Compactness, Optimization and Risk

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Computational and Analytical Mathematics in honour of Jonathan Borwein's 60th Birthday

The coauthors

- B. Cascales, V. Fonf, S. Troyanski and J.O. *Boundaries of Asplund spaces* Journal Functional Analysis 259 (2010) 1346–1368
- M. Ruiz Galán and J.O. *A coercive and nonlinear James's weak compactness theorem*
- M. Ruiz Galán and J.O. *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces*

Contents

Weak Compactness Theorem of R.C. James

Theorem

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The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact

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Risk meausures

Definition

A monetary utility function is a concave non-decreasing map

$$U : \mathbb{L}^\infty(\Omega, \mathcal{F}\mathbb{P}) \to [-\infty, +\infty)$$

with $\mathrm{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset$ and

$$U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^\infty, c \in \mathbb{R}$$

Defining $\rho(X) = -U(X)$ the above definition of monetary utility function yields the definition of a convex risk measure

The space of financial positions $\mathcal{X}$ verifies $\mathbb{L}^\infty \subseteq \mathcal{X} \subseteq L^0$ and monetary risk measures ρ are defined on $\mathcal{X}$

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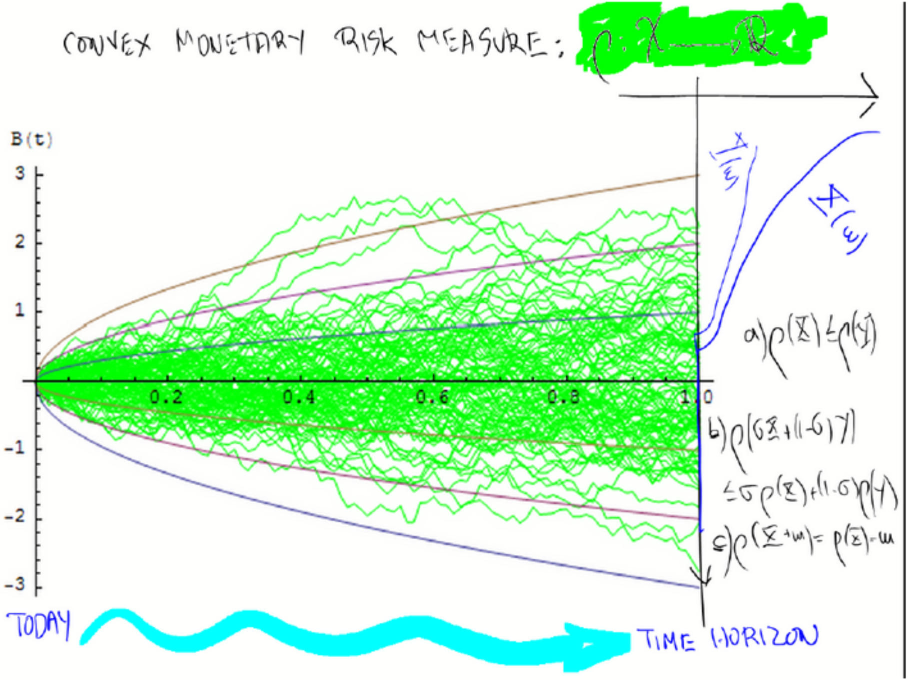
CONVEX MONETARY RISK MEASURE: $\rho : \mathcal{X} \to \mathbb{R}$
B(t)
3
2
1
-1
-2
-3
0.2
0.4
0.6
0.8
1.0
$Y(\omega)$
$X(\omega)$
a) $\rho(X) \leq \rho(Y)$
b) $\rho(\sigma X + (1-\sigma)Y)$
$\leq \sigma\rho(X) + (1-\sigma)\rho(Y)$
c) $\rho(X + m) = \rho(X) - m$
TODAY
TIME HORIZON

Minimizing $\{V(Y) + \mathbb{E}(X \cdot Y) : Y \in \mathbb{L}^1\}$

Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. *$\{U^* \leq c\}$ is $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$-compact subset for all $c \in \mathbb{R}$*
2. *For every $X \in \mathbb{L}^\infty$ the infimum in the equality*
$$U(X) = \inf_{Y \in \mathbb{L}^1} \{U^*(Y) + \mathbb{E}[XY]\},$$
is attained
3. *For every uniformly bounded sequence (X_n) tending a.s. to X we have*
$$\lim_{n \to \infty} U(X_n) = U(X).$$

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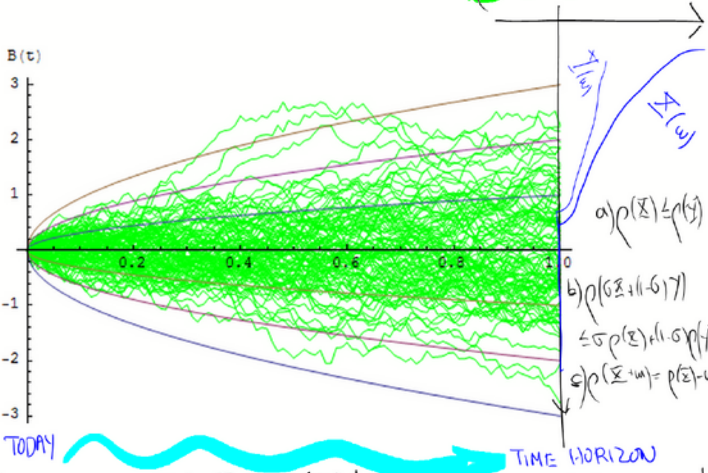

a) $\rho(X) \le \rho(Y)$

b) $\rho(\sigma X + (1-\sigma)Y) \le \sigma\rho(X) + (1-\sigma)\rho(Y)$

c) $\rho(X + m) = \rho(X) - m$

ρ has Fatou if $X_n \downarrow X \Rightarrow \rho(X_n) \uparrow \rho(X) \Leftrightarrow \sigma(L^\infty, L^1)$ lower semicont.

ρ is order sequentially continuous $\Leftrightarrow$ $|X_n| \le Z \ \forall n \quad X_n \xrightarrow{a.s.} X$

$\Downarrow$

$\lim_{n \to \infty} \rho(X_n) = \rho(X)$

ρ has Lebesgue property

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Tools for the proof

- The proof in [JST] is for separable $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. The separability is needed to show 2) $\Rightarrow$ 1) with a variant of the separable James' compactness Theorem.
- Delbaen has given a proof for general non separable spaces using an homogenisation trick. He shows how to apply directly the non separable James' compactness Theorem in the duality $\langle \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rangle$.

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Minimizing $\{V(y) + x^*(y) : y \in E\}$

Theorem (J. Orihuela)

Let E be a separable Banach space,

$$\alpha : E \to \mathbb{R} \cup \{\infty\}$$

proper, convex l.s.c. with dom$(\alpha) = \{\alpha < \infty\}$ a bounded subset of E. Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be weakly compact. Then there is $x^* \in E^*$ such that,the infimum

$$\inf_{x \in E} \{\langle x, x^* \rangle + \alpha(x)\}$$

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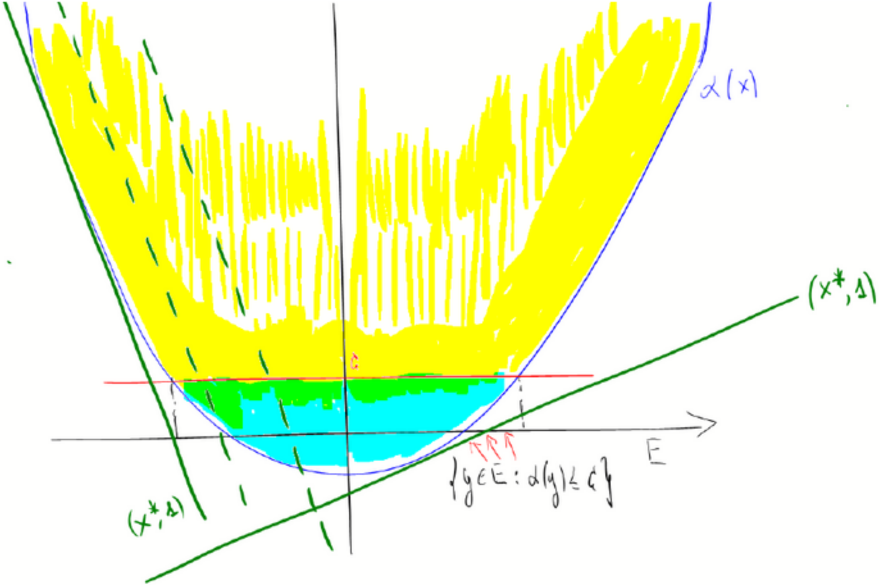
α(x)
(x*, 1)
c
E
{y∈E : α(y) ≤ c}
(x*, 1)

Simons' inequality

Lemma (Simons)

Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^\infty \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^\infty \lambda_n z_n(b),$$

then

$$\sup_{b\in\Lambda}\{\limsup_{n\to\infty} z_n(b)\} \geq \inf\left\{\sup_\Gamma w : w \in \operatorname{co}\{z_n : n \in \mathbb{N}\}\right\}$$

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Weak Compactness through inequalities

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k \in K}\{\limsup_{n \to \infty} x_n^*(k)\} \geq \inf\left\{ \sup_{\kappa \in \overline{K}^{w^*}} w(\kappa) : w \in \mathrm{co}\{x_n^* : n \in \mathbb{N}\}\right\}$$

Simons inequality $\Rightarrow$ Compactness

- If (2) happens and K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} \geq$$

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Weak Compactness through I-generation

Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For any covering $K \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of closed convex subsets $D_n \subset K$, we have*

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I-generation ⇒ Weak Compactness

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- $B_m := \overline{\mathrm{co}(\{x_n : n \leq m\})}^{\|\cdot\|}$ is finite dimensional closed compact set
- $D_m := B_m + \delta B_{E^{**}}$ for $\delta > 0$ fixed
- Since $K \subset \bigcup_{m=1}^{\infty} D_m$, the I-generation says that

$$\overline{\bigcup_{m}^{\infty} \overline{D_m}^{w^*}}^{\|\cdot\|} = \overline{K}^{w^*}.$$

- So $(\bigcup_m^{\infty} B_m) + 2\delta B_{E^{**}} \supset \overline{K}^{w^*}$.
- Finally $\bigcap_{\delta>0}(\bigcup_m^{\infty} B_m) + 2\delta B_{E^{**}} = \overline{K}^{\|\cdot\|} = K = \overline{K}^{w^*}$.

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- $B_m := \overline{\mathrm{co}(\{x_n : n \leq m\})}^{\|\cdot\|}$ is finite dimensional closed compact set
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- So $(\bigcup_m^{\infty} B_m) + 2\delta B_{E^{**}} \supset \overline{K}^{w^*}$.
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Simons versus Fonf-Lindenstrauss

Theorem (Cascales, Fonf, Troyanski and Orihuela, J.F.A.-2010)

Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

1. *For any covering $B \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of convex subsets $D_n \subset K$, we have*

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2. $\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in K} (\limsup_k g(x_k))$ *for every sequence $\{x_k\} \subset B_X$.*
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Theorem (Inf-liminf Theorem in $\mathbb{R}^\Gamma$)

Let $\{\Phi_k\}_{k\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^\Gamma$. We set $\Lambda \subseteq \Gamma$ satisfying the following boundary condition:
For all $\Phi = \sum_{i=1}^{\infty} \lambda_i \Phi_i, \sum_{i=1}^{\infty} \lambda_i = 1, 0 \leq \lambda_i \leq 1$, there exists

$$\lambda_0 \in \Lambda \text{ with } \Phi(\lambda_0) = \inf\{\Phi(\gamma) : \gamma \in \Gamma\}$$

Then

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A Nonlinear James Theorem

Theorem

Let E be a Banach space with B_{E^*} convex-block compact for $\sigma(E^*, E)$. If

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

is a proper map such that for every $x^* \in E^*$ the minimization problem

$$\inf\{\alpha(y) + x^*(y) : y \in E\}$$

is attained at some point of E, then the level sets

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Questions we answer

- The former Theorem applies to arbitrary $L^1(\Omega)$ including Delbaen-JST Theorem.
- The former Theorem extends the separable case of S.Calvert-Fitzpatrick's work for arbitrary maps
- B. Calvert and S. Fitzpatrick proved, in a 1985 paper:

Theorem (Calvert, Fitzpatrick)

If the subdifferential of a proper, convex and lower semicontinuous map $f : E \to \mathbb{R} \cup \{\infty\}$, with $\mathrm{dom(f)} \neq \emptyset$, is such that $\partial f(E) = E^*$, then the Banach space E must be reflexive.

Erratum

- S. Simons showed omissions in their proof and the authors presented an Erratum in 2000. The paper reduce its generality assuming coercitivity everywhere. It become more difficult to read since referenced lemmas must be adjusted too.
- **Conjecture:**The Nonlinear James Theorem is true in arbitrary Banach spaces without any control on the sequential compactness of the dual unit ball .

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Namioka-Klee Theorem

Theorem

Any linear and positive functional $\varphi : \mathcal{X} \to \mathbb{R}$ on a Fréchet lattice $\mathcal{X}$ is continuous

Theorem (S.Biagini and M.Fritelli 2009)

Any proper convex monotone increasing functional $U : \mathcal{X} \to (-\infty, +\infty]$ on a Frechet lattice $(\mathcal{X}, \mathcal{T})$ is continuous and subdifferentiable on the interior of its domain. Moreover, it admits a dual representation as

$$U(x) = \max_{y' \in \mathcal{X}'_+} \{y'(x) - U^*(y')\}$$

for all $x \in$ int(Dom(U))

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Order continuos lattices

Theorem (S. Biagini and M. Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ be an order continuous Frechet lattice. Any convex monotone increasing functional $U : \mathcal{X} \to \mathbb{R}$ is order continuous and it admits a dual representation as

$$U(x) = \max_{y' \in (\mathcal{X}_n^{\sim})_+} \{y'(x) - U^*(y')\}$$

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C-Property

Definition

A linear topology $\mathcal{T}$ on a Riesz space $\mathcal{X}$ has the C-property if for every $A \subset X$ and every $x \in \overline{A}^{\mathcal{T}}$ there is a sequence $(x_n) \in A$ together with $z_n \in \mathrm{co}\{x_p : p \geq n\}$ such that (z_n) is order convergent to x.

Theorem (S.Biagini and M.Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ a locally convex Frechet lattice and $U : \mathcal{X} \to (-\infty, +\infty]$ proper and convex. If $\sigma(\mathcal{X}, \mathcal{X}_n^{\sim})$ has the C-property then U is order lower semicontinuous if, and only if

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An even, convex function $\Psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

1. $\Psi(0) = 0$
2. $\lim_{x\to\infty} \Psi(x) = +\infty$
3. $\Psi < +\infty$ in a neighbourhood of 0

is called a Young function

1. $\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathbb{L}^0 : \exists \alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\}$
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Theorem (Lebesgue Risk Measures)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}[XY] - \alpha(Y)\}$ *be a strong convex risk measure on* L^Ψ *with* $\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to \mathbb{R} \cup \{+\infty\}$ *T.F.A.E.:*

(i) *For all* $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ *is a relatively weakly compact subset of* $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.

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Theorem (Nonlinear James Theorem)

Let E be a real Banach space,

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a proper, coercive and weakly lower semicontinuous function. Then

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Galán Simons Theorem

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Let A be a weakly closed subset of a real Banach space and let

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be a bounded function such that for all $x^* \in E^*$ the function

$$x^* - \psi, \text{ when restricted to } A, \text{ attains its supremum.}$$

Then A is weakly compact.

Nonlinear Variational Problems

Theorem (Reflexivity frame)

Let E be a real Banach space and

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A real Banach space with dual ball w^*-convex-block compact, for instance without copies of l^1, is reflexive whenever there exists a monotone, symmetric and surjective operator

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