Analysis, Stochastics and Aplications, held at Viena in July 2010, to celebrate Walter Schachermayer's 60th Birthday, see http://www.mat.univie.ac.at/ anstap10/slides/Orihuela.pdf, for Banach spaces E with weak*-sequentially compact dual unit ball, as they are the Orlicz spaces that contain the domain of a penalty function α in our applications. Our approach contains classical James' result with new Simons's inequalities for unbounded sets as the epigraph

$$\mathrm{Epi}(\alpha) = \{(x, t) \in E \times \mathbb{R} : \alpha(x) \le t\}$$

of the penalty function α. Indeed we prove the following:

Theorem 5 (Inf-liminf Theorem in $\mathbb{R}^T$) *Let $\{\Phi_k\}_{k \ge 1}$ be a pointwise bounded sequence in $\mathbb{R}^T$ and suppose that Y is a subset of T satisfying the following boundary condition:*

for every $\Phi \in \mathrm{co}_\sigma\{\Phi_k : k \ge 1\}$ there exists $y \in Y$ with $\Phi(y) = \inf_T(\Phi)$.

where $\mathrm{co}_\sigma\{\Phi_n : n \ge 1\} := \{\sum_{n=1}^\infty \lambda_n \Phi_n :$ for all $n \ge 1$, $\lambda_n \ge 0$ and $\sum_{n=1}^\infty \lambda_n = 1\}$. Then

$$\inf_Y \left(\liminf_{k \ge 1} \Phi_k\right) = \inf_T \left(\liminf_{k \ge 1} \Phi_k\right).$$

Corollary 6 *If $Y \subset T$ are nonempty subsets of E^*, and $\{x_n\}_{n \ge 1}$ is a bounded sequence in the Banach space E such that*

for every $x \in \mathrm{co}_\sigma\{x_n : n \ge 1\}$ there exists $y^* \in Y$ with $y^*(x) = \inf_T(x)$,

(resp. $y^*(x) = \sup_T(x)$)
then

$$\inf_Y \left(\liminf_{n \ge 1} x_n\right) = \inf_T \left(\liminf_{n \ge 1} x_n\right).$$

(resp.

$$\sup_Y \left(\limsup_{n \ge 1} x_n\right) = \sup_T \left(\limsup_{n \ge 1} x_n\right))$$

Poster based on a joint work wit M. Ruiz Galán from Granada University (Spain). Mathematics and Financial Economics: Volume 6, Issue 1 (2012), 15-35. DOI 10.1007/s11579-012-0058-5

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Partially supported by the research grant MTM2011-25377, Ministerio Ciencia e Innovación, Spain