A NOTE ON SUMMABILITY IN BANACH SPACES

JOSÉ RODRÍGUEZ

Abstract. Let Z and X be Banach spaces. Suppose that X is Asplund. Let $\mathcal{M}$ be a bounded set of operators from Z to X with the following property: a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in Z is weakly null if, for each $M \in \mathcal{M}$, the sequence $(M(z_n))_{n\in\mathbb{N}}$ is weakly null. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z such that: (a) for each $n \in \mathbb{N}$, the set $\{M(z_n) : M \in \mathcal{M}\}$ is relatively norm compact; (b) for each sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, the series $\sum_{n=1}^{\infty} M_n(z_n)$ is weakly unconditionally Cauchy. We prove that if $T \in \mathcal{M}$ is Dunford-Pettis and $\inf_{n\in\mathbb{N}} \|T(z_n)\|\|z_n\|^{-1} > 0$, then the series $\sum_{n=1}^{\infty} T(z_n)$ is absolutely convergent. As an application, we provide another proof of the fact that a countably additive vector measure taking values in an Asplund Banach space has finite variation whenever its integration operator is Dunford-Pettis.

1. Introduction

Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a countably additive vector measure. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be *ν-integrable* if: (a) f is $|x^*\nu|$-integrable for all $x^* \in X^*$; (b) for each $A \in \Sigma$ there is $\int_A f\, d\nu \in X$ such that $x^*\left(\int_A f\, d\nu\right) = \int_A f\, d(x^*\nu)$ for all $x^* \in X^*$. By identifying ν-a.e. equal functions, the set $L_1(\nu)$ of all (equivalence classes of) ν-integrable functions is a Banach lattice with the ν-a.e. order and the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|x^*\nu|.$$

We refer to [14] for basic information on these spaces, which play a relevant role in Banach lattices and operator theory. The *integration operator* of ν is the (norm one) operator $I_\nu : L_1(\nu) \to X$ defined by

$$I_\nu(f) := \int_\Omega f\, d\nu \quad \text{for all } f \in L_1(\nu).$$

Certain properties of I_ν have strong consequences on the structure of $L_1(\nu)$. For instance, ν has finite variation and the inclusion operator $\iota_\nu : L_1(|\nu|) \to L_1(\nu)$ is a lattice-isomorphism in each of the following cases:

(i) I_ν is compact, [11, Theorem 1] (cf. [13, Theorem 2.2] and [3, Theorem 3.3]);
(ii) I_ν is absolutely p-summing for some $1 \leq p < \infty$, [12, Theorem 2.2];

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(iii) I_ν is Dunford-Pettis and Asplund, [16, Theorem 3.3].

Note that case (iii) generalizes both (i) and (ii) because weakly compact operators are Asplund. The proof of (iii) given in [16] (cf. [15, Section 3.3]) is based on the Davis-Figiel-Johnson-Pełczyński factorization procedure and the following result obtained in [3, Theorem 1.3]:

Theorem 1.1. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a countably additive vector measure. If I_ν is Dunford-Pettis and X is Asplund, then ν has finite variation.*

The particular case of Theorem 1.1 when X has an unconditional Schauder basis and no subspace isomorphic to ℓ_1 had been proved earlier in [12, Theorem 1.2]. The question of whether the statement of Theorem 1.1 holds for arbitrary Banach spaces not containing subspaces isomorphic to ℓ_1 seems to be still open.

In this note we elaborate an abstract framework that allows to provide a simpler proof of Theorem 1.1. The following concept will be important along this way. Given two Banach spaces Z and X, we denote by $\mathcal{L}(Z, X)$ the Banach space of all operators from Z to X, equipped with the operator norm.

Definition 1.2. *Let Z and X be Banach spaces. We say that a set $\mathcal{M} \subseteq \mathcal{L}(Z, X)$ has the* Rainwater property *if the following holds: a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in Z is weakly null if, for each $M \in \mathcal{M}$, the sequence $(M(z_n))_{n\in\mathbb{N}}$ is weakly null.*

The Rainwater-Simons theorem (see, e.g., [6, Theorem 3.134]) states that, for an arbitrary Banach space Z, any James boundary of Z has the Rainwater property (with $X = \mathbb{R}$). More generally, James boundaries are (I)-generating, [7, Theorem 2.3], and all (I)-generating sets have the Rainwater property, see [9].

The main result of this note is the following:

Theorem 1.3. *Let Z and X be Banach spaces. Suppose that X is Asplund. Let $\mathcal{M}$ be a bounded subset of $\mathcal{L}(Z, X)$ having the Rainwater property. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z such that:*

(a) *for each $n \in \mathbb{N}$, the set $\{M(z_n) : M \in \mathcal{M}\}$ is relatively norm compact;*
(b) *for each sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, the series $\sum_{n=1}^\infty M_n(z_n)$ is weakly unconditionally Cauchy.*

Let $T \in \mathcal{M}$ such that:

(c) *T is Dunford-Pettis;*
(d) $\inf_{n\in\mathbb{N}} \|T(z_n)\| \|z_n\|^{-1} > 0$.

Then the series $\sum_{n=1}^\infty T(z_n)$ is absolutely convergent.

The paper is organized as follows. Section 2 is devoted to proving Theorem 1.3. In Section 3 we focus on the L_1 space of a vector measure and we get Theorem 1.1 as an application of Theorem 1.3.

Terminology. All our Banach spaces are real. By an *operator* we mean a continuous linear map between Banach spaces. An operator is called *Dunford-Pettis* if it maps weakly null sequences to norm null ones. By a *subspace* of a Banach space we mean a closed linear subspace. Let Z be a Banach space. We denote its norm by $\|\cdot\|_Z$ or simply $\|\cdot\|$. Given a set $C \subseteq Z$, we write $\|C\| := \sup\{\|z\| : z \in C\}$. The closed unit ball of Z is denoted by B_Z. The subspace of Z generated by a set $H \subseteq Z$ is denoted by $\overline{\operatorname{span}}(H)$. We write Z^* for the dual of Z. A set $B \subseteq B_{Z^*}$ is said to be a *James boundary* of Z if for every $z \in Z$ there is $z^* \in B$ such that $\|z\| = z^*(z)$. The space Z is said to be *Asplund* if every separable subspace of Z has separable dual.

2. Main result

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. For each $k \in \mathbb{N}$, we have an operator $P_k : X \to X$ defined by $P_k(x) := \sum_{n=1}^{k} e_n^*(x)e_n$ for all $x \in X$, where $(e_n^*)_{n\in\mathbb{N}}$ is the sequence in X^* of biorthogonal functionals associated with $(e_n)_{n\in\mathbb{N}}$. The operators of this form are called the *partial sum operators* on X associated with $(e_n)_{n\in\mathbb{N}}$. They satisfy $\sup_{k\in\mathbb{N}} \|P_k\| < \infty$.

The following lemma uses some ideas of the proof of [15, Lemma 3.4].

Lemma 2.1. *Let X be a Banach space with a Schauder basis and let $(P_k)_{k\in\mathbb{N}}$ be the associated sequence of partial sum operators on X. Write $\alpha := \sup_{k\in\mathbb{N}} \|P_k\|$. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of relatively norm compact subsets of X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \in K_n$ for all $n \in \mathbb{N}$. Suppose that:*

(a) *the series $\sum_{n=1}^{\infty} x_n$ is not absolutely convergent;*
(b) *$\sum_{n=1}^{\infty} \|P_k(K_n)\| < \infty$ for every $k \in \mathbb{N}$.*

Then there exist two strictly increasing sequences $(k_j)_{j\in\mathbb{N}}$ and $(l_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that, if $w_j \in \|x_{l_j}\|^{-1}K_{l_j}$ for all $j \in \mathbb{N}$, then:

(i) *$\|w_j - (P_{k_{j+1}} - P_{k_j})(w_j)\| \leq 2^{-j}$ for every $j \in \mathbb{N}$;*
(ii) *$\|w_j - w_{j'}\| \geq \alpha^{-1}\|w_j\| - 2^{-j}$ whenever $j' > j$.*

Proof. We can assume without loss of generality that $x_n \neq 0$ for all $n \in \mathbb{N}$. Write $Q_k := \mathrm{id}_X - P_k$ for all $k \in \mathbb{N}$, where id_X stands for the identity operator on X. Since $\|Q_k\| \leq 1 + \alpha$ for all $k \in \mathbb{N}$ and $\|Q_k(x)\| \to 0$ as $k \to \infty$ for every $x \in X$, the sequence of operators $(Q_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on each relatively norm compact subset of X.

We will construct by induction strictly increasing sequences $(k_j)_{j\in\mathbb{N}}$ and $(\tilde{l}_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ in such a way that, for each $j \in \mathbb{N}$, we have

$$\text{(c)} \quad \left\|P_{k_j}(K_{\tilde{l}_{j+1}})\right\| \leq \frac{\|x_{\tilde{l}_{j+1}}\|}{2^{j+1}} \qquad \text{and} \qquad \text{(d)} \quad \left\|Q_{k_j}(K_{\tilde{l}_j})\right\| \leq \frac{\|x_{\tilde{l}_j}\|}{2^j}.$$

Set $\tilde{l}_1 := 1$ and choose $k_1 \in \mathbb{N}$ such that $\|Q_{k_1}(K_1)\| \leq \frac{1}{2}\|x_1\|$. Suppose that $k_N, \tilde{l}_N \in \mathbb{N}$ are already chosen for some $N \in \mathbb{N}$. By (a) and (b), there is $\tilde{l}_{N+1} \in \mathbb{N}$ with $\tilde{l}_{N+1} > \tilde{l}_N$ such that

$$\left\|P_{k_N}(K_{\tilde{l}_{N+1}})\right\| \leq \frac{\|x_{\tilde{l}_{N+1}}\|}{2^{N+1}}.$$

Now, we take $k_{N+1} \in \mathbb{N}$ with $k_{N+1} > k_N$ such that

$$\left\|Q_{k_{N+1}}(K_{\tilde{l}_{N+1}})\right\| \leq \frac{\|x_{\tilde{l}_{N+1}}\|}{2^{N+1}}.$$

This finishes the construction of $(k_j)_{j\in\mathbb{N}}$ and $(\tilde{l}_j)_{j\in\mathbb{N}}$.

Define $l_j := \tilde{l}_{j+1}$ for all $j \in \mathbb{N}$. Take $(z_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} K_{l_j}$ and, for each $j \in \mathbb{N}$, define $w_j := \|x_{l_j}\|^{-1} z_j$. Then

$$\begin{aligned}\|w_j - (P_{k_{j+1}} - P_{k_j})(w_j)\| &= \|Q_{k_{j+1}}(w_j) + P_{k_j}(w_j)\| \\ &\le \|Q_{k_{j+1}}(w_j)\| + \|P_{k_j}(w_j)\| \\ &\overset{\text{(c)\&(d)}}{\le} \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} = \frac{1}{2^j}\end{aligned}$$

for every $j \in \mathbb{N}$. This proves (i).

To check property (ii), take $j' > j$ in $\mathbb{N}$. Then

$$\begin{aligned}\|w_j - w_{j'}\| &\ge \alpha^{-1}\|P_{k_{j+1}}(w_j - w_{j'})\| \\ &= \alpha^{-1}\|w_j - Q_{k_{j+1}}(w_j) - P_{k_{j+1}}(w_{j'})\| \\ &\ge \alpha^{-1}\Big(\|w_j\| - \|Q_{k_{j+1}}(w_j)\| - \|P_{k_{j+1}}(w_{j'})\|\Big) \\ &= \alpha^{-1}\Big(\|w_j\| - \|Q_{k_{j+1}}(w_j)\| - \|P_{k_{j+1}}(P_{k_{j'}}(w_{j'}))\|\Big) \\ &\ge \alpha^{-1}\Big(\|w_j\| - \|Q_{k_{j+1}}(w_j)\| - \alpha\|P_{k_{j'}}(w_{j'})\|\Big) \\ &\overset{(\alpha\ge 1)}{\ge} \alpha^{-1}\|w_j\| - \|Q_{k_{j+1}}(w_j)\| - \|P_{k_{j'}}(w_{j'})\| \\ &\overset{\text{(c)\&(d)}}{\ge} \alpha^{-1}\|w_j\| - \frac{1}{2^{j+1}} - \frac{1}{2^{j'+1}} \ \ge\ \alpha^{-1}\|w_j\| - \frac{1}{2^j}.\end{aligned}$$

The proof is finished. □

Corollary 2.2. *Let X be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\sum_{n=1}^\infty x_n$ is weakly unconditionally Cauchy and $\{\|x_n\|^{-1}x_n : n \in \mathbb{N},\, x_n \neq 0\}$ is relatively norm compact. Then $\sum_{n=1}^\infty x_n$ is absolutely convergent.*

Proof. The subspace $\overline{\operatorname{span}}(\{x_n : n \in \mathbb{N}\}) \subseteq X$ is separable, so it embeds isometrically into the Banach space $C([0,1])$. Hence, we can assume without loss of generality that $X = C([0,1])$. Since this space has a Schauder basis, the conclusion follows from Lemma 2.1(ii) by taking $K_n := \{x_n\}$ for all $n \in \mathbb{N}$. Indeed, if $(P_k)_{k\in\mathbb{N}}$ is the sequence of partial sum operators on $C([0,1])$ associated with a given Schauder basis, then for each $k \in \mathbb{N}$ the series $\sum_{n=1}^\infty P_k(x_n)$ is absolutely convergent, because it is weakly unconditionally Cauchy and $P_k(X)$ is finite-dimensional. □

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. By a *block sequence with respect to* $(e_n)_{n\in\mathbb{N}}$ we mean a sequence $(x_j)_{j\in\mathbb{N}}$ in X for which there exist a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $(I_j)_{j\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}$ such that $\max(I_j) < \min(I_{j+1})$ and $x_j = \sum_{n\in I_j} a_n e_n$ for all $j \in \mathbb{N}$. Recall that the Schauder basis $(e_n)_{n\in\mathbb{N}}$ is said to be *shrinking* if its sequence of biorthogonal functionals $(e_n^*)_{n\in\mathbb{N}}$ satisfies $X^* = \overline{\operatorname{span}}(\{e_n^* : n \in \mathbb{N}\})$.

We can now prove our main result.

Proof of Theorem 1.3. Clearly, we can suppose that $\|M\| \le 1$ for every $M \in \mathcal{M}$.

Let us consider the subspace $Z_0 := \overline{\operatorname{span}}(\{z_n : n \in \mathbb{N}\}) \subseteq Z$. The set of restrictions $\{M|_{Z_0} : M \in \mathcal{M}\} \subseteq B_{\mathcal{L}(Z_0,X)}$ has the Rainwater property and fulfills conditions (a) and (b). Obviously, the restriction $T|_{Z_0}$ also satisfies conditions (c) and (d). The subspace

$$X_0 := \overline{\operatorname{span}}\left(\bigcup_{n\in\mathbb{N}} \{M(z_n) : M \in \mathcal{M}\}\right) \subseteq X$$

is separable (thanks to (a)) and we have $M(Z_0) \subseteq X_0$ for every $M \in \mathcal{M}$. Since X is Asplund and X_0 is separable, X_0^* is separable. Therefore, we can assume without loss of generality that X^* is separable.

A result of Zippin [18] (cf. [5, Chapter 5]) states that every Banach space with separable dual embeds isomorphically into a Banach space with a shrinking Schauder basis. Therefore, we can assume further that X has a shrinking Schauder basis, say $(e_n)_{n\in\mathbb{N}}$. Let $(P_k)_{k\in\mathbb{N}}$ be the sequence of partial sum operators on X associated with $(e_n)_{n\in\mathbb{N}}$.

For each $n \in \mathbb{N}$, write $x_n := T(z_n) \in X$ and consider the relatively norm compact set

$$K_n := \{M(z_n) : M \in \mathcal{M}\} \subseteq X.$$

Observe that for each $k \in \mathbb{N}$ we have $\sum_{n=1}^{\infty} \|P_k(K_n)\| < \infty$. Indeed, for every $n \in \mathbb{N}$ we choose $M_n \in \mathcal{M}$ such that

$$\|P_k(K_n)\| \leq \|P_k(M_n(z_n))\| + \frac{1}{2^n}. \tag{2.1}$$

Since $\sum_{n=1}^{\infty} M_n(z_n)$ is weakly unconditionally Cauchy (by condition (b)) and $P_k(X)$ is finite-dimensional, the series $\sum_{n=1}^{\infty} P_k(M_n(z_n))$ is absolutely convergent and so inequality (2.1) yields $\sum_{n=1}^{\infty} \|P_k(K_n)\| < \infty$, as claimed.

Suppose, by contradiction, that $\sum_{n=1}^{\infty} x_n$ is not absolutely convergent and apply Lemma 2.1. Let $(k_j)_{j\in\mathbb{N}}$ and $(l_j)_{j\in\mathbb{N}}$ be as in Lemma 2.1. Define

$$R_j := P_{k_{j+1}} - P_{k_j} \in \mathcal{L}(X,X) \quad \text{and} \quad u_j := \|x_{l_j}\|^{-1} z_{l_j} \in Z \quad \text{for all } j \in \mathbb{N}.$$

Write $\beta := \inf_{n\in\mathbb{N}} \|T(z_n)\|\|z_n\|^{-1} > 0$. Fix $M \in \mathcal{M}$ and define

$$w_j^M := M(u_j) = \|x_{l_j}\|^{-1} M(z_{l_j}) \in \|x_{l_j}\|^{-1} K_{l_j} \quad \text{for all } j \in \mathbb{N}.$$

Note that $\|u_j\| \leq \beta^{-1}$ and so $\|w_j^M\| \leq \|M\|\beta^{-1} \leq \beta^{-1}$ for all $j \in \mathbb{N}$. Observe that $(R_j(w_j^M))_{j\in\mathbb{N}}$ is a block sequence with respect to $(e_n)_{n\in\mathbb{N}}$ which is bounded, because the sequence $(w_j^M)_{j\in\mathbb{N}}$ is bounded and $\|R_j\| \leq 2\sup_{k\in\mathbb{N}} \|P_k\| < \infty$ for all $j \in \mathbb{N}$. Since $(e_n)_{n\in\mathbb{N}}$ is shrinking, we deduce that $(R_j(w_j^M))_{j\in\mathbb{N}}$ is weakly null (see, e.g., [1, Proposition 3.2.7]). Since

$$\|w_j^M - R_j(w_j^M)\| \leq \frac{1}{2^j} \quad \text{for all } j \in \mathbb{N}$$

(by part (i) of Lemma 2.1), we conclude that $(w_j^M)_{j\in\mathbb{N}}$ is weakly null as well.

As $M \in \mathcal{M}$ is arbitrary, the Rainwater property of $\mathcal{M}$ implies that the sequence $(u_j)_{j\in\mathbb{N}}$ is weakly null in Z. This is a contradiction, because T is Dunford-Pettis and $\|T(u_j)\| = 1$ for every $j \in \mathbb{N}$. □

A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space is said to be an ℓ_1-*sequence* if it is bounded and there is a constant $c>0$ such that

$$\left\|\sum_{n=1}^{N} a_n x_n\right\| \geq c\sum_{n=1}^{N}|a_n|$$

for every $N\in\mathbb{N}$ and for all $a_1,\dots,a_N\in\mathbb{R}$. That is, $(x_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence if and only if it is a basic sequence which is equivalent to the usual Schauder basis of ℓ_1 (see, e.g., [1, Section 1.3]).

Corollary 2.3. *Let Z and X be Banach spaces. Suppose that X is Asplund. Let $\mathcal{M}$ be a bounded subset of $\mathcal{L}(Z,X)$ having the Rainwater property. Let $(e_n)_{n\in\mathbb{N}}$ be a seminormalized basic sequence in Z such that:*

(a) *for each $n\in\mathbb{N}$, the set $\{M(e_n): M\in\mathcal{M}\}$ is relatively norm compact;*
(b) *for each sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that the series $\sum_{n=1}^\infty a_n e_n$ is convergent and for each sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, the series $\sum_{n=1}^\infty a_n M_n(e_n)$ is weakly unconditionally Cauchy.*

Let $T\in\mathcal{M}$ such that:

(c) *T is Dunford-Pettis;*
(d) $\inf_{n\in\mathbb{N}}\|T(e_n)\|\|e_n\|^{-1}>0$.

Then $(e_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $\sum_{n=1}^\infty |a_n|\|e_n\|<\infty$ if (and only if) the series $\sum_{n=1}^\infty a_n e_n$ is convergent. To check this, we can assume without loss of generality that $a_n\neq 0$ for all $n\in\mathbb{N}$. Now, Theorem 1.3 (applied to $z_n:=a_n e_n$) ensures that if $\sum_{n=1}^\infty a_n e_n$ is convergent, then we have $\sum_{n=1}^\infty |a_n|\|T(e_n)\|<\infty$ and so $\sum_{n=1}^\infty |a_n|\|e_n\|<\infty$ (by (d)). This shows that $(\|e_n\|^{-1}e_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence. Since $(e_n)_{n\in\mathbb{N}}$ is seminormalized, it is an ℓ_1-sequence as well. □

We finish this section with a few remarks on sets of operators having the Rainwater property and some examples. The first one is an immediate consequence of the aforementioned Rainwater-Simons theorem (see, e.g., [6, Theorem 3.134]).

Corollary 2.4. *Let Z and X be Banach spaces and let $\mathcal{M}\subseteq B_{\mathcal{L}(Z,X)}$. The following statements are equivalent and imply that $\mathcal{M}$ has the Rainwater property:*

(i) *the set $\bigcup_{M\in\mathcal{M}} M^*(B_{X^*})\subseteq B_{Z^*}$ is a James boundary of Z;*
(ii) *for every $z\in Z$ there is $M\in\mathcal{M}$ such that $\|z\|=\|M(z)\|$.*

Definition 2.5. *Let Z and X be Banach spaces. We say that a set $\mathcal{M}\subseteq B_{\mathcal{L}(Z,X)}$ has the* James boundary property *if it satisfies conditions (i)-(ii) of Corollary 2.4.*

Example 2.6. *Let X be a Banach space and let E be a Banach space with a normalized 1-unconditional Schauder basis $(e_n)_{n\in\mathbb{N}}$. Let Z be the E-sum of countably many copies of X, that is, Z is the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$ in X such that the series $\sum_{n=1}^\infty \|x_n\|e_n$ converges in E, equipped with the norm*

$$\|(x_n)_{n\in\mathbb{N}}\|_Z := \left\|\sum_{n=1}^\infty \|x_n\|\, e_n\right\|_E .$$

Let $\mathcal{M} \subseteq B_{\mathcal{L}(Z,X)}$ be the set of all coordinate projections.

(i) *If E^* is separable, then $\mathcal{M}$ has the Rainwater property (see, e.g.,* [17, Lemma 3.22]*).*

(ii) *If $E = c_0$, then $\mathcal{M}$ has the James boundary property.*

(iii) *If $E = \ell_p$ for some $1 < p < \infty$, then $\mathcal{M}$ has the Rainwater property but fails to have the James boundary property (unless $X = \{0\}$). Indeed, bear in mind that c_0 does not contain subspaces isomorphic to ℓ_p (see, e.g.,* [1, Corollary 2.1.6]*).*

It is natural to wonder when a single operator has the Rainwater property. An obvious necessary condition is that such an operator must be injective. In fact:

Remark 2.7. Let Z and X be Banach spaces and let $\mathcal{M} \subseteq \mathcal{L}(Z,X)$ be a set having the Rainwater property. Then $\bigcap_{M\in\mathcal{M}} \ker M = \{0\}$. Indeed, if $z \in \bigcap_{M\in\mathcal{M}} \ker M$, then the Rainwater property of $\mathcal{M}$ implies that the constant sequence $(z, z, \dots)$ is weakly null in Z, which is equivalent to saying that $z = 0$.

Let Z and X be Banach spaces. An operator $T : Z \to X$ is called *tauberian* if its second adjoint satisfies $(T^{**})^{-1}(X) \subseteq Z$. This is equivalent to saying that a bounded set $C \subseteq Z$ is relatively weakly compact if (and only if) $T(C)$ is relatively weakly compact (see, e.g., [8, Corollary 2.2.5]). As a consequence, we have:

Remark 2.8. *Let Z and X be Banach spaces and let $T \in \mathcal{L}(Z,X)$ be injective.*

(a) *If T is tauberian, then $\{T\}$ has the Rainwater property.*

(b) *If $\{T\}$ has the Rainwater property and Z is weakly sequentially complete, then T is tauberian.*

In part (b) of the previous remark, the additional assumption on Z cannot be dropped in general:

Example 2.9. *Let $T : c_0 \to \ell_1$ be the injective operator defined by*

$$T((a_n)_{n\in\mathbb{N}}) := (2^{-n}a_n)_{n\in\mathbb{N}} \quad \text{for all } (a_n)_{n\in\mathbb{N}} \in c_0.$$

Then $\{T\}$ has the Rainwater property, but T is not tauberian. Indeed, any tauberian operator maps the closed unit ball of the domain space to a closed set (see, e.g., [8, Theorem 2.1.7]*). However, $T(B_{c_0})$ is not closed. For instance, it is easy to check that $x = (2^{-n})_{n\in\mathbb{N}} \in \ell_1$ satisfies $x \in \overline{T(B_{c_0})} \setminus T(B_{c_0})$.*

The previous example is a particular case of a more general construction:

Proposition 2.10. *Let Z and X be Banach spaces and let $\mathcal{M} \subseteq \mathcal{L}(Z,X)$ be a countable set having the Rainwater property. Let E be a Banach space with a normalized 1-unconditional Schauder basis $(e_n)_{n\in\mathbb{N}}$ and let Y be the E-sum of countably many copies of X. Then there is an injective operator $T : Z \to Y$ such that $\{T\}$ has the Rainwater property.*

Proof. Enumerate $\mathcal{M} = \{M_n : n \in \mathbb{N}\}$. If we multiply each M_n by a non-zero constant, the resulting set also has the Rainwater property. So, we can assume that the series $\sum_{n=1}^{\infty} \|M_n\| e_n$ converges E. Now, the map $T : Z \to Y$ defined by $T(z) := (M_n(z))_{n\in\mathbb{N}}$ for all $z \in Z$ satisfies the requirements. □

3. Application to the L_1 space of a vector measure

Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a countably additive vector measure. The variation and semivariation of ν are denoted by $|\nu|$ and $\|\nu\|$, respectively. Given $x^* \in X^*$, we denote by $x^*\nu : \Sigma \to \mathbb{R}$ the composition of ν with x^* and we denote by $|x^*\nu|$ its variation. We say that $A \in \Sigma$ is *ν-null* if $\|\nu\|(A) = 0$ or, equivalently, $\nu(B) = 0$ for every $B \in \Sigma$ contained in A. The subset of Σ consisting of all ν-null sets is denoted by $\mathcal{N}(\nu)$.

Every ν-essentially bounded Σ-measurable function $f : \Omega \to \mathbb{R}$ is ν-integrable. By identifying ν-a.e. equal functions, the set $L_\infty(\nu)$ of all (equivalence classes of) ν-essentially bounded Σ-measurable functions is a Banach lattice with the ν-a.e. order and the ν-essential supremum norm $\|\cdot\|_{L_\infty(\nu)}$. For each $g \in L_\infty(\nu)$, we denote by $M_g : L_1(\nu) \to X$ the operator defined by

$$M_g(f) := \int_\Omega f g \, d\nu \quad \text{for all } f \in L_1(\nu),$$

which satisfies $\|M_g\| \le \|g\|_{L_\infty(\nu)}$. It is known that

$$\|f\|_{L_1(\nu)} = \sup_{g \in B_{L_\infty(\nu)}} \|M_g(f)\| \quad \text{for all } f \in L_1(\nu) \tag{3.1}$$

(see, e.g., [14, Proposition 3.31]).

The following lemma can be found in [12, Lemma 3.3] and [2, Corollary 4.2]. Note that part (ii) follows at once from part (i) and (3.1). It is worth pointing out that in (ii) the set $B_{L_\infty(\nu)}$ can be replaced by its extreme points, that is, the subset $\{\chi_A - \chi_{\Omega \setminus A} : A \in \Sigma\}$, see [4, Corollary 2.4].

Lemma 3.1. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a countably additive vector measure such that the set $\{\nu(A) : A \in \Sigma\}$ is relatively norm compact. Then:*

(i) *for each $f \in L_1(\nu)$, the set $\{M_g(f) : g \in B_{L_\infty(\nu)}\}$ is norm compact;*
(ii) *the set $\{M_g : g \in B_{L_\infty(\nu)}\}$ has the James boundary property.*

In particular, $\{M_g : g \in B_{L_\infty(\nu)}\}$ has the Rainwater property.

Lemma 3.2. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a countably additive vector measure. Let $(f_n)_{n \in \mathbb{N}}$ be sequence of pairwise disjoint non-zero elements of $L_1(\nu)$. Then $(f_n)_{n \in \mathbb{N}}$ is a 1-unconditional basic sequence in $L_1(\nu)$.*

Proof. It suffices to check that

$$\left\| \sum_{k=1}^{n} a_k f_k \right\|_{L_1(\nu)} \le \left\| \sum_{k=1}^{m} b_k f_k \right\|_{L_1(\nu)} \tag{3.2}$$

for all sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in $\mathbb{R}$ such that $|a_k| \le |b_k|$ for every $k \in \mathbb{N}$ and for all $n \le m$ in $\mathbb{N}$ (see, e.g., [1, Propositions 1.1.9 and 3.1.3]). Fix $x^* \in B_{X^*}$.

Since the f_k's are pairwise disjoint, we have

$$\int_\Omega \left|\sum_{k=1}^n a_k f_k\right| d|x^*\nu| = \sum_{k=1}^n |a_k| \int_\Omega |f_k|\, d|x^*\nu|$$
$$\leq \sum_{k=1}^m |b_k| \int_\Omega |f_k|\, d|x^*\nu| = \int_\Omega \left|\sum_{k=1}^m b_k f_k\right| d|x^*\nu| \leq \left\|\sum_{k=1}^m b_k f_k\right\|_{L_1(\nu)}.$$

By taking the supremum when x^* runs over all B_{X^*}, we get (3.2). □

We can now prove Theorem 1.1 by using Corollary 2.3.

Proof of Theorem 1.1. It suffices to show that $\sum_{n=1}^\infty \|\nu(C_n)\| < \infty$ for every sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\Sigma \setminus \mathcal{N}(\nu)$ (see, e.g., [10, Corollary 2]).

Fix $\rho > 2$ and $n \in \mathbb{N}$. We can take $A_n \in \Sigma \setminus \mathcal{N}(\nu)$ such that $A_n \subseteq C_n$ and $\rho\|\nu(A_n)\| \geq \|\nu\|(C_n)$. Define $f_n := \|\nu\|(C_n)^{-1}\chi_{A_n} \in L_1(\nu)$ and note that

$$\frac{1}{\rho} \leq \frac{\|\nu(A_n)\|}{\|\nu\|(C_n)} \leq \|f_n\|_{L_1(\nu)} = \frac{\|\nu\|(A_n)}{\|\nu\|(C_n)} \leq 1. \tag{3.3}$$

Hence, $(f_n)_{n\in\mathbb{N}}$ is a seminormalized 1-unconditional basic sequence in $L_1(\nu)$ (apply Lemma 3.2).

We will show that $(f_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence via Corollary 2.3 applied to the operator $T := M_{\chi_\Omega} = I_\nu$ and the family $\mathcal{M} := \{M_g : g \in B_{L_\infty(\nu)}\}$. Since I_ν is Dunford-Pettis, the set $\{\nu(A) : A \in \Sigma\}$ is relatively norm compact (see [2, Theorem 5.8], cf. [15, Proposition 2.6]). Hence, $\mathcal{M}$ has the Rainwater property and condition (a) of Corollary 2.3 holds (apply Lemma 3.1). Condition (d) holds because

$$\|I_\nu(f_n)\|_X \|f_n\|^{-1}_{L_1(\nu)} = \frac{\|\nu(A_n)\|}{\|\nu\|(A_n)} \geq \frac{\|\nu(A_n)\|}{\|\nu\|(C_n)} \overset{(3.3)}{\geq} \frac{1}{\rho} \quad \text{for all } n \in \mathbb{N}.$$

To check condition (b), let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\sum_{n=1}^\infty a_n f_n$ is convergent in $L_1(\nu)$ and let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $B_{L_\infty(\nu)}$. Since the A_n's are pairwise disjoint, we can find $g \in B_{L_\infty(\nu)}$ such that $g|_{A_n} = g_n|_{A_n}$ for every $n \in \mathbb{N}$. Since $(f_n)_{n\in\mathbb{N}}$ is an unconditional basic sequence, $\sum_{n=1}^\infty a_n f_n$ is unconditionally convergent. Then the series

$$\sum_{n=1}^\infty a_n M_{g_n}(f_n) = \sum_{n=1}^\infty a_n \int_\Omega f_n g_n\, d\nu = \sum_{n=1}^\infty a_n \int_\Omega f_n g\, d\nu = \sum_{n=1}^\infty M_g(a_n f_n)$$

is unconditionally convergent in X (because M_g is an operator) and, therefore, it is weakly unconditionally Cauchy. So, condition (b) of Corollary 2.3 holds too. From that result it follows that $(f_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence.

Let $c > 0$ such that

$$\sum_{n=1}^N |a_n| \leq c \left\|\sum_{n=1}^N a_n f_n\right\|_{L_1(\nu)}$$

for every $N \in \mathbb{N}$ and for all $a_1, \dots, a_N \in \mathbb{R}$. The previous inequality applied to $a_n := \|\nu\|(C_n)$ yields

$$\sum_{n=1}^{N} \|\nu(C_n)\| \leq \sum_{n=1}^{N} \|\nu\|(C_n)$$
$$\leq c \left\| \sum_{n=1}^{N} \chi_{A_n} \right\|_{L_1(\nu)} = c \left\| \chi_{\bigcup_{n=1}^{N} A_n} \right\|_{L_1(\nu)} = c \|\nu\| \left(\bigcup_{n=1}^{N} A_n \right) \leq c \|\nu\| (\Omega)$$

for every $N \in \mathbb{N}$. It follows that $\sum_{n=1}^{\infty} \|\nu(C_n)\| \leq c\|\nu\|(\Omega) < \infty$, as required. □

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Dpto. de Matemáticas, E.T.S. de Ingenieros Industriales de Albacete, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

Email address: jose.rodriguezruiz@uclm.es