ON VECTOR MEASURES WITH VALUES IN $c_0(\kappa)$

JOSÉ RODRÍGUEZ

Abstract. Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. We prove that if ν is homogeneous and $L_1(\nu)$ is non-separable, then there is a vector measure $\tilde{\nu} : \Sigma \to c_0(\kappa)$ such that $L_1(\nu) = L_1(\tilde{\nu})$ with equal norms, where κ is the density character of $L_1(\nu)$. This is a non-separable version of a result of [G.P. Curbera, Pacific J. Math. 162 (1994), no. 2, 287–303].

1. Introduction

Spaces of integrable functions with respect to a vector measure play an important role in Banach lattices and operator theory. Every Banach lattice with order continuous norm and a weak unit is lattice-isometric to the L_1 space of some vector measure, [2, Theorem 8] (cf. [6, Proposition 2.4]). Such a representation is not unique, in the sense that a Banach lattice can be lattice-isometric to the L_1 spaces of completely different vector measures. The following result was proved in [3, Theorem 1] (cf. [14, Theorem 5] for a different proof):

Theorem 1.1 (G. P. Curbera). *Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. If ν is atomless and $L_1(\nu)$ is separable, then there is a vector measure $\tilde{\nu} : \Sigma \to c_0$ such that $L_1(\nu) = L_1(\tilde{\nu})$ with equal norms.*

In general, this result is not valid for vector measures with atoms, as shown in [3, pp. 294–295]. It is natural to ask about non-separable versions of Theorem 1.1 by using $c_0(\kappa)$ as target space for a large enough cardinal κ. This question was posed by Z. Lipecki at the conference "Integration, Vector Measures and Related Topics VI" (Będlewo, June 2014). In [19] we provided some partial answers by using a certain superspace of $c_0(\kappa)$, namely, the so-called Pełczyński-Sudakov space. In the particular case $\kappa = \aleph_1$ (the first uncountable cardinal), this is the Banach space $\ell_\infty^c(\aleph_1)$ of all bounded real-valued functions on $\aleph_1$ with countable support.

In this note we refine the results of [19] by proving the following:

Theorem 1.2. *Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. If ν is homogeneous and $L_1(\nu)$ is non-separable, then*

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there is a vector measure $\tilde{\nu} : \Sigma \to c_0(\kappa)$ such that $L_1(\nu) = L_1(\tilde{\nu})$ with equal norms, where κ is the density character of $L_1(\nu)$.

The proof of Theorem 1.2 uses some ideas of [19, Example 2.6]. There we showed that, for an arbitrary uncountable cardinal κ and $1 < p < \infty$, the L_p space of the usual product probability measure on the Cantor cube $\{-1, 1\}^\kappa$ is equal to $L_1(\nu)$ for some $c_0(\kappa)$-valued vector measure ν. We stress that one cannot arrive at the same conclusion by using a c_0-valued vector measure, see [17, Example 4.16].

The paper is organized as follows. In Section 2 we fix the terminology and include some preliminary facts on L_1 spaces of a vector measure. In Section 3 we prove Theorem 1.2 and a similar result for non-homogeneous vector measures (Theorem 3.2). We finish the paper with further remarks on $c_0(\kappa)$-valued vector measures which might be of independent interest (Theorem 3.3).

2. Preliminaries

Our notation is standard as can be found in [5] and [16]. We write $\mathbb{N} = \{1, 2, \dots\}$. The *density character* of a topological space T, denoted by $\mathrm{dens}(T)$, is the minimal cardinality of a dense subset of T.

Given a non-empty set I, we denote by Λ_I the σ-algebra on $\{-1, 1\}^I$ generated by all the sets of the form

$$\{x \in \{-1, 1\}^I : x(i) = y(i) \text{ for all } i \in J\},$$

where $J \subseteq I$ is finite and $y \in \{-1, 1\}^J$. Every closed-and-open subset of $\{-1, 1\}^I$ is a finite union of sets as above. The symbol λ_I stands for the usual product probability measure on $(\{-1, 1\}^I, \Lambda_I)$. For each $i \in I$ we denote by

$$\pi_i^I : \{-1, 1\}^I \to \{-1, 1\}$$

the ith-coordinate projection and, for each non-empty set $J \subseteq I$, we denote by

$$\rho_J^I : \{-1, 1\}^I \to \{-1, 1\}^J$$

the canonical projection.

All our Banach spaces are real. The closed unit ball of a Banach space X is denoted by B_X and the dual of X is denoted by X^*. The symbol $\|\cdot\|_X$ stands for the norm of X. Given a non-empty set Γ, we denote by $c_0(\Gamma)$ the Banach space of all bounded functions $\varphi : \Gamma \to \mathbb{R}$ such that $\{\gamma \in \Gamma : |\varphi(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$, equipped with the supremum norm.

Let (Ω, Σ) be a measurable space. Given a Banach space X, we denote by $\mathrm{ca}(\Sigma, X)$ the set of all X-valued vector measures defined on Σ. Unless stated otherwise, our *measures* are meant to be countably additive.

Let $\nu \in \mathrm{ca}(\Sigma, X)$. Given $A \in \Sigma$, we denote by ν_A the restriction of ν to

$$\Sigma_A := \{B \in \Sigma : B \subseteq A\}$$

(which is a σ-algebra on A). The set A is called *ν-null* if $\nu(B) = 0$ for every $B \in \Sigma_A$ or, equivalently, $\|\nu\|(A) = 0$, where $\|\nu\|$ is the semivariation of ν. The family of all ν-null sets is denoted by $\mathcal{N}(\nu)$. We say that ν is *atomless* if for every $A \in \Sigma \setminus \mathcal{N}(\nu)$ there is $B \in \Sigma_A$ such that neither B nor $A \setminus B$ is ν-null.

By a *Rybakov control measure* of ν we mean a finite non-negative measure of the form $\mu = |x^*\nu|$ for some $x^* \in X^*$ such that $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ (see, e.g., [5, p. 268, Theorem 2]); here $|x^*\nu|$ is the variation of the signed measure $x^*\nu : \Sigma \to \mathbb{R}$ obtained as the composition of ν with x^*.

A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be *ν-integrable* if it is $|x^*\nu|$-integrable for all $x^* \in X^*$ and, for each $A \in \Sigma$, there is $\int_A f\, d\nu \in X$ such that

$$x^*\left(\int_A f\, d\nu\right) = \int_A f\, d(x^*\nu) \quad \text{for all } x^* \in X^*.$$

By identifying functions which coincide ν-a.e., the set $L_1(\nu)$ of all (equivalence classes of) ν-integrable functions is a Banach lattice with the ν-a.e. order and the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|x^*\nu|.$$

We write $\mathrm{sim}(\Sigma)$ to denote the linear subspace of $L_1(\nu)$ consisting of all (equivalence classes of) *simple functions*, that is, linear combinations of characteristic functions χ_A where $A \in \Sigma$. The set $\mathrm{sim}(\Sigma)$ is norm dense in $L_1(\nu)$. As in the case of finite non-negative measures, if $L_1(\nu)$ is infinite-dimensional, then its density character coincides with the minimal cardinality of a set $\mathcal{C} \subseteq \Sigma$ satisfying that $\inf_{C \in \mathcal{C}} \|\nu\|(A \triangle C) = 0$ for all $A \in \Sigma$. We say that ν is *homogeneous* if it is atomless and

$$\mathrm{dens}(L_1(\nu)) = \mathrm{dens}(L_1(\nu_A)) \quad \text{for every } A \in \Sigma \setminus \mathcal{N}(\nu).$$

In this case, the cardinal $\mathrm{dens}(L_1(\nu))$ is called the *Maharam type* of ν. It is easy to check that: (i) $\mathrm{dens}(L_1(\nu)) = \mathrm{dens}(L_1(\mu))$ for any Rybakov control measure μ of ν; (ii) ν is atomless (resp., homogeneous) if and only if some/any Rybakov control measure of ν is atomless (resp., homogeneous).

As a Banach lattice, $L_1(\nu)$ has order continuous norm and a weak unit (the function χ_Ω). If μ is a Rybakov control measure of ν, then $L_1(\nu)$ is a *Köthe function space* over (Ω, Σ, μ) and we can consider its *Köthe dual*

$$L_1(\nu)' := \{g \in L_1(\mu) : \, fg \in L_1(\mu) \text{ for all } f \in L_1(\nu)\}.$$

For each $g \in L_1(\nu)'$ we have a functional $\varphi_g \in L_1(\nu)^*$ defined by

$$\varphi_g(f) := \int_\Omega fg\, d\mu \quad \text{for all } f \in L_1(\nu).$$

Since $L_1(\nu)$ has order continuous norm, the equality

$$L_1(\nu)^* = \{\varphi_g : \, g \in L_1(\nu)'\} \tag{2.1}$$

holds (see, e.g., [13, p. 29]).

We will also need the following two auxiliary results, which can be found in [19, Lemma 2.3] and [15, Lemma 3.6], respectively.

Lemma 2.1. *Let Γ be a non-empty set and let Z be a closed subspace of $\ell_\infty(\Gamma)$. For each $\gamma \in \Gamma$, denote by $e_\gamma^* \in B_{\ell_\infty(\Gamma)^*}$ the γ-th coordinate projection. Let (Ω, Σ)*

be a measurable space and let $\nu \in ca(\Sigma, Z)$*. Then*

$$\|f\|_{L_1(\nu)} = \sup_{\gamma\in\Gamma} \int_\Omega |f|\, d|e_\gamma^*\nu| \quad \textit{for every } f \in L_1(\nu).$$

Lemma 2.2. *Let* X *and* Y *be Banach spaces, let* (Ω, Σ) *be a measurable space and let* $\nu \in \mathrm{ca}(\Sigma, X)$ *and* $\tilde{\nu} \in \mathrm{ca}(\Sigma, Y)$ *such that* $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$*. Suppose that there is a constant* $c > 0$ *such that* $\|f\|_{L_1(\nu)} \leq c\|f\|_{L_1(\tilde{\nu})}$ *for every* $f \in \mathrm{sim}(\Sigma)$*. Then* $L_1(\tilde{\nu})$ *embeds continuously into* $L_1(\nu)$ *with norm* $\leq c$*.*

3. Results

Given a probability space (Ω, Σ, μ), we consider the equivalence relation on Σ defined by $A \sim B$ if and only if $\mu(A \triangle B) = 0$. The set $\Sigma/\mathcal{N}(\mu)$ of equivalence classes becomes a *measure algebra* with the usual Boolean algebra operations and the functional defined by $\mu^\bullet(A^\bullet) := \mu(A)$ for all $A \in \Sigma$, where $A^\bullet \in \Sigma/\mathcal{N}(\mu)$ is the equivalence class of A. We refer to [9] for more information on measure algebras.

We now proceed with the proof of our main result.

Proof of Theorem 1.2. We divide the proof into several steps.

Step 1. Let (Ω, Σ) be the underlying measurable space and let μ be a Rybakov control measure of ν. Suppose without loss of generality that $\mu(\Omega) = 1$. Since μ is homogeneous and has Maharam type κ, Maharam's theorem (see, e.g., [9, Section 3] or [12, §14]) ensures that the measure algebras of μ and λ_κ are isomorphic, that is, there is a Boolean algebra isomorphism

$$\theta : \Sigma/\mathcal{N}(\mu) \to \Lambda_\kappa/\mathcal{N}(\lambda_\kappa)$$

such that $\lambda_\kappa^\bullet \circ \theta = \mu^\bullet$. This isomorphism induces a lattice isometry

$$\Phi : L_1(\mu) \to L_1(\lambda_\kappa)$$

such that for every $f \in L_1(\mu)$ we have

$$\int_\Omega f\, d\mu = \int_{\{-1,1\}^\kappa} \Phi(f)\, d\lambda_\kappa \tag{3.1}$$

and

$$\Phi(f\chi_A) = \Phi(f)\chi_C \quad \text{whenever } A \in \Sigma \text{ and } C \in \Lambda_\kappa \text{ satisfy } \theta(A^\bullet) = C^\bullet. \tag{3.2}$$

Step 2. It is well known that for an arbitrary Banach space Y the inequalities

$$\mathrm{dens}(Y^*, \mathrm{weak}^*) \leq \mathrm{dens}(B_{Y^*}, \mathrm{weak}^*) \leq \mathrm{dens}(Y)$$

hold. If, in addition, Y is weakly compactly generated, then they turn out to be equalities (see, e.g., [7, Theorem 13.3]). Therefore, since $L_1(\nu)$ is weakly compactly generated, [2, Theorem 2] (cf. [1, p. 193]), we have

$$\mathrm{dens}(B_{L_1(\nu)^*}, \mathrm{weak}^*) = \mathrm{dens}(L_1(\nu)) = \kappa.$$

Let $H \subseteq B_{L_1(\nu)^*}$ be a weak*-dense subset of $B_{L_1(\nu)^*}$ with cardinality κ. Let us write $H = \{\varphi_{h_\alpha} : \alpha < \kappa\}$ where $h_\alpha \in L_1(\nu)'$ for all $\alpha < \kappa$ (see equality (2.1) at page 3). Then

$$\|f\|_{L_1(\nu)} = \sup_{\alpha<\kappa} \varphi_{h_\alpha}(f) \quad \text{for all } f \in L_1(\nu).$$

Since $|h_\alpha| \in L_1(\nu)'$ and $\varphi_{|h_\alpha|} \in B_{L_1(\nu)^*}$ for all $\alpha < \kappa$, the previous equality yields

$$\|f\|_{L_1(\nu)} = \sup_{\alpha<\kappa} \varphi_{|h_\alpha|}(|f|) \quad \text{for all } f \in L_1(\nu). \tag{3.3}$$

Fix $\alpha < \kappa$. Then $h_\alpha \in L_1(\mu)$ and so we can consider $\Phi(h_\alpha) \in L_1(\lambda_\kappa)$. Hence, there exist a countable set $I_\alpha \subseteq \kappa$ and $\tilde{h}_\alpha \in L_1(\lambda_{I_\alpha})$ such that

$$\Phi(h_\alpha) = \tilde{h}_\alpha \circ \rho^\kappa_{I_\alpha} \tag{3.4}$$

(see, e.g., [10, 254Q]).

Let $\psi : \kappa \to \kappa$ be an injective map such that $\psi(\alpha) \not\in I_\alpha$ for all $\alpha < \kappa$. Note that such a map can be constructed by transfinite induction. Indeed, take $\alpha < \kappa$ and suppose that $\psi(\beta)$ is already defined for all $\beta < \alpha$. Then $\{\psi(\beta) : \beta < \alpha\} \cup I_\alpha$ has cardinality strictly less than κ (bear in mind that κ is uncountable and I_α is countable). Hence, we can pick $\psi(\alpha) \in \kappa \setminus \{\psi(\beta) : \beta < \alpha\} \cup I_\alpha$.

STEP 3. Fix $\alpha < \kappa$. We write

$$\pi^\kappa_{\psi(\alpha)} = \chi_{C_{\psi(\alpha)}} - \chi_{\{-1,1\}^\kappa \setminus C_{\psi(\alpha)}},$$

where $C_{\psi(\alpha)} := (\pi^\kappa_{\psi(\alpha)})^{-1}(\{1\}) \in \Lambda_\kappa$. Then

$$\Phi^{-1}(\pi^\kappa_{\psi(\alpha)}) = \chi_{A_{\psi(\alpha)}} - \chi_{\Omega \setminus A_{\psi(\alpha)}},$$

where $A_{\psi(\alpha)}$ is some element of Σ with $\theta(A^\bullet_{\psi(\alpha)}) = C^\bullet_{\psi(\alpha)}$. Since $|\Phi^{-1}(\pi^\kappa_{\psi(\alpha)})| = \chi_\Omega$ and $\varphi_{h_\alpha} \in B_{L_1(\nu)^*}$, we have

$$g_\alpha := h_\alpha \Phi^{-1}(\pi^\kappa_{\psi(\alpha)}) \in L_1(\nu)'$$

with $\varphi_{g_\alpha} \in B_{L_1(\nu)^*}$ and so

$$\Big| \int_A g_\alpha \, d\mu \Big| = |\varphi_{g_\alpha}(\chi_A)| \leq \|\chi_A\|_{L_1(\nu)} \|\varphi_{g_\alpha}\|_{L_1(\nu)^*} \leq \|\chi_A\|_{L_1(\nu)} = \|\nu\|(A)$$

for every $A \in \Sigma$. Hence, we have

$$\tilde{\nu}(A) := \left(\int_A g_\alpha \, d\mu \right)_{\alpha<\kappa} \in \ell_\infty(\kappa)$$

and

$$\|\tilde{\nu}(A)\|_{\ell_\infty(\kappa)} \leq \|\nu\|(A) \tag{3.5}$$

for every $A \in \Sigma$. Clearly, $\tilde{\nu} : \Sigma \to \ell_\infty(\kappa)$ is finitely additive. Since $\|\nu\|(A) \to 0$ as $\mu(A) \to 0$ (see, e.g., [5, p. 10, Theorem 1]), inequality (3.5) ensures that $\tilde{\nu}$ is countably additive, that is, $\tilde{\nu} \in \mathrm{ca}(\Sigma, \ell_\infty(\kappa))$.

STEP 4. Let $C \subseteq \{-1,1\}^\kappa$ be an arbitrary closed-and-open set (in particular, $C \in \Lambda_\kappa$) and let $A \in \Sigma$ such that $\theta(A^\bullet) = C^\bullet$. We claim that for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of pairwise distinct elements of κ we have $\int_A g_{\alpha_n} \, d\mu = 0$ for n large enough.

Indeed, since C is closed-and-open, there exist a finite set $I \subseteq \kappa$ and a set $B \subseteq \{-1,1\}^I$ such that $C = B \times \{-1,1\}^{\kappa\setminus I}$. For each $n \in \mathbb{N}$ we define $J_n := I \cup I_{\alpha_n}$ and, bearing in mind (3.4), we write

$$\Phi(h_{\alpha_n})\chi_C = \hat{h}_{\alpha_n} \circ \rho^\kappa_{J_n} \tag{3.6}$$

for some $\hat{h}_{\alpha_n} \in L_1(\lambda_{J_n})$. Since I is finite, ψ is injective and the α_n's are pairwise distinct, there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\psi(\alpha_n) \notin J_n$, thus

$$\int_{\{-1,1\}^{\kappa \setminus J_n}} \pi^{\kappa \setminus J_n}_{\psi(\alpha_n)} \, d\lambda_{\kappa \setminus J_n} = 0 \tag{3.7}$$

and so

$$\begin{aligned}
\int_A g_{\alpha_n} \, d\mu &= \int_\Omega h_{\alpha_n} (\chi_{A_{\psi(\alpha_n)} \cap A} - \chi_{A \setminus A_{\psi(\alpha_n)}}) \, d\mu \\
&\overset{(3.1)\ \&\ (3.2)}{=} \int_{\{-1,1\}^\kappa} \Phi(h_{\alpha_n}) (\chi_{C_{\psi(\alpha_n)} \cap C} - \chi_{C \setminus C_{\psi(\alpha_n)}}) \, d\lambda_\kappa \\
&= \int_{\{-1,1\}^\kappa} \Phi(h_{\alpha_n}) \chi_C \, \pi^\kappa_{\psi(\alpha_n)} \, d\lambda_\kappa \\
&\overset{(3.6)}{=} \int_{\{-1,1\}^\kappa} \left(\hat{h}_{\alpha_n} \circ \rho^\kappa_{J_n}\right) \left(\pi^{\kappa \setminus J_n}_{\psi(\alpha_n)} \circ \rho^\kappa_{\kappa \setminus J_n}\right) d\lambda_\kappa \\
&\overset{(*)}{=} \left(\int_{\{-1,1\}^{J_n}} \hat{h}_{\alpha_n} \, d\lambda_{J_n} \right) \left(\int_{\{-1,1\}^{\kappa \setminus J_n}} \pi^{\kappa \setminus J_n}_{\psi(\alpha_n)} \, d\lambda_{\kappa \setminus J_n} \right) \overset{(3.7)}{=} 0,
\end{aligned}$$

where equality (*) follows from Fubini's theorem.

STEP 5. We claim that

$$\tilde{\nu}(A) \in c_0(\kappa) \quad \text{for every } A \in \Sigma$$

and so $\tilde{\nu} \in \mathrm{ca}(\Sigma, c_0(\kappa))$.

Indeed, fix $A \in \Sigma$ and $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\|\tilde{\nu}(B)\|_{\ell_\infty(\kappa)} \leq \frac{\varepsilon}{2} \quad \text{for every } B \in \Sigma \text{ with } \mu(B) \leq \delta$$

(see STEP 3). Take $C \in \Lambda_\kappa$ such that $\theta(A^\bullet) = C^\bullet$. There is a closed-and-open set $C_\varepsilon \subseteq \{-1,1\}^\kappa$ such that $\lambda_\kappa(C \triangle C_\varepsilon) \leq \delta$. Take $A_\varepsilon \in \Sigma$ such that $\theta(A_\varepsilon^\bullet) = C_\varepsilon^\bullet$. By STEP 4, we have $\tilde{\nu}(A_\varepsilon) \in c_0(\kappa)$. Since $\mu(A \triangle A_\varepsilon) = \lambda_\kappa(C \triangle C_\varepsilon) \leq \delta$, we have

$$\begin{aligned}
\|\tilde{\nu}(A) - \tilde{\nu}(A_\varepsilon)\|_{\ell_\infty(\kappa)} &= \|\tilde{\nu}(A \setminus A_\varepsilon) - \tilde{\nu}(A_\varepsilon \setminus A)\|_{\ell_\infty(\kappa)} \\
&\leq \|\tilde{\nu}(A \setminus A_\varepsilon)\|_{\ell_\infty(\kappa)} + \|\tilde{\nu}(A_\varepsilon \setminus A)\|_{\ell_\infty(\kappa)} \leq \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $\tilde{\nu}(A_\varepsilon) \in c_0(\kappa)$ and $c_0(\kappa)$ is a closed subspace of $\ell_\infty(\kappa)$, it follows that $\tilde{\nu}(A) \in c_0(\kappa)$. This proves the claim.

STEP 6. Fix $f \in \mathrm{sim}(\Sigma)$. By Lemma 2.1 and the very definition of $\tilde{\nu}$, we have

$$\|f\|_{L_1(\tilde{\nu})} = \sup_{\alpha < \kappa} \int_\Omega |f g_\alpha| \, d\mu = \sup_{\alpha < \kappa} \int_\Omega |f h_\alpha| \, d\mu = \sup_{\alpha < \kappa} \varphi_{|h_\alpha|}(|f|) \overset{(3.3)}{=} \|f\|_{L_1(\nu)}.$$

In particular, $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$ and we can apply Lemma 2.2 twice to infer that $L_1(\nu) = L_1(\tilde{\nu})$ with equal norms. The proof is finished. □

The following lemma will be useful when dealing with non-homogeneous vector measures. Let us recall first a standard renorming for the L_1 space of a vector measure. Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Then the formula

$$|||f|||_{L_1(\nu)} := \sup_{A \in \Sigma} \left\| \int_A f \, d\nu \right\|_X, \quad f \in L_1(\nu),$$

defines an equivalent norm on $L_1(\nu)$ and, in fact, one has

$$|||f|||_{L_1(\nu)} \leq \|f\|_{L_1(\nu)} \leq 2|||f|||_{L_1(\nu)} \quad \text{for all } f \in L_1(\nu) \tag{3.8}$$

(see, e.g., [16, p. 112]).

Lemma 3.1. *Let X, X_1 and X_2 be Banach spaces, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Let $A_1, A_2 \in \Sigma$ be disjoint with $\Omega = A_1 \cup A_2$ and let $\nu_i \in \mathrm{ca}(\Sigma_{A_i}, X_i)$ such that $L_1(\nu_{A_i}) = L_1(\nu_i)$ with equivalent norms for $i \in \{1, 2\}$. Define $\tilde{\nu} : \Sigma \to X_1 \oplus_\infty X_2$ by*

$$\tilde{\nu}(A) := (\nu_1(A \cap A_1), \nu_2(A \cap A_2)) \quad \text{for all } A \in \Sigma.$$

Then $\tilde{\nu} \in \mathrm{ca}(\Sigma, X_1 \oplus_\infty X_2)$ and $L_1(\nu) = L_1(\tilde{\nu})$ with equivalent norms.

Proof. Write $Z := X_1 \oplus_\infty X_2$. Clearly, $\tilde{\nu} \in \mathrm{ca}(\Sigma, Z)$. Let c and d be positive constants such that

$$c^{-1}\|f|_{A_1}\|_{L_1(\nu_1)} \leq \|f|_{A_1}\|_{L_1(\nu_{A_1})} \leq c\|f|_{A_1}\|_{L_1(\nu_1)} \tag{3.9}$$

and

$$d^{-1}\|f|_{A_2}\|_{L_1(\nu_2)} \leq \|f|_{A_2}\|_{L_1(\nu_{A_2})} \leq d\|f|_{A_2}\|_{L_1(\nu_2)} \tag{3.10}$$

for every $f \in \mathrm{sim}(\Sigma)$.

On the one hand, we have

$$\|f\|_{L_1(\nu)} \leq 2(c + d)\|f\|_{L_1(\tilde{\nu})} \tag{3.11}$$

for every $f \in \mathrm{sim}(\Sigma)$. Indeed, note that

$$\int_A f \, d\tilde{\nu} = \left(\int_{A \cap A_1} f|_{A_1} \, d\nu_1, \int_{A \cap A_2} f|_{A_2} \, d\nu_2 \right) \quad \text{for all } A \in \Sigma \tag{3.12}$$

and so for each $i \in \{1, 2\}$ we have

(3.13)

$$\|f|_{A_i}\|_{L_1(\nu_i)} \overset{(3.8)}{\leq} 2 \sup_{A \in \Sigma} \left\| \int_{A \cap A_i} f|_{A_i} \, d\nu_i \right\|_{X_i} \overset{(3.12)}{\leq} 2 \sup_{A \in \Sigma} \left\| \int_A f \, d\tilde{\nu} \right\|_Z \overset{(3.8)}{\leq} 2\|f\|_{L_1(\tilde{\nu})}.$$

It follows that

$$\begin{aligned} \|f\|_{L_1(\nu)} &\leq \|f\chi_{A_1}\|_{L_1(\nu)} + \|f\chi_{A_2}\|_{L_1(\nu)} \\ &= \|f|_{A_1}\|_{L_1(\nu_{A_1})} + \|f|_{A_2}\|_{L_1(\nu_{A_2})} \\ &\overset{(3.9)\ \&\ (3.10)}{\leq} c\|f|_{A_1}\|_{L_1(\nu_1)} + d\|f|_{A_2}\|_{L_1(\nu_2)} \overset{(3.13)}{\leq} 2(c + d)\|f\|_{L_1(\tilde{\nu})}. \end{aligned}$$

This proves inequality (3.11).

On the other hand, we have

$$\|f\|_{L_1(\tilde{\nu})} \leq 2 \max\{c, d\}\|f\|_{L_1(\nu)} \tag{3.14}$$

for every $f \in \mathrm{sim}(\Sigma)$. Indeed, observe that

$$\begin{aligned}
\|f\|_{L_1(\tilde{\nu})} &\overset{(3.8)}{\leq} 2 \sup_{A\in\Sigma} \left\| \int_A f \, d\tilde{\nu} \right\|_Z \\
&\overset{(3.12)}{=} 2 \sup_{A\in\Sigma} \max \left\{ \left\| \int_{A\cap A_1} f|_{A_1} \, d\nu_1 \right\|_{X_1}, \left\| \int_{A\cap A_2} f|_{A_2} \, d\nu_2 \right\|_{X_2} \right\} \\
&\overset{(3.8)}{\leq} 2 \max\{ \|f|_{A_1}\|_{L_1(\nu_1)}, \|f|_{A_2}\|_{L_1(\nu_2)} \} \\
&\overset{(3.9)\ \&\ (3.10)}{\leq} 2 \max\{ c\|f|_{A_1}\|_{L_1(\nu_{A_1})}, d\|f|_{A_2}\|_{L_1(\nu_{A_2})} \} \\
&\leq 2 \max\{c, d\} \|f\|_{L_1(\nu)},
\end{aligned}$$

as claimed.

Finally, inequalities (3.11) and (3.14) allow to apply Lemma 2.2 to deduce that $L_1(\nu) = L_1(\tilde{\nu})$ with equivalent norms. □

Theorem 3.2. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. If ν is atomless and $L_1(\nu)$ has density character $\aleph_k$ for some $k \in \mathbb{N}$, then there is $\tilde{\nu} \in \mathrm{ca}(\Sigma, c_0(\aleph_k))$ such that $L_1(\nu) = L_1(\tilde{\nu})$ with equivalent norms.*

Proof. Let μ be a Rybakov control measure of ν. Then μ is atomless and $L_1(\mu)$ has density character $\aleph_k$. Therefore, there exists a finite partition $\{A_1, \ldots, A_n\}$ of Ω consisting of elements of Σ such that, for each $i \in \{1, \ldots, n\}$, the restriction of μ to Σ_{A_i} is homogeneous and has Maharam type $\aleph_{m_i}$ for some $m_i \in \mathbb{N} \cup \{0\}$ satisfying $m_1 < m_2 < \ldots < m_n = k$ (see, e.g., [9, Section 3] or [12, p. 122, Theorem 7]). Now, for each $i \in \{1, \ldots, n\}$ we can apply either Theorem 1.1 or Theorem 1.2 to ν_{A_i} in order to get $\nu_i \in \mathrm{ca}(\Sigma_{A_i}, c_0(\aleph_{m_i}))$ such that $L_1(\nu_{A_i}) = L_1(\nu_i)$ with equal norms. Let us consider the Banach space

$$Y := \left(\bigoplus_{i=1}^{n} c_0(\aleph_{m_i}) \right)_{\infty},$$

which is isometric to $c_0(\aleph_k)$. Finally, we can apply inductively Lemma 3.1 to get $\tilde{\nu} \in \mathrm{ca}(\Sigma, Y)$ such that $L_1(\nu) = L_1(\tilde{\nu})$ with equivalent norms. □

Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a map. The Orlicz-Pettis theorem (see, e.g., [5, p. 22, Corollary 4]) implies that $\nu \in \mathrm{ca}(\Sigma, X)$ if and only if the composition of ν with each $x^* \in X^*$ is countably additive. Diestel and Faires [4] (cf. [5, p. 23, Corollary 7]) proved that if X contains no closed subspace isomorphic to ℓ_∞ and $\Delta \subseteq X^*$ is a total set (i.e., $\bigcap_{x^* \in \Delta} \ker x^* = \{0\}$), then $\nu \in \mathrm{ca}(\Sigma, X)$ if and only if the composition of ν with each $x^* \in \Delta$ is countably additive. As an application, we get part (i) of the following result, which also collects further properties of $c_0(\kappa)$-valued vector measures.

Theorem 3.3. *Let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to c_0(\kappa)$ be a map, where κ is a cardinal. For each $\alpha < \kappa$, let $e_\alpha^* \in c_0(\kappa)^*$ be the αth-coordinate projection and let $\nu_\alpha : \Sigma \to \mathbb{R}$ be the composition of ν with e_α^*. The following statements hold:*

(i) $\nu \in \mathrm{ca}(\Sigma, c_0(\kappa))$ *if and only if* $\nu_\alpha \in \mathrm{ca}(\Sigma, \mathbb{R})$ *for all* $\alpha < \kappa$.

(ii) *If* $\nu \in \mathrm{ca}(\Sigma, c_0(\kappa))$*, then:*

(ii.a) *There is a countable set* $\Gamma \subseteq \kappa$ *such that* $\bigcap_{\alpha\in\Gamma} \mathcal{N}(\nu_\alpha) \subseteq \mathcal{N}(\nu)$.

(ii.b) *For each* $\varepsilon > 0$ *there is a countable partition* $\kappa = \bigcup_{n\in\mathbb{N}} \Gamma_{n,\varepsilon}$ *such that for every* $n \in \mathbb{N}$ *and for every* $A \in \Sigma$ *the set* $\{\alpha \in \Gamma_{n,\varepsilon} : |\nu_\alpha(A)| > \varepsilon\}$ *has cardinality less than* n.

Proof. (i) This follows from the aforementioned result of Diestel and Faires applied to the total set $\Delta := \{e^*_\alpha : \alpha < \kappa\} \subseteq c_0(\kappa)^*$ (bear in mind that weakly compactly generated Banach spaces, like $c_0(\kappa)$, contain no closed subspace isomorphic to ℓ_∞).

(ii.a) Let μ be a Rybakov control measure of ν. Since $c_0(\kappa)^* = \ell_1(\kappa)$, there is $\varphi \in \ell_1(\kappa)$ such that $\mu = |\varphi \circ \nu|$. The set $\Gamma := \{\alpha < \kappa : \varphi(\alpha) \neq 0\}$ is countable and $(\varphi \circ \nu)(A) = \sum_{\alpha\in\Gamma} \varphi(\alpha)\nu_\alpha(A)$ for every $A \in \Sigma$, the series being absolutely convergent. Clearly, the inclusion $\bigcap_{\alpha\in\Gamma} \mathcal{N}(\nu_\alpha) \subseteq \mathcal{N}(\mu) = \mathcal{N}(\nu)$ holds.

(ii.b) Let $K \subseteq c_0(\kappa)$ be the weak closure of the set $\{\nu(A) : A \in \Sigma\}$. Then K is weakly compact (see, e.g., [5, p. 14, Corollary 7]) and, in fact, it is uniform Eberlein compact (i.e., it is homeomorphic to a weakly compact subset of a Hilbert space) when equipped with the weak topology, [18, Corollary 2.3]. The conclusion follows from Farmaki's characterization of those compact subsets of sigma-products which are uniform Eberlein compact, see [8] (cf. [11, Corollary 6.33(i)]). □

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References

[1] A. V. Bukhvalov, A. I. Veksler, and G. Ya. Lozanovskij, *Banach lattices - some Banach aspects of the theory*, Russ. Math. Surv. **34** (1979), no. 2, 159–212.

[2] G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), no. 2, 317–330.

[3] G. P. Curbera, *When L^1 of a vector measure is an AL-space*, Pacific J. Math. **162** (1994), no. 2, 287–303.

[4] J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc. **198** (1974), 253–271.

[5] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[6] P. G. Dodds, B. de Pagter, and W. Ricker, *Reflexivity and order properties of scalar-type spectral operators in locally convex spaces*, Trans. Amer. Math. Soc. **293** (1986), no. 1, 355–380.

[7] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics, Springer, New York, 2011.

[8] V. Farmaki, *The structure of Eberlein, uniformly Eberlein and Talagrand compact spaces in* $\Sigma(\mathbf{R}^\Gamma)$, Fund. Math. **128** (1987), no. 1, 15–28.

[9] D. H. Fremlin, *Measure algebras*, Handbook of Boolean algebras, Vol. 3, North-Holland, Amsterdam, 1989, pp. 877–980.

[10] D. H. Fremlin, *Measure theory. Vol. 2. Broad foundations*, Torres Fremlin, Colchester, 2003.

[11] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics, Springer, New York, 2008.

[12] H. E. Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, Band 208, Springer-Verlag, New York, 1974.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II. Function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer-Verlag, Berlin, 1979.

[14] Z. Lipecki, *Semivariations of a vector measure*, Acta Sci. Math. (Szeged) **76** (2010), no. 3-4, 411–425.

[15] O. Nygaard and J. Rodríguez, *Isometric factorization of vector measures and applications to spaces of integrable functions*, J. Math. Anal. Appl. **508** (2022), no. 1, paper no. 125857, 16 p.

[16] S. Okada, W. J. Ricker, and E. A. Sánchez Pérez, *Optimal domain and integral extension of operators. Acting in function spaces*, Operator Theory: Advances and Applications, vol. 180, Birkhäuser Verlag, Basel, 2008.

[17] S. Okada, J. Rodríguez, and E. A. Sánchez-Pérez, *On vector measures with values in* ℓ_∞, Studia Math. **274** (2024), no. 2, 173–199.

[18] J. Rodríguez, *Factorization of vector measures and their integration operators*, Colloq. Math. **144** (2016), no. 1, 115–125.

[19] J. Rodríguez, *On non-separable* L^1*-spaces of a vector measure*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 4, 1039–1050.

Dpto. de Matemáticas, E.T.S. de Ingenieros Industriales de Albacete, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

Email address: jose.rodriguezruiz@uclm.es, joserr@um.es