Interplay between Functional Analysis Optimality and Risk

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The coauthors

- B. Cascales, V. Fonf, S. Troyanski and J.O. *Boundaries of Asplund spaces* Journal Functional Analysis 259 (2010) 1346–1368
- M. Ruiz Galán and J.O. *A nonlinear James compactness theorem*

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Weak Compactness Theorem of R.C. James

Theorem

A Banach space is reflexive if and only if each continuous linear functional attains its supremum on the unit ball

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A bounded and weakly closed subset K of a Banach space is weakly compact if and only if each continuous linear functional attains its supremum on K

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The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

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Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *be a monetary utility function with the Fatou property and* $V : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ *its Fenchel-Legendre transform. They are equivalent:*

1. $\{V \leq c\}$ *is* $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$*-compact subset of* $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ *for all* $c \in \mathbb{R}$
2. *For every* $X \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *the infimum in the equality*
$$U(X) = \inf_{Y \in \mathbb{L}^1} \{V(Y) + \mathbb{E}[XY]\},$$
is attained
3. *For every uniformly bounded sequence* (X_n) *tending a.s. to* X *we have*
$$\lim_{n\to\infty} U(X_n) = U(X).$$

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Tools for the proof

- The proof in [JST] is for separable $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. The separability is needed to show 2) $\Rightarrow$ 1) with a variant of the separable James' compactness Theorem.
- Delbaen has given a proof for general non separable spaces using an homogenisation trick. He shows how to apply directly the non separable James' compactness Theorem in the duality $\langle \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rangle$.

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Theorem

Let E be a separable Banach space,

$$V : E \to \mathbb{R} \cup \{\infty\}$$

proper, convex l.s.c. with $\mathrm{dom}(V) = \{V < \infty\}$ *a bounded subset of* E*. Suppose that there is* $c \in \mathbb{R}$ *such that the level set* $\{V \leq c\}$ *fails to be weakly compact. Then there is* $x^* \in E^*$ *such that,the infimum*

$$\inf_{x \in E} \{\langle x, x^* \rangle + V(x)\}$$

is not attained.

Minimizing $\{V(y) + x^*(y) : y \in E\}$

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Simons' inequality

Lemma (Simons)

Let Γ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Λ is a subset of Γ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then

$$\sup_{b\in\Lambda}\{\limsup_{n\to\infty} z_n(b)\} \geq \inf\left\{ \sup_{\Gamma} w : w \in \mathrm{co}\{z_n : n \in \mathbb{N}\}\right\}$$

Weak Compactness through inequalities

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k \in K}\{\limsup_{n \to \infty} x_n^*(k)\} \geq \inf\left\{ \sup_{\kappa \in \overline{K}^{w^*}} w(\kappa) : w \in \mathrm{co}\{x_n^* : n \in \mathbb{N}\}\right\}$$

Simons inequality $\Rightarrow$ Compactness

- If (2) happens and K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} \geq$$

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Weak Compactness through I-generation

Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

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- Take $\{x_n : n \in \mathbb{N}\}$ norm dense in K
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- $D_m := B_m + \delta B_{E^{**}}$ for $\delta > 0$ fixed
- Since $K \subset \bigcup_{m=1}^{\infty} D_m$, the I-generation says that

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- So $(\bigcup_m^{\infty} B_m) + 2\delta B_{E^{**}} \supset \overline{K}^{w^*}$.
- Finally $\bigcap_{\delta>0}(\bigcup_m^{\infty} B_m) + 2\delta B_{E^{**}} = \overline{K}^{\|\cdot\|} = K = \overline{K}^{w^*}$.

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Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

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Simons versus Fonf-Lindenstrauss

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and B a subset of E (not necessarely bounded) such that

- *For every $x^* \in C$ there is $b_0 \in B$ with*

$$x^*(b_0) = \inf\{x^*(b) : b \in B\}$$

Then we have:

- $\inf_{b \in B} \liminf_{n \to \infty} x_n^*(b) \leq \sup_{x^* \in C} \inf_{b \in B} x^*(b)$
- *If $D = \overline{B}^{\sigma(E^{**}, E^*)}$ we have* $\inf_{b \in B} \liminf_{n \to \infty} x_n^*(b) = \inf_{y \in D} \liminf_{n \to \infty} x_n^*(y)$

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- $\inf_{b \in B} \liminf_{n \to \infty} x_n^*(b) \leq \sup_{x^* \in C} \inf_{b \in B} x^*(b)$
- *If $D = \overline{B}^{\sigma(E^{**}, E^*)}$ we have* $\inf_{b \in B} \liminf_{n \to \infty} x_n^*(b) = \inf_{y \in D} \liminf_{n \to \infty} x_n^*(y)$

Theorem

Let E be a Banach space, $\{x_n^*\} \subset B_{E^*}$,

$$C = \{\sum_{n=1}^{\infty} \lambda_n x_n^* : \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\}$$

and B a subset of E (not necessarely bounded) such that

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Simon's inequality on unbounded sets

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Theorem (Birthday's Theorem)

Let E be a Banach space with B_{E^*} sequentially compact in the $\sigma(E^*, E)$-topology. Let

$$V : E \to \mathbb{R} \cup \{+\infty\}$$

be a proper, convex and lower semicontinuous map. If for every $x^* \in E^*$ the minimization problem

$$\inf\{V(y) + x^*(y) : y \in E\}$$

is attained at some point of E, then every level set

$$\{y \in E : V(y) \leq c\}$$

is weakly compact for every $c \in \mathbb{R}$. When $Dom(V)$ is bounded the reverse implication is always true.

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A nonlinear James Theorem

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Weak solutions are necessary in non reflexive spaces

Corollary

Let E be a Banach space with B_{E^*} sequentially compact in the $\sigma(E^*, E)$-topology. Let

$$V : E \to \mathbb{R} \cup \{+\infty\}$$

be a proper, convex and lower semicontinuous map. If

$$\partial V(E) = E^*$$

and the $Dom(V)$ has non empty interior, then E is a reflexive Banach space.

Questions we answer

- Birthday's Theorem partially answer a question of B. Calvert, S. Fitzpatrick and S. Simons.
- B. Calvert and S. Fitzpatrick proved, in a 1985 paper, that when the subdifferential of a proper convex and lower semicontinuous map V, with non void interior of its domain, is such that $\partial V(E) = E^*$, then the Banach space E must be reflexive.
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Open Problem

- Conjecture: Birthday's Theorem is valid in every Banach space.

To Walter with admiration : MY CONGRATULATIONS with...

- Cultivo la rosa blanca
- tanto en Julio como en Enero,
- para el amigo sincero
- que me da su mano franca.
- Y para el cruel que arranca
- el corazón con que vivo,
- cardo ni oruga cultivo
- cultivo la rosa blanca

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