DUNFORD-PETTIS TYPE PROPERTIES IN L_1 OF A VECTOR MEASURE

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Abstract. Let ν be a countably additive vector measure defined on a σ-algebra and taking values in a Banach space. In this paper we deal with the following three properties for the Banach lattice $L_1(\nu)$ of all ν-integrable real-valued functions: the Dunford-Pettis property, the positive Schur property and being lattice-isomorphic to an AL-space. We give new results and we also provide alternative proofs of some already known ones.

1. Introduction

Let X be a Banach space with (topological) dual X^*, let (Ω, Σ) be a measurable space and let $\nu : \Sigma \to X$ be a (countably additive) vector measure. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is called *ν-integrable* if it is $|x^*\nu|$-integrable for all $x^* \in X^*$ and, for each $A \in \Sigma$, there is $\int_A f \, d\nu \in X$ such that

$$x^* \left(\int_A f \, d\nu \right) = \int_A f \, d(x^*\nu) \quad \text{for all } x^* \in X^*.$$

Here $x^*\nu$ is the signed measure obtained as the composition of ν with x^* and $|x^*\nu|$ denotes its variation. By identifying functions which coincide except to a ν-null set (where $A \in \Sigma$ is said to be *ν-null* if $\nu(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$), the set $L_1(\nu)$ of all (equivalence classes of) ν-integrable functions is a Banach lattice with the ν-a.e. order and the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|x^*\nu|.$$

Here B_{X^*} denotes the closed unit ball of X^*. Let us agree to say that $L_1(\nu)$ is the L_1 space of the vector measure ν.

Every Banach lattice with order continuous norm and a weak unit is lattice-isometric to the L_1 space of a vector measure, [11, Theorem 8] (cf. [19, Proposition 2.4]). Such a representation is not unique. For instance, the usual space $L_1[0,1]$ is equal to $L_1(\nu_i)$ for each one of the following X_i-valued vector measures ν_i defined on the Borel σ-algebra of $[0,1]$:

- $X_1 := \mathbb{R}$ and $\nu_1(A) := \lambda(A)$ (the Lebesgue measure of A);

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- $X_2 := L_1[0,1]$ and $\nu_2(A) := \chi_A$ (the characteristic function of A);
- $X_3 := c_0$ and $\nu_3(A) := (\int_A r_n \, d\lambda)_{n\in\mathbb{N}}$, where $(r_n)_{n\in\mathbb{N}}$ is the sequence of Rademacher functions.

The structure of the space $L_1(\nu)$ can be greatly conditioned by certain properties of ν. For complete information on these spaces and their important role in Banach lattices and operator theory, we refer the reader to the monograph [33] and the papers [6, 14, 15, 28, 34, 36, 38].

The inclusion map

$$\iota_\nu : L_1(|\nu|) \to L_1(\nu)$$

is a well-defined injective lattice-homomorphism, where $|\nu|$ is the variation of ν (see, e.g., [33, Lemma 3.14]). If ι_ν is surjective, then it is a lattice-isomorphism and, moreover, we have $|\nu|(\Omega) < \infty$. Curbera [12] addressed the question of when the L_1 space of a vector measure is lattice-isomorphic to an AL-space. Recall that a Banach lattice E is said to be an *AL-space* if its norm satisfies $\|x+y\| = \|x\|+\|y\|$ whenever $x, y \in E$ are disjoint, which is equivalent to saying that E is lattice-isometric to the usual space $L_1(\mu)$ of a non-negative measure μ (see, e.g., [3, Theorem 4.27]). It turns out that $L_1(\nu)$ is lattice-isomorphic to an AL-space if and only if ι_ν is surjective, [12, Proposition 2]. This is also equivalent to the fact that the *integration operator* of ν, that is, the norm 1 operator

$$I_\nu : L_1(\nu) \to X, \quad I_\nu(f) := \int_\Omega f \, d\nu \quad \text{for all } f \in L_1(\nu),$$

is *cone absolutely summing* (i.e., the series $\sum_{n=1}^\infty I_\nu(f_n)$ is absolutely convergent whenever $\sum_{n\in\mathbb{N}} f_n$ is unconditionally convergent and $f_n \in L_1(\nu)^+$ for all $n \in \mathbb{N}$), [10, Proposition 3.1]. As usual, given a Banach lattice E, we denote by E^+ its positive cone, that is, $E^+ := \{x \in E : x \geq 0\}$. At this point we should stress that if a Banach lattice is isomorphic (just as a Banach space) to an AL-space, then it is lattice-isomorphic to an AL-space [1] (cf. [16, Proposition 2.1]).

An operator between Banach spaces is said to be *Dunford-Pettis* (or *completely continuous*) if it maps weakly null sequences to norm null ones. The space $L_1(\mu)$ of a non-negative measure μ has the *Dunford-Pettis property*, that is, every weakly compact operator from $L_1(\mu)$ to an arbitrary Banach space is Dunford-Pettis (see, e.g., [2, Theorem 5.4.5] or [3, Theorem 5.85]). In general, this is not true for the L_1 space of a vector measure. Indeed, reflexive infinite-dimensional Banach spaces fail the Dunford-Pettis property and, as we have already mentioned, spaces like ℓ_p and $L_p[0,1]$ for $1 < p < \infty$ can be seen as L_1 spaces of a vector measure. On the other side, there are L_1 spaces of a vector measure having the Dunford-Pettis property which are not lattice-isomorphic to an AL-space, like c_0. Curbera showed in [13, Theorem 4] that $L_1(\nu)$ has the Dunford-Pettis property if ν has σ-finite variation and X has the Schur property (i.e., every weakly null sequence in X is norm null). In fact, he proved that:

(i) $L_1(\nu)$ has the positive Schur property whenever X has the Schur property.
(ii) If $L_1(\nu)$ has the positive Schur property and ν has σ-finite variation, then $L_1(\nu)$ has the Dunford-Pettis property (cf. [6, Section 3.2]).

Recall that a Banach lattice E is said to have the *positive Schur property* if every weakly null sequence in E^+ is norm null. Note that statement (i) can be deduced at once from the fact that $L_1(\nu)$ has the positive Schur property if and only if the integration operator I_ν is *almost Dunford-Pettis* (i.e., $(I_\nu(f_n))_{n\in\mathbb{N}}$ is norm null for every weakly null sequence $(f_n)_{n\in\mathbb{N}}$ in $L_1(\nu)^+$), see [6, Theorem 5.12].

The integration operator is undoubtedly a key point in the theory of L_1 spaces of a vector measure. Note that its properties depend on ν rather than on the space $L_1(\nu)$ itself. For instance, going back to the example at the beginning, we have:

- I_{ν_1} is the functional given by $I_{\nu_1}(f) = \int_{[0,1]} f\, d\lambda$;
- I_{ν_2} is the identity operator on $L_1[0,1]$;
- $I_{\nu_3} : L_1[0,1] \to c_0$ is the operator given by $I_{\nu_3}(f) = (\int_{[0,1]} r_n f\, d\lambda)_{n\in\mathbb{N}}$, which is strictly singular but fails to be weakly compact.

It is known that $L_1(\nu)$ is lattice-isomorphic to an AL-space whenever I_ν is compact (see [30, Theorem 1], cf. [32, Theorem 2.2] and [7, Theorem 3.3]), absolutely p-summing for $1 \le p < \infty$ (see [31, Theorem 2.2]) or, more generally, Dunford-Pettis and Asplund (see [35, Theorem 3.3]). Recall that an operator between Banach spaces is said to be *Asplund* if it factors through a Banach space which is Asplund (i.e., all of its separable subspaces have separable dual). In particular, $L_1(\nu)$ is lattice-isomorphic to an AL-space if I_ν is Dunford-Pettis and X is Asplund, [7, Theorem 1.3]. This is a partial answer to the following question posed by Okada, Ricker and Rodríguez-Piazza [31]:

Question 1.1. *Suppose that I_ν is Dunford-Pettis and that X contains no subspace isomorphic to ℓ_1. Is $L_1(\nu)$ lattice-isomorphic to an AL-space?*

They showed that this is the case if, in addition, X has an unconditional Schauder basis, [31, Theorem 1.2]. Note that any Banach space with an unconditional Schauder basis and no subspace isomorphic to ℓ_1 has separable dual (see, e.g., [2, Theorem 3.3.1]). To the best of our knowledge, Question 1.1 remains open.

In this paper we deal with L_1 spaces of a vector measure with focus on the property of being isomorphic to an AL-space, the positive Schur property and the Dunford-Pettis property. Our aim is twofold: we include new results and we also present alternative proofs of some already known ones which hopefully might led to a better understanding of the theory. The structure of the paper is as follows.

In Section 2 we collect some known preliminary facts on L_1 spaces of a vector measure that will be needed later.

In Section 3 we revisit the aforementioned positive answer to Question 1.1 for Asplund spaces (Corollary 3.8) and the related result for integration operators which are Dunford-Pettis and Asplund (Corollary 3.11).

In Section 4 we show that the positive Schur property of $L_1(\nu)$ can be characterized by means of a Dunford-Pettis type property with respect to the so-called "vector duality" induced by the integration operator, that is, the continuous bilinear map

$$L_1(\nu) \times L_\infty(\nu) \to X, \qquad (f,g) \mapsto I_\nu(fg) = \int_\Omega fg\, d\nu$$

(Theorem 4.3). We also give another proof of the aforementioned result of [13] stating that $L_1(\nu)$ has the Dunford-Pettis property if it has the positive Schur property and ν has σ-finite variation (Corollary 4.5). It seems to be an open question whether the assumption on the variation can be dropped, namely:

Question 1.2. *Suppose that $L_1(\nu)$ has the positive Schur property. Does $L_1(\nu)$ have the Dunford-Pettis property?*

Finally, in Example 4.6 we discuss a class of vector measures ν such that $L_1(\nu)$ has the positive Schur property and the Dunford-Pettis property, but fails to be lattice-isomorphic to an AL-space, among other interesting properties.

2. Preliminaries

All Banach spaces considered in this paper are real. An *operator* is a continuous linear map between Banach spaces. Given an operator T, its adjoint is denoted by T^*. By a *subspace* of a Banach space we mean a norm closed linear subspace. Let Z be a Banach space. The norm of Z is denoted by $\|\cdot\|_Z$, or simply $\|\cdot\|$, and we write $B_Z := \{z \in Z : \|z\| \leq 1\}$ (the closed unit ball of Z). The evaluation of $z^* \in Z^*$ at $z \in Z$ is denoted by either $z^*(z)$ or $\langle z^*, z\rangle$. By a *projection* from Z onto a subspace $Y \subseteq Z$ we mean an operator $P : Z \to Z$ such that $P(Z) = Y$ and P is the identity when restricted to Y. The subspace of Z generated by a set $H \subseteq Z$ is denoted by $\overline{\mathrm{span}}(H)$.

In this section we gather, for the reader's convenience, some known facts on L_1 spaces of a vector measure. A basic reference on this topic is [33, Chapter 3].

Throughout this section X is a Banach space, (Ω, Σ) is a measurable space and $\nu \in \mathrm{ca}(\Sigma, X)$. As usual, we denote by $\mathrm{ca}(\Sigma, X)$ the set of all countably additive X-valued vector measures defined on Σ. The *range* of ν is the set

$$\mathcal{R}(\nu) := \{\nu(A) : A \in \Sigma\} \subseteq X.$$

The variation and semivariation of ν are denoted by $|\nu|$ and $\|\nu\|$, respectively. The family of all ν-null sets is denoted by $\mathcal{N}(\nu)$. By a *Rybakov control measure* of ν we mean a finite non-negative measure of the form $\mu = |x^*\nu|$ for some $x^* \in X^*$ such that $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ (see, e.g., [18, p. 268, Theorem 2]). Throughout this section μ is a fixed Rybakov control measure of ν.

2.1. **L_∞ of a vector measure.** A function $f : \Omega \to \mathbb{R}$ is called Σ-*simple* if it is a linear combination of functions of the form χ_A, where $A \in \Sigma$. Clearly, all Σ-simple functions are ν-integrable. The set of all Σ-simple functions is norm dense in $L_1(\nu)$ (see, e.g., [33, Theorem 3.7(ii)]), so one has

$$\overline{I_\nu(L_1(\nu))} = \overline{\mathrm{span}}(\mathcal{R}(\nu)). \tag{2.1}$$

More generally, every ν-essentially bounded Σ-measurable function $f : \Omega \to \mathbb{R}$ is ν-integrable. By identifying functions which coincide ν-a.e., the set $L_\infty(\nu)$ of all (equivalence classes of) ν-essentially bounded Σ-measurable functions is a Banach lattice with the ν-a.e. order and the ν-essential supremum norm $\|\cdot\|_{L_\infty(\nu)}$. Of course, $L_\infty(\nu)$ is equal to the usual spaces $L_\infty(|\nu|)$ and $L_\infty(\mu)$. The inclusion map

$$j_\nu : L_\infty(\nu) \to L_1(\nu)$$

is an injective operator. Moreover, it is weakly compact. Indeed, $j_\nu(B_{L_\infty(\nu)})$ coincides with the order interval $[-\chi_\Omega, \chi_\Omega]$ in $L_1(\nu)$, so it is weakly compact as $L_1(\nu)$ has order continuous norm (see, e.g., [3, Theorem 4.9]). Hence, $I_\nu(j_\nu(B_{L_\infty(\nu)}))$ is weakly compact in X. We have the following characterization of relative norm compactness of $\mathcal{R}(\nu)$ (see, e.g., [33, Proposition 2.41]):

Proposition 2.1. *The following statements are equivalent:*

(i) $\mathcal{R}(\nu)$ *is relatively norm compact.*
(ii) $I_\nu(j_\nu(B_{L_\infty(\nu)}))$ *is norm compact.*

2.2. **Composition of a vector measure with an operator.** We will use several times the following fact (see, e.g., [33, Lemma 3.27]):

Proposition 2.2. *Let* $T : X \to Y$ *be an operator between Banach spaces. Then:*

(i) *The composition* $\tilde{\nu} := T \circ \nu : \Sigma \to Y$ *is a countably additive vector measure.*
(ii) *Every* ν*-integrable function is* $\tilde{\nu}$*-integrable.*
(iii) *The inclusion map* $u : L_1(\nu) \to L_1(\tilde{\nu})$ *is an operator and* $I_{\tilde{\nu}} \circ u = T \circ I_\nu$.

2.3. **L-weakly compact sets and the positive Schur property.** Let E be a Banach lattice. Given a set $W \subseteq E$, we denote by $\mathrm{Sol}(W)$ its *solid hull*, that is, the set of all $x \in E$ such that $|x| \leq |y|$ for some $y \in W$. It is known that if W is relatively weakly compact, then every disjoint sequence in $\mathrm{Sol}(W)$ is weakly null (see, e.g., [3, Theorem 4.34]). The set W is said to be *L-weakly compact* if it is bounded and every disjoint sequence in $\mathrm{Sol}(W)$ is norm null. Every L-weakly compact set is relatively weakly compact (see, e.g., [3, Theorem 5.55]), but the converse does not hold in general. The following result is well-known (see [26, Corollaries 2.3.5 and 3.6.8], [39, Theorem 1.16] and [41, Lemma 3]):

Proposition 2.3. *Let* E *be a Banach lattice. The following statements are equivalent:*

(i) E *has the positive Schur property.*
(ii) *Every disjoint weakly null sequence in* E *is norm null.*
(iii) *Every disjoint weakly null sequence in* E^+ *is norm null.*
(iv) *Every relatively weakly compact subset of* E *is L-weakly compact.*

Proposition 2.4 below characterizes L-weakly compact sets in the L_1 space of a vector measure. We first need to introduce some terminology. Given $f \in L_1(\nu)$, the map $\nu_f : \Sigma \to X$ defined by

$$\nu_f(A) := I_\nu(f\chi_A) = \int_A f \, d\nu \quad \text{for all } A \in \Sigma$$

is a countably additive vector measure by the Orlicz-Pettis theorem (see, e.g., [18, p. 22, Corollary 4]). Note that $\|\nu_f\|(A) = \|f\chi_A\|_{L_1(\nu)}$ for all $A \in \Sigma$. Moreover, ν_f is μ*-continuous*, that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\nu_f(A)\| \leq \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \leq \delta$ (see, e.g., [18, p. 10, Theorem 1]). A set $F \subseteq L_1(\nu)$ is said to be *equi-integrable* if the set $\{\nu_f : f \in F\}$ is *uniformly μ-continuous*, that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{f \in F} \|\nu_f(A)\| \leq \varepsilon \quad \text{for every } A \in \Sigma \text{ with } \mu(A) \leq \delta.$$

The following result can be found in [26, Proposition 3.6.2] and [33, Lemma 2.37] within a more general framework.

Proposition 2.4. *Let $F \subseteq L_1(\nu)$ be a set. The following statements are equivalent:*

(i) *F is L-weakly compact.*
(ii) *F is bounded and equi-integrable.*
(iii) *F is* approximately order bounded*, i.e., for every $\varepsilon > 0$ there is $\rho > 0$ such that $F \subseteq j_\nu(\rho B_{L_\infty(\nu)}) + \varepsilon B_{L_1(\nu)}$.*

We refer the reader to the recent works [4, 9, 22] for further results related to the positive Schur property in Banach lattices.

2.4. **Characterization of Dunford-Pettis integration operators.** Proposition 2.6 below was first proved in [6, Theorem 5.8]. We include here a more direct proof for the reader's convenience. One part follows the argument used in [13, Theorem 4] to show that $L_1(\nu)$ has the positive Schur property if X has the Schur property. The following auxiliary lemma will also be used later.

Lemma 2.5. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\nu)$ such that the sequence $(\nu_{f_n}(A))_{n\in\mathbb{N}}$ is norm convergent for every $A \in \Sigma$. Then $(f_n)_{n\in\mathbb{N}}$ is equi-integrable.*

Proof. This follows from the Vitali-Hahn-Saks theorem (see, e.g., [18, p. 24, Corollary 10]) applied to the sequence of μ-continuous vector measures $(\nu_{f_n})_{n\in\mathbb{N}}$. □

Proposition 2.6. *The following statements are equivalent:*

(i) *$L_1(\nu)$ has the positive Schur property and $\mathcal{R}(\nu)$ is relatively norm compact.*
(ii) *I_ν is Dunford-Pettis.*

Proof. (i)⇒(ii): Let $F \subseteq L_1(\nu)$ be a relatively weakly compact set. We will show that $I_\nu(F)$ is relatively norm compact by checking that for each $\varepsilon > 0$ there is a norm compact set $K_\varepsilon \subseteq X$ such that $I_\nu(F) \subseteq K_\varepsilon + \varepsilon B_X$. Fix $\varepsilon > 0$. Since F is approximately order bounded (by Propositions 2.3 and 2.4), there is $\rho > 0$ such that $F \subseteq j_\nu(\rho B_{L_\infty(\nu)}) + \varepsilon B_{L_1(\nu)}$. Therefore, $I_\nu(F) \subseteq K_\varepsilon + \varepsilon B_X$, where $K_\varepsilon := I_\nu(j_\nu(\rho B_{L_\infty(\nu)}))$ is norm compact by Proposition 2.1.

(ii)⇒(i): Since $j_\nu(B_{L_\infty(\nu)})$ is weakly compact in $L_1(\nu)$ (see the paragraph preceding Proposition 2.1) and I_ν is Dunford-Pettis, the set $I_\nu(j_\nu(B_{L_\infty(\nu)}))$ is norm compact and so $\mathcal{R}(\nu)$ is relatively norm compact (Proposition 2.1). To prove that $L_1(\nu)$ has the positive Schur property it suffices to check that every weakly convergent sequence is equi-integrable (Propositions 2.3 and 2.4). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\nu)$ which converges weakly to some $f \in L_1(\nu)$. Then for each $A \in \Sigma$ the sequence $(f_n\chi_A)_{n\in\mathbb{N}}$ converges weakly to $f\chi_A$ in $L_1(\nu)$ (bear in mind that the map $h \mapsto h\chi_A$ is an operator on $L_1(\nu)$). Since I_ν is Dunford-Pettis, for each $A \in \Sigma$

the sequence $(I_\nu(f_n\chi_A))_{n\in\mathbb{N}} = (\nu_{f_n}(A))_{n\in\mathbb{N}}$ converges in norm to $I_\nu(f\chi_A) = \nu_f(A)$. Now, Lemma 2.5 applies to conclude that $(f_n)_{n\in\mathbb{N}}$ is equi-integrable. □

2.5. **The "vector duality" induced by the integration operator.** The following result (see, e.g., [33, Proposition 3.31]) shows, in particular, that we have a continuous bilinear map

$$L_1(\nu) \times L_\infty(\nu) \to X$$

defined by

$$(f,g) \mapsto I_\nu(fg) = \int_\Omega fg\, d\nu.$$

Proposition 2.7. *Let $f \in L_1(\nu)$. Then:*

(i) *For every $g \in L_\infty(\nu)$ the product $fg \in L_1(\nu)$ and*

$$\|fg\|_{L_1(\nu)} \leq \|f\|_{L_1(\nu)} \|g\|_{L_\infty(\nu)}.$$

(ii) *The norm of f in $L_1(\nu)$ is*

$$\|f\|_{L_1(\nu)} = \sup_{g\in B_{L_\infty(\nu)}} \|I_\nu(fg)\|_X.$$

There are some elements of $L_1(\nu)^*$ which admit a simple description and are helpful for dealing with the weak topology of $L_1(\nu)$. By Proposition 2.7, for each $(g,x^*) \in B_{L_\infty(\nu)} \times B_{X^*}$ we can define a functional $\gamma_{(g,x^*)} \in B_{L_1(\nu)^*}$ by the formula

$$\gamma_{(g,x^*)}(f) := x^*(I_\nu(fg)) = \int_\Omega fg\, d(x^*\nu) \quad \text{for all } f \in L_1(\nu),$$

and the set

$$\Gamma_\nu := \{\gamma_{(g,x^*)} : (g,x^*) \in B_{L_\infty(\nu)} \times B_{X^*}\} \subseteq B_{L_1(\nu)^*}$$

is norming for $L_1(\nu)$, that is,

$$\|f\|_{L_1(\nu)} = \sup_{\gamma\in\Gamma_\nu} \gamma(f) \quad \text{for all } f \in L_1(\nu). \tag{2.2}$$

Let $\sigma(L_1(\nu),\Gamma_\nu)$ be the (locally convex Hausdorff) topology on $L_1(\nu)$ of pointwise convergence on Γ_ν, which is weaker than the weak topology. Proposition 2.8 below was first proved in [29, Proposition 17]. It was pointed out in [25, Section 4.7] that it can also be seen as a corollary of the Rainwater-Simons theorem (see, e.g., [20, Theorem 3.134]) and the fact that Γ_ν is a James boundary for $L_1(\nu)$ (i.e., the supremum in (2.2) is a maximum) whenever $\mathcal{R}(\nu)$ is relatively norm compact. We refer the reader to [6, Section 4] and [8, Section 2] for more information on this topic.

Proposition 2.8. *Suppose that $\mathcal{R}(\nu)$ is relatively norm compact. Then every bounded and $\sigma(L_1(\nu),\Gamma_\nu)$-convergent sequence in $L_1(\nu)$ is weakly convergent.*

3. Dunford-Pettis integration operators

Let $\mathcal{A}$ be an operator ideal. Following [31], a Banach space X is said to be *$\mathcal{A}$-variation admissible* if for every measurable space (Ω, Σ) and for every $\nu \in \mathrm{ca}(\Sigma, X)$ such that $I_\nu \in \mathcal{A}$, we have $|\nu|(\Omega) < \infty$. The interest of this concept is based on the following result, [31, Proposition 1.1], which provides a tool for proving that L_1 is lattice-isomorphic to an AL-space under some additional assumptions.

Proposition 3.1. *Let $\mathcal{A}$ be an operator ideal and let X be a Banach space. If X is $\mathcal{A}$-variation admissible, then for every measurable space (Ω, Σ) and for every $\nu \in \mathrm{ca}(\Sigma, X)$ such that $I_\nu \in \mathcal{A}$, the inclusion map $\iota_\nu : L_1(|\nu|) \to L_1(\nu)$ is a lattice-isomorphism.*

The next proposition is elementary.

Proposition 3.2. *Let $\mathcal{A}$ be an operator ideal and let X and Y be Banach spaces.*

(i) *If X and Y are isomorphic, then X is $\mathcal{A}$-variation admissible if and only if Y is $\mathcal{A}$-variation admissible.*

(ii) *If X is $\mathcal{A}$-variation admissible, then every subspace of X is $\mathcal{A}$-variation admissible.*

Proof. Let (Ω, Σ) be a measurable space.

(i) If $T : X \to Y$ is an isomorphism and $\nu \in \mathrm{ca}(\Sigma, X)$, then we can apply Proposition 2.2 to deduce that a function is ν-integrable if and only if it is $\tilde{\nu}$-integrable, where we write $\tilde{\nu} := T \circ \nu \in \mathrm{ca}(\Sigma, Y)$, and that the identity map $u : L_1(\nu) \to L_1(\tilde{\nu})$ is an isomorphism satisfying $I_{\tilde{\nu}} \circ u = T \circ I_\nu$. Hence, $I_\nu \in \mathcal{A}$ if and only if $I_{\tilde{\nu}} \in \mathcal{A}$. Moreover, we have $\|T^{-1}\|^{-1} \cdot |\nu|(\Omega) \leq |\tilde{\nu}|(\Omega) \leq \|T\| \cdot |\nu|(\Omega)$.

(ii) Let $Z \subseteq X$ be a subspace and let $i : Z \to X$ be the inclusion operator. Fix $\nu \in \mathrm{ca}(\Sigma, Z)$ such that $I_\nu \in \mathcal{A}$ and define $\tilde{\nu} := i \circ \nu \in \mathrm{ca}(\Sigma, X)$. Then a function is ν-integrable if and only if it is $\tilde{\nu}$-integrable, the identity map $u : L_1(\nu) \to L_1(\tilde{\nu})$ is an isometry and $I_{\tilde{\nu}} = i \circ I_\nu \circ u^{-1} \in \mathcal{A}$ (Proposition 2.2). Since X is $\mathcal{A}$-variation admissible, we have $|\nu|(\Omega) = |\tilde{\nu}|(\Omega) < \infty$. □

In this section we focus on the operator ideal $\mathcal{A}_{cc}$ of all Dunford-Pettis operators. Our main goal is to provide a somehow simpler proof of the fact that Asplund spaces are $\mathcal{A}_{cc}$-variation admissible, [7, Theorem 1.3], see Corollary 3.8 below.

Part (i) of the following result was already pointed out in [31]:

Proposition 3.3. *Let X be a Banach space.*

(i) *If X is $\mathcal{A}_{cc}$-variation admissible, then X contains no subspace isomorphic to ℓ_1.*

(ii) *If every separable subspace of X is $\mathcal{A}_{cc}$-variation admissible, then X is $\mathcal{A}_{cc}$-variation admissible.*

Proof. (i) By Proposition 3.2, its suffices to check that ℓ_1 is not $\mathcal{A}_{cc}$-variation admissible. Since ℓ_1 is infinite-dimensional, the Dvoretzky-Rogers theorem (see, e.g., [17, Theorem 1.2]) ensures the existence of an unconditionally convergent series $\sum_{n=1}^\infty x_n$ in ℓ_1 which is not absolutely convergent. Now, define $\nu : \mathcal{P}(\mathbb{N}) \to \ell_1$ by

$\nu(A) := \sum_{n\in A} x_n$ for all $A \in \mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$). Then $\nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}), \ell_1)$ satisfies $|\nu|(\mathcal{P}(\mathbb{N})) = \sum_{n=1}^{\infty} \|x_n\| = \infty$, while I_ν is Dunford-Pettis by the Schur property of ℓ_1 (see, e.g., [2, Theorem 2.3.6]). Hence, ℓ_1 is not $\mathcal{A}_{cc}$-variation admissible.

(ii) Let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$ such that I_ν is Dunford-Pettis. Then $\mathcal{R}(\nu)$ is relatively norm compact (Proposition 2.6), so it is separable. Hence, the subspace $Z := \overline{\mathrm{span}}(\mathcal{R}(\nu)) = \overline{I_\nu(L_1(\nu))}$ (see (2.1) at page 4) is separable. Then Z is $\mathcal{A}_{cc}$-variation admissible by assumption. Since ν takes values in Z and I_ν is Dunford-Pettis, we deduce that $|\nu|(\Omega) < \infty$. $\square$

3.1. **Schauder decompositions and the variation of a vector measure.** Let X be a Banach space. A *Schauder decomposition* of X is a sequence $(X_n)_{n\in\mathbb{N}}$ of (non-zero) subspaces of X such that each $x \in X$ can be written in a unique way as a convergent series of the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for all $n \in \mathbb{N}$. In this case, for each $n \in \mathbb{N}$ there is a projection S_n from X onto X_n such that $x = \sum_{n=1}^{\infty} S_n(x)$ for all $x \in X$. For each $k \in \mathbb{N}$, the operator $P_k := \sum_{n=1}^{k} S_n$ is a projection from X onto the subspace $\bigoplus_{n=1}^{k} X_n$, we have $\sup_{k\in\mathbb{N}} \|P_k\| < \infty$ and the formula $|||x||| = \sup_{k\in\mathbb{N}} \|P_k(x)\|$ defines an equivalent norm on X. Note that $P_{k'} \circ P_k = P_k \circ P_{k'} = P_{\min\{k,k'\}}$ for all $k, k' \in \mathbb{N}$ and $\|P_k(x) - x\| \to 0$ as $k \to \infty$ for every $x \in X$. Of course, if X has a Schauder basis $(e_n)_{n\in\mathbb{N}}$, then the sequence of 1-dimensional subspaces generated by each e_n is a Schauder decomposition of X.

The following lemma will be a key tool for proving Theorem 3.6 below.

Lemma 3.4. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Suppose that:*

- *$|\nu|(\Omega) = \infty$ and $\mathcal{R}(\nu)$ is relatively norm compact.*
- *X has a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ such that $|P_k \circ \nu|(\Omega) < \infty$ for all $k \in \mathbb{N}$, where P_k is the associated projection from X onto $\bigoplus_{n=1}^{k} X_n$.*

Then there exist a sequence $(B_j)_{j\in\mathbb{N}}$ of pairwise disjoint elements of $\Sigma \setminus \mathcal{N}(\nu)$, a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ and $\varepsilon > 0$ in such a way that the functions $f_j := \frac{1}{\|\nu\|(B_j)} \chi_{B_j} \in L_1(\nu)$ and the projections $R_j := P_{k_{j+1}} - P_{k_j}$ satisfy:

(i) *$\|I_\nu(f_j g) - R_j(I_\nu(f_j g))\| \le 2^{-j}$ for all $g \in B_{L_\infty(\nu)}$ and $j \in \mathbb{N}$.*
(ii) *There is $j_0 \in \mathbb{N}$ such that $\|I_\nu(f_j) - I_\nu(f_{j'})\| \ge \varepsilon$ for all distinct $j, j' \ge j_0$.*

Proof. Write $Q_k := \mathrm{id}_X - P_k$ for all $k \in \mathbb{N}$ (where id_X stands for the identity operator on X). Since $\sup_{k\in\mathbb{N}} \|Q_k\| < \infty$ and $\|Q_k(x)\| \to 0$ as $k \to \infty$ for every $x \in X$, the sequence $(Q_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on each norm compact subset of X. By renorming, we can assume without loss of generality that $\|P_k\| = 1$ for all $k \in \mathbb{N}$.

Since $|\nu|(\Omega) = \infty$, there is a sequence $(C_l)_{l\in\mathbb{N}}$ of pairwise disjoint elements of $\Sigma \setminus \mathcal{N}(\nu)$ such that $\sum_{l=1}^{\infty} \|\nu(C_l)\| = \infty$, [27, Corollary 2]. Fix $\rho > 2$ and, for each $l \in \mathbb{N}$, take $A_l \in \Sigma \setminus \mathcal{N}(\nu)$ such that $A_l \subseteq C_l$ and $\rho\|\nu(A_l)\| \ge \|\nu\|(C_l)$. Then $\sum_{l=1}^{\infty} \|\nu(A_l)\| = \infty$ and

$$\|\nu(A_l)\| \ge \rho^{-1}\|\nu\|(A_l) \quad \text{for all } l \in \mathbb{N}. \tag{3.1}$$

Claim. There exist two strictly increasing sequences $(k_j)_{j\in\mathbb{N}}$ and $(l_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that for every $j \in \mathbb{N}$ we have:

(α) $\|P_{k_j}(I_\nu(\chi_{A_{l_{j+1}}} g))\| \leq 2^{-j-1}\|\nu\|(A_{l_{j+1}})$ for all $g \in B_{L_\infty(\nu)}$;
(β) $\|Q_{k_j}(I_\nu(\chi_{A_{l_j}} g))\| \leq 2^{-j}\|\nu\|(A_{l_j})$ for all $g \in B_{L_\infty(\nu)}$.

Indeed, we proceed by induction. Set $l_1 := 1$ and consider

$$K_1 := \left\{ I_\nu(\chi_{A_1} g) : \, g \in B_{L_\infty(\nu)} \right\} \subseteq X.$$

Since $\mathcal{R}(\nu)$ is relatively norm compact, so is K_1 (Proposition 2.1) and therefore we can pick $k_1 \in \mathbb{N}$ such that

$$\sup_{x \in K_1} \|Q_{k_1}(x)\| \leq \frac{\|\nu\|(A_1)}{2}.$$

Hence, (β) holds for $j = 1$. Suppose now that $k_N, l_N \in \mathbb{N}$ are already chosen for some $N \in \mathbb{N}$. Since $\tilde{\nu} := P_{k_N} \circ \nu$ satisfies $|\tilde{\nu}|(\Omega) < \infty$ and $\sum_{l=1}^\infty \|\nu\|(A_l) = \infty$, there is $l_{N+1} \in \mathbb{N}$ with $l_{N+1} > l_N$ such that

$$|\tilde{\nu}|(A_{l_{N+1}}) \leq 2^{-N-1}\|\nu\|(A_{l_{N+1}}). \tag{3.2}$$

Observe that for each $g \in B_{L_\infty(\nu)}$ we have

$$\begin{aligned} \|P_{k_N}(I_\nu(\chi_{A_{l_{N+1}}} g))\| &\overset{(*)}{=} \|I_{\tilde{\nu}}(\chi_{A_{l_{N+1}}} g)\| \leq \|\chi_{A_{l_{N+1}}} g\|_{L_1(\tilde{\nu})} \overset{(**)}{\leq} \|\chi_{A_{l_{N+1}}}\|_{L_1(\tilde{\nu})} \\ &= \|\tilde{\nu}\|(A_{l_{N+1}}) \leq |\tilde{\nu}|(A_{l_{N+1}}) \overset{(3.2)}{\leq} 2^{-N-1}\|\nu\|(A_{l_{N+1}}), \end{aligned}$$

where $(*)$ and $(**)$ follow from Propositions 2.2 and 2.7(i), respectively. Hence, (α) holds for $j = N$. Now, we consider the relatively norm compact subset of X defined by

$$K_{N+1} := \left\{ I_\nu(\chi_{A_{l_{N+1}}} g) : \, g \in B_{L_\infty(\nu)} \right\}$$

(apply Proposition 2.1 again) and we choose $k_{N+1} \in \mathbb{N}$ with $k_{N+1} > k_N$ such that

$$\sup_{x \in K_{N+1}} \|Q_{k_{N+1}}(x)\| \leq 2^{-N-1}\|\nu\|(A_{l_{N+1}}).$$

Therefore, (β) holds for $j = N + 1$. This finishes the proof of the *Claim.*

For each $j \in \mathbb{N}$, define $B_j := A_{l_{j+1}}$ and let f_j and R_j be as in the statement. To check property (i), take $j \in \mathbb{N}$ and $g \in B_{L_\infty(\nu)}$. Then (α) and (β) imply

$$\begin{aligned} \|I_\nu(f_j g) - R_j(I_\nu(f_j g))\| &= \|Q_{k_{j+1}}(I_\nu(f_j g)) + P_{k_j}(I_\nu(f_j g))\| \\ &\leq \|Q_{k_{j+1}}(I_\nu(f_j g))\| + \|P_{k_j}(I_\nu(f_j g))\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} = \frac{1}{2^j}. \end{aligned}$$

Finally, we will check that (ii) holds for an arbitrary $0 < \varepsilon < \rho^{-1}$. Choose $j_0 \in \mathbb{N}$ large enough such that $\rho^{-1} - 2^{-j_0} \geq \varepsilon$. Take $j' > j \geq j_0$ in $\mathbb{N}$. Then

$$\begin{aligned}
\|I_\nu(f_j) - I_\nu(f_{j'})\| &\geq \|P_{k_{j+1}}(I_\nu(f_j) - I_\nu(f_{j'}))\| \\
&\quad \text{(because } \|P_{k_{j+1}}\| = 1) \\
&= \|I_\nu(f_j) - Q_{k_{j+1}}(I_\nu(f_j)) - P_{k_{j+1}}(I_\nu(f_{j'}))\| \\
&\geq \|I_\nu(f_j)\| - \|Q_{k_{j+1}}(I_\nu(f_j))\| - \|P_{k_{j+1}}(I_\nu(f_{j'}))\| \\
&= \|I_\nu(f_j)\| - \|Q_{k_{j+1}}(I_\nu(f_j))\| - \|P_{k_{j+1}}(P_{k_{j'}}(I_\nu(f_{j'})))\| \\
&\quad \text{(because } P_{k_{j+1}} \circ P_{k_{j'}} = P_{k_{j+1}}) \\
&\geq \rho^{-1} - \|Q_{k_{j+1}}(I_\nu(f_j))\| - \|P_{k_{j'}}(I_\nu(f_{j'}))\| \\
&\quad \text{(by (3.1) and } \|P_{k_{j+1}}\| = 1) \\
&\geq \rho^{-1} - \frac{1}{2^{j+1}} - \frac{1}{2^{j'+1}} \\
&\quad \text{(by } (\alpha) \text{ and } (\beta) \text{ with } g = \chi_\Omega) \\
&> \rho^{-1} - \frac{1}{2^{j_0}} \geq \varepsilon.
\end{aligned}$$

The proof is finished. □

3.2. **Asplund spaces are $\mathcal{A}_{cc}$-variation admissible.** Let $(X_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of a Banach space X. By a *block sequence with respect to* $(X_n)_{n\in\mathbb{N}}$ we mean a sequence $(x_j)_{j\in\mathbb{N}}$ in X for which there is a sequence $(I_j)_{j\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}$ such that $\max(I_j) < \min(I_{j+1})$ and $x_j \in \bigoplus_{n\in I_j} X_n$ for all $j \in \mathbb{N}$. We say that $(X_n)_{n\in\mathbb{N}}$ is *shrinking* if $\|P_k^*(x^*) - x^*\| \to 0$ as $k \to \infty$ for every $x^* \in X^*$, where P_k is the associated projection from X onto $\bigoplus_{n=1}^k X_n$. When X has a Schauder basis $(e_n)_{n\in\mathbb{N}}$ and each X_n is the subspace generated by e_n, then $(X_n)_{n\in\mathbb{N}}$ is shrinking if and only if $(e_n)_{n\in\mathbb{N}}$ is shrinking in the usual sense.

The following fact belongs to the folklore and can be proved as in the case of Schauder bases (see, e.g., [2, Proposition 3.2.7]).

Proposition 3.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of a Banach space X. The following statements are equivalent:*

(i) *$(X_n)_{n\in\mathbb{N}}$ is shrinking.*
(ii) *Every bounded block sequence with respect to $(X_n)_{n\in\mathbb{N}}$ is weakly null.*

Theorem 3.6. *Let X be a Banach space having a shrinking Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ such that X_n is finite-dimensional for all $n \in \mathbb{N}$. Then X is $\mathcal{A}_{cc}$-variation admissible. In particular, every Banach space having a shrinking Schauder basis is $\mathcal{A}_{cc}$-variation admissible.*

Proof. Let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$ such that I_ν is Dunford-Pettis. Then $\mathcal{R}(\nu)$ is relatively norm compact (Proposition 2.6). Fix $k \in \mathbb{N}$ and denote by P_k the associated projection from X onto $\bigoplus_{n=1}^k X_n$. Since $\bigoplus_{n=1}^k X_n$ is finite-dimensional, we have $|P_k \circ \nu|(\Omega) < \infty$. By renorming, we can assume that $\|P_k\| = 1$ for all $k \in \mathbb{N}$.

Suppose, by contradiction, that $|\nu|(\Omega) = \infty$. Let $(f_j)_{j\in\mathbb{N}}$ and $(R_j)_{j\in\mathbb{N}}$ be as in Lemma 3.4. Since $(I_\nu(f_j))_{j\in\mathbb{N}}$ is not norm convergent (by property (ii) in Lemma 3.4) and I_ν is Dunford-Pettis, the sequence $(f_j)_{j\in\mathbb{N}}$ is not weakly convergent in $L_1(\nu)$. In addition, $\|f_j\|_{L_1(\nu)} = 1$ for all $j \in \mathbb{N}$. By Proposition 2.8, there is $g \in B_{L_\infty(\nu)}$ such that the sequence $(I_\nu(f_j g))_{j\in\mathbb{N}}$ is not weakly null in X. Then $(R_j(I_\nu(f_j g)))_{j\in\mathbb{N}}$ is a bounded block sequence with respect to $(X_n)_{n\in\mathbb{N}}$ which cannot be weakly null, by property (i) in Lemma 3.4. This contradicts that $(X_n)_{n\in\mathbb{N}}$ is shrinking (Proposition 3.5). □

The last ingredient of our proof that Asplund spaces are $\mathcal{A}_{cc}$-variation admissible is the following deep result of Zippin [43] (cf. [21, Theorem III.1] and [40]):

Theorem 3.7 (Zippin)**.** *Every Banach space having separable dual is isomorphic to a subspace of a Banach space having a shrinking Schauder basis.*

Corollary 3.8. *Every Asplund space is $\mathcal{A}_{cc}$-variation admissible.*

Proof. By Proposition 3.3(ii), it suffices to prove that every Banach space having separable dual is $\mathcal{A}_{cc}$-admissible. Since every Banach space having a shrinking Schauder basis is $\mathcal{A}_{cc}$-variation admissible (Theorem 3.6), the conclusion follows from Theorem 3.7 and Proposition 3.2(ii). □

3.3. **An application of the Davis-Figiel-Johnson-Pełczyński factorization.** We begin by recalling the refinement of the DFJP factorization developed by Lima, Nygaard and Oja in [23]. Let Z and X be Banach spaces, let $T : Z \to X$ be a (non-zero) operator and consider the set

$$K := \frac{1}{\|T\|}\overline{T(B_Z)} \subseteq B_X.$$

Fix $a \in (1, \infty)$ and write

$$f(a) := \left(\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2}\right)^{1/2}.$$

For each $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the Minkowski functional of $K_n := a^{n/2}K + a^{-n/2}B_X$, that is,

$$\|x\|_n := \inf\{t > 0 : x \in tK_n\} \quad \text{for all } x \in X.$$

The following theorem can be found in [23, Lemmas 1.1 and 2.1, Theorem 2.2], with the exception of part (vi), which can be obtained similarly as for the usual DFJP factorization (see, e.g., [5, §3]).

Theorem 3.9. *Under the previous assumptions, the following statements hold:*

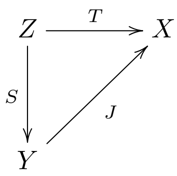
(i) $Y := \{x \in X : \sum_{n=1}^{\infty} \|x\|_n^2 < \infty\}$ *is a Banach space with the norm*

$$\|x\|_Y := \left(\sum_{n=1}^{\infty} \|x\|_n^2\right)^{1/2}.$$

(ii) $K \subseteq f(a)B_Y$ *and the identity map* $J : Y \to X$ *is an operator.*

(iii) *T factors as*

(3.3)



where S is an operator.

(iv) *J is a norm-to-norm homeomorphism when restricted to K. In fact:*

$$\|x\|_Y^2 \leq \Big(\frac{1}{4} + \frac{1}{\ln a}\Big)\|x\| \quad \textit{for all } x \in K.$$

Therefore, if T is Dunford-Pettis, then S is Dunford-Pettis as well.

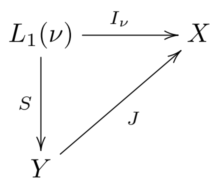
(v) *If T is weakly compact, then Y is reflexive.*

(vi) *If T is Asplund, then Y is Asplund.*

(vii) *If a is the unique element of $(1,\infty)$ satisfying $f(a) = 1$, then $\|S\| = \|T\|$ and $\|J\| = 1$. In this case,* (3.3) *is called the* DFJP-LNO factorization *of T.*

In [28] the DFJP-LNO factorization was applied to the integration operator of a vector measure. Our next proposition gathers some of the results obtained in [28, Theorems 3.7 and 4.5]:

Proposition 3.10. *Let X be a Banach space, let (Ω, Σ) be a measurable space, let $\nu \in \mathrm{ca}(\Sigma, X)$ and let*



be the DFJP-LNO factorization of I_ν. Define $\tilde{\nu} : \Sigma \to Y$ by $\tilde{\nu}(A) := S(\chi_A)$ for all $A \in \Sigma$. Then:

(i) *$\tilde{\nu} \in \mathrm{ca}(\Sigma, Y)$, $\nu = J \circ \tilde{\nu}$ and $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$.*

(ii) *$L_1(\tilde{\nu}) = L_1(\nu)$, with $\|f\|_{L_1(\nu)} = \|f\|_{L_1(\tilde{\nu})}$ for all $f \in L_1(\nu)$, and $S = I_{\tilde{\nu}}$.*

(iii) *$\tilde{\nu}$ has finite (resp., σ-finite) variation whenever ν does.*

Corollary 3.11. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. If I_ν is Asplund and Dunford-Pettis, then $|\nu|(\Omega) < \infty$ and the inclusion map $\iota_\nu : L_1(|\nu|) \to L_1(\nu)$ is a lattice-isomorphism.*

Proof. Let Y, J and $\tilde{\nu}$ be as in Proposition 3.10. Since $I_{\tilde{\nu}}$ is Dunford-Pettis and Y is Asplund (Theorem 3.9, parts (iv) and (vi)), we can apply Corollary 3.8 to get $|\tilde{\nu}|(\Omega) < \infty$, hence $|\nu|(\Omega) = |J \circ \tilde{\nu}|(\Omega) \leq |\tilde{\nu}|(\Omega) < \infty$. This shows that every Banach space is $\mathcal{A}$-variation admissible, where $\mathcal{A}$ denotes the operator ideal of all Asplund and Dunford-Pettis operators. The last statement follows from Proposition 3.1. □

4. Dunford-Pettis type properties

4.1. **A remark on equimeasurability.** Let (Ω, Σ, μ) be a finite measure space. A set $H \subseteq L_\infty(\mu)$ is said to be *equimeasurable* if for every $\varepsilon > 0$ there is $A \in \Sigma$ with $\mu(\Omega \setminus A) \leq \varepsilon$ such that $\{h\chi_A : h \in H\}$ is relatively norm compact in $L_\infty(\mu)$. Theorem 4.1 below is a particular case of [5, Theorem 5.5.4]. We include a direct proof for the sake of completeness.

Theorem 4.1. *Let (Ω, Σ, μ) be a finite measure space. If $H \subseteq L_\infty(\mu)$ is relatively weakly compact, then it is equimeasurable.*

Proof. By the Davis-Figiel-Johnson-Pełczyński factorization (see, e.g., [3, Theorem 5.37]), there exist a reflexive Banach space Y and an operator $T : Y \to L_\infty(\mu)$ such that $T(B_Y) \supseteq H$. Let $i : L_1(\mu) \to L_\infty(\mu)^*$ be the inclusion operator and let $S := T^* \circ i : L_1(\mu) \to Y^*$. Since Y^* is reflexive, S is representable, that is, there is $g \in L_\infty(\mu, Y^*)$ such that

$$S(f) = (\text{Bochner})\text{-}\int_\Omega f g \, d\mu \quad \text{for all } f \in L_1(\mu)$$

(see, e.g., [18, p. 75, Theorem 12]).

Fix $\varepsilon > 0$. Since g is strongly μ-measurable, Egorov's theorem ensures the existence of $A \in \Sigma$ with $\mu(\Omega \setminus A) \leq \varepsilon$ and a sequence $g_n : \Omega \to Y^*$ of Σ-simple Y^*-valued functions such that

$$\|g(t) - g_n(t)\| \leq \frac{1}{n} \quad \text{for every } t \in A \text{ and for every } n \in \mathbb{N}. \tag{4.1}$$

For each $n \in \mathbb{N}$, let us consider the operator $S_n : L_1(\mu) \to Y^*$ defined by

$$S_n(f) = (\text{Bochner})\text{-}\int_A f g_n \, d\mu \quad \text{for all } f \in L_1(\mu).$$

Note that S_n is a finite-rank operator, because g_n is the sum of finitely many functions of the form $y^*\chi_B$, where $y^* \in Y^*$ and $B \in \Sigma$. Hence, S_n is compact. Moreover, if $P_A : L_1(\mu) \to L_1(\mu)$ is the projection defined by $P_A(f) := f\chi_A$ for all $f \in L_1(\mu)$, then the operator $S \circ P_A : L_1(\mu) \to Y^*$ satisfies

$$\|S \circ P_A - S_n\| = \sup_{f \in B_{L_1(\mu)}} \left\| (\text{Bochner})\text{-}\int_A f(g - g_n) \, d\mu \right\| \overset{(4.1)}{\leq} \frac{1}{n}.$$

It follows that $(S_n)_{n\in\mathbb{N}}$ converges to $S \circ P_A$ in the operator norm. In particular, $S \circ P_A$ is compact and, therefore, $(S \circ P_A)^* : Y \to L_\infty(\mu)$ is compact as well (by Schauder's theorem).

For every $y \in Y$ and for every $f \in L_1(\mu)$ we have

$$\begin{aligned} \langle (S \circ P_A)^*(y), f \rangle &= \langle y, (S \circ P_A)(f) \rangle = \langle y, T^*(i(f\chi_A)) \rangle \\ &= \langle T(y), f\chi_A \rangle = \int_A f T(y) \, dy = \langle T(y)\chi_A, f \rangle. \end{aligned}$$

Therefore $(S \circ P_A)^*(y) = T(y)\chi_A$ for all $y \in Y$. It follows that

$$\{h\chi_A : h \in H\} \subseteq \{T(y)\chi_A : y \in B_Y\} = (S \circ P_A)^*(B_Y)$$

and so $\{h\chi_A : h \in H\}$ is relatively norm compact in $L_\infty(\mu)$. □

4.2. **A Dunford-Pettis type property for L_1 of a vector measure.** Recall that a Banach space Z has the Dunford-Pettis property if and only if $z_n^*(z_n) \to 0$ as $n \to \infty$ for all weakly null sequences $(z_n)_{n\in\mathbb{N}}$ and $(z_n^*)_{n\in\mathbb{N}}$ in Z and Z^*, respectively (see, e.g., [2, Theorem 5.4.4]).

We next show that the L_1 space of an arbitrary vector measure enjoys a Dunford-Pettis type property with respect to the "vector duality" induced by the integration operator (Subsection 2.5).

Theorem 4.2. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\nu)$ and let $(g_n)_{n\in\mathbb{N}}$ be a weakly null sequence in $L_\infty(\nu)$.*

(i) *If $(f_n)_{n\in\mathbb{N}}$ is weakly null, then $(I_\nu(f_n g_n))_{n\in\mathbb{N}}$ is weakly null.*
(ii) *If $(f_n)_{n\in\mathbb{N}}$ is bounded and equi-integrable, then $(I_\nu(f_n g_n))_{n\in\mathbb{N}}$ is norm null.*

Proof. (i) Fix $x^* \in X^*$. Let $h_{x^*} \in L_\infty(|x^*\nu|)$ be the Radon-Nikodým derivative of $x^*\nu$ with respect to $|x^*\nu|$. For each $n \in \mathbb{N}$ we have

$$x^*\big(I_\nu(f_n g_n)\big) = \int_\Omega f_n g_n \, d(x^*\nu) = \int_\Omega f_n h_{x^*} g_n \, d|x^*\nu|. \tag{4.2}$$

Since $(f_n)_{n\in\mathbb{N}}$ is weakly null in $L_1(\nu)$ and the inclusion map $L_1(\nu) \to L_1(|x^*\nu|)$ is an operator, $(f_n)_{n\in\mathbb{N}}$ is weakly null in $L_1(|x^*\nu|)$ and so the same holds for $(f_n h_{x^*})_{n\in\mathbb{N}}$. In the same way, $(g_n)_{n\in\mathbb{N}}$ is weakly null in $L_\infty(|x^*\nu|)$, so we can apply the Dunford-Pettis property of $L_1(|x^*\nu|)$ and (4.2) to conclude that $x^*(I_\nu(f_n g_n)) \to 0$ as $n \to \infty$. Since $x^* \in X^*$ is arbitrary, $(I_\nu(f_n g_n))_{n\in\mathbb{N}}$ is weakly null.

(ii) Define $\alpha := \sup_{n\in\mathbb{N}} \|f_n\|_{L_1(\nu)}$ and $\beta := \sup_{n\in\mathbb{N}} \|g_n\|_{L_\infty(\nu)}$. Let μ be a Rybakov control measure of ν. Fix $\varepsilon > 0$. Since $(f_n)_{n\in\mathbb{N}}$ is equi-integrable, we can choose $\delta > 0$ such that

$$\sup_{f\in F} \|f\chi_B\|_{L_1(\nu)} \le \varepsilon \quad \text{for every } B \in \Sigma \text{ with } \mu(B) \le \delta. \tag{4.3}$$

By Theorem 4.1, the set $\{g_n : n \in \mathbb{N}\}$ is equimeasurable, so there is $A \in \Sigma$ with $\mu(\Omega \setminus A) \le \delta$ such that $\{g_n\chi_A : n \in \mathbb{N}\}$ is relatively norm compact in $L_\infty(\nu)$. Since the sequence $(g_n\chi_A)_{n\in\mathbb{N}}$ is weakly null in $L_\infty(\nu)$ (bear in mind that the map $g \mapsto g\chi_A$ is an operator on $L_\infty(\nu)$), we conclude that $(g_n\chi_A)_{n\in\mathbb{N}}$ is norm null in $L_\infty(\nu)$. Choose $n_0 \in \mathbb{N}$ such that

$$\sup_{n\ge n_0} \|g_n\chi_A\|_{L_\infty(\nu)} \le \varepsilon. \tag{4.4}$$

Now, for every $f \in F$ and for every $n \in \mathbb{N}$ with $n \ge n_0$ we have

$$\begin{aligned}
\big\|I_\nu(fg_n)\big\| &\le \big\|I_\nu(fg_n\chi_{\Omega\setminus A})\big\| + \big\|I_\nu(fg_n\chi_A)\big\| \\
&\overset{\text{(Prop. 2.7(i))}}{\le} \|f\chi_{\Omega\setminus A}\|_{L_1(\nu)}\|g_n\|_{L_\infty(\nu)} + \|f\|_{L_1(\nu)}\|g_n\chi_A\|_{L_\infty(\nu)} \\
&\overset{\text{(4.3) \& (4.4)}}{\le} (\beta+\alpha)\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the sequence $(I_\nu(f_n g_n))_{n\in\mathbb{N}}$ is norm null. $\square$

4.3. **The positive Schur property as a Dunford-Pettis type property.** As a natural outcome of our previous work we get the following characterization:

Theorem 4.3. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$. The following statements are equivalent:*

(i) *$L_1(\nu)$ has the positive Schur property.*
(ii) *For all weakly null sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $L_1(\nu)$ and $L_\infty(\nu)$, respectively, the sequence $(I_\nu(f_n g_n))_{n\in\mathbb{N}}$ is norm null.*

Proof. (i)⇒(ii): This follows from Theorem 4.2, because the positive Schur property of $L_1(\nu)$ is equivalent to the fact that every relatively weakly compact subset of $L_1(\nu)$ is equi-integrable (Propositions 2.3 and 2.4).

(ii)⇒(i): By Propositions 2.3 and 2.4, it suffices to prove that *every disjoint weakly null sequence $(f_n)_{n\in\mathbb{N}}$ in $L_1(\nu)$ is equi-integrable.* Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of Σ such that $f_n\chi_{A_n} = f_n$ for all $n \in \mathbb{N}$. Observe that $(\chi_{A_n})_{n\in\mathbb{N}}$ is weakly null in $L_\infty(\nu)$. Indeed, we can assume without loss of generality that $A_n \not\in \mathcal{N}(\nu)$ for all $n \in \mathbb{N}$. Then $(\chi_{A_n})_{n\in\mathbb{N}}$ is a basic sequence in $L_\infty(\nu)$ which is equivalent to the usual basis of c_0. In particular, $(\chi_{A_n})_{n\in\mathbb{N}}$ is weakly null in $L_\infty(\nu)$.

Fix $A \in \Sigma$. Define $\tilde{f}_n := f_n\chi_A$ for all $n \in \mathbb{N}$. Note that

$$I_\nu(\tilde{f}_n\chi_{A_n}) = I_\nu(\tilde{f}_n) = \nu_{f_n}(A) \quad \text{for all } n \in \mathbb{N}. \tag{4.5}$$

Since $(\tilde{f}_n)_{n\in\mathbb{N}}$ is weakly null in $L_1(\nu)$ (because $(f_n)_{n\in\mathbb{N}}$ is weakly null and the map $h \mapsto h\chi_A$ is an operator on $L_1(\nu)$) and $(\chi_{A_n})_{n\in\mathbb{N}}$ is weakly null in $L_\infty(\nu)$, condition (ii) and (4.5) imply that the sequence $(\nu_{f_n}(A))_{n\in\mathbb{N}}$ is norm null. As $A \in \Sigma$ is arbitrary, we can apply Lemma 2.5 to conclude that $(f_n)_{n\in\mathbb{N}}$ is equi-integrable. □

Of course, Theorems 4.2 and 4.3 provide another point of view for the positive Schur property of the L_1 space of a vector measure taking values in a Banach space with the Schur property, [13, proof of Theorem 4].

4.4. **Vector measures with σ-finite variation.** The analysis of the Dunford-Pettis property is simpler for L_1 spaces of a vector measure with σ-finite variation.

Proposition 4.4. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$ with σ-finite variation. If $(f_n)_{n\in\mathbb{N}}$ is a bounded and equi-integrable sequence in $L_1(\nu)$ and $(\varphi_n)_{n\in\mathbb{N}}$ is a weakly null sequence in $L_1(\nu)^*$, then $\varphi_n(f_n) \to 0$ as $n \to \infty$.*

Proof. The sequence $(f_n)_{n\in\mathbb{N}}$ is approximately order bounded (Proposition 2.4). Hence, we can assume without loss of generality that $f_n \in j_\nu(B_{L_\infty(\nu)})$ for all $n \in \mathbb{N}$. Define $\alpha := \sup_{n\in\mathbb{N}} \|\varphi_n\|_{L_1(\nu)^*}$. Let $(A_m)_{m\in\mathbb{N}}$ be an increasing sequence in Σ such that $\Omega = \bigcup_{m\in\mathbb{N}} A_m$ and $|\nu|(A_m) < \infty$ for all $m \in \mathbb{N}$. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ large enough such that

$$\|\nu\|(\Omega \setminus A_m) \le \varepsilon. \tag{4.6}$$

Define $\mu(A) := |\nu|(A\cap A_m)$ for all $A \in \Sigma$, so that μ is a finite non-negative measure. Consider the inclusion operator $\iota : L_1(\mu) \to L_1(\nu)$ (see, e.g., [33, Lemma 3.14]) and

$\iota^* : L_1(\nu)^* \to L_\infty(\mu)$. Define $g_n := \iota^*(\varphi_n) \in L_\infty(\mu)$ for all $n \in \mathbb{N}$, so that $(g_n)_{n\in\mathbb{N}}$ is weakly null in $L_\infty(\mu)$.

The sequence $(f_n\chi_{A_m})_{n\in\mathbb{N}}$ is bounded and equi-integrable in $L_1(\mu)$ and

$$\langle g_n, f_n\chi_{A_m}\rangle = \int_{A_m} f_n g_n \, d\mu = \varphi_n(f_n\chi_{A_m}) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the Dunford-Pettis property of $L_1(\mu)$ (cf. Theorem 4.2(ii)) ensures that $\varphi_n(f_n\chi_{A_m}) \to 0$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that

$$|\varphi_n(f_n\chi_{A_m})| \leq \varepsilon \quad \text{whenever } n \geq n_0.$$

Since

$$|\varphi_n(f_n\chi_{\Omega\setminus A_m})| \leq \alpha\|f_n\chi_{\Omega\setminus A_m}\|_{L_1(\nu)} \leq \alpha\|\nu\|(\Omega\setminus A_m) \leq \alpha\varepsilon \quad \text{for all } n \in \mathbb{N}$$

(by Proposition 2.7(i) and (4.6)), we have

$$|\varphi_n(f_n)| \leq |\varphi_n(f_n\chi_{A_m})| + |\varphi_n(f_n\chi_{\Omega\setminus A_m})| \leq (1+\alpha)\varepsilon \quad \text{whenever } n \geq n_0.$$

This shows that $\varphi_n(f_n) \to 0$ as $n \to \infty$. □

By putting together Propositions 2.3, 2.4 and 4.4, we get the already mentioned result from [13]:

Corollary 4.5. *Let X be a Banach space, let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, X)$ with σ-finite variation. If $L_1(\nu)$ has the positive Schur property, then it has the Dunford-Pettis property.*

Let E be a Banach space with a normalized 1-unconditional Schauder basis, say $(e_n)_{n\in\mathbb{N}}$. The E-sum of countably many copies of $L_1[0,1]$ is the Banach lattice Z of all sequences $(h_n)_{n\in\mathbb{N}}$ in $L_1[0,1]$ such that the series $\sum_{n=1}^\infty \|h_n\|_{L_1[0,1]}\, e_n$ converges unconditionally in E, equipped with the norm

$$\|(h_n)_{n\in\mathbb{N}}\|_Z := \left\| \sum_{n=1}^\infty \|h_n\|_{L_1[0,1]}\, e_n \right\|_E$$

and the coordinatewise order. If E has the the Schur property, then Z has the positive Schur property, but it is not lattice-isomorphic to an AL-space unless E is isomorphic to ℓ_1, [42, Section 3].

The following construction provides more examples of Banach lattices with such features:

Example 4.6. *Let X be a Banach space and let $\sum_{n=1}^\infty x_n$ be an unconditionally convergent series in X with $x_n \neq 0$ for all $n \in \mathbb{N}$. Let λ be the Lebesgue measure on the σ-algebra Σ of all Borel subsets of $[0,1]$. Write $I_n := (2^{-n}, 2^{-n+1}]$ for all $n \in \mathbb{N}$. Then:*

(i) *The formula*

$$\nu(A) := \sum_{n=1}^\infty 2^n \lambda(A \cap I_n) x_n, \quad A \in \Sigma,$$

defines a vector measure $\nu \in \mathrm{ca}(\Sigma, X)$.

(ii) *$\mathcal{N}(\nu) = \mathcal{N}(\lambda)$. Hence, ν is atomless and $L_1(\nu)$ is separable.*

(iii) *$\mathcal{R}(\nu)$ is relatively norm compact.*
(iv) *$|\nu|$ is σ-finite and $|\nu|([0,1]) = \sum_{n=1}^{\infty} \|x_n\|$.*
(v) *If $\sum_{n=1}^{\infty} x_n$ is not absolutely convergent, then $L_1(\nu)$ is not lattice-isomorphic to an AL-space.*
(vi) *If X has the Schur property, then $L_1(\nu)$ has the positive Schur property and the Dunford-Pettis property.*
(vii) *If $\sum_{n=1}^{\infty} x_n$ is not absolutely convergent and X has the Schur property, then $L_1(\nu)$ is not lattice-isomorphic to $L_1(\tilde{\nu})$ for any σ-algebra $\tilde{\Sigma}$ and any $\tilde{\nu} \in \mathrm{ca}(\tilde{\Sigma}, c_0)$ such that $\mathcal{R}(\tilde{\nu})$ is relatively norm compact.*

Proof. Since $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, for every $(a_n)_{n\in\mathbb{N}} \in \ell_\infty$ the series $\sum_{n=1}^{\infty} a_n x_n$ is unconditionally convergent and the map

$$T : \ell_\infty \to X, \quad T((a_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} a_n x_n,$$

is a compact operator (see, e.g., [17, Theorem 1.9]). This shows that the map ν is well-defined and has relatively norm compact range (note that $2^n \lambda(A \cap I_n) \leq 1$ for all $n \in \mathbb{N}$). Since the map $\Sigma \ni A \mapsto 2^n \lambda(A \cap I_n) x_n \in X$ is countably additive for each $n \in \mathbb{N}$, the Vitali-Hahn-Saks theorem (see, e.g., [18, p. 24, Corollary 10]) ensures that ν is countably additive. This proves parts (i) and (iii).

(ii) The equality $\mathcal{N}(\nu) = \mathcal{N}(\lambda)$ is obvious. Since λ is atomless, so is ν. Let $\mathcal{C} \subseteq \Sigma$ be a countable set such that for every $A \in \Sigma$ we have $\inf_{C \in \mathcal{C}} \lambda(A \triangle C) = 0$. Then for every $A \in \Sigma$ we also have $\inf_{C\in\mathcal{C}} \|\nu\|(A \triangle C) = 0$ (notice that ν is λ-continuous). This implies that $L_1(\nu)$ is separable, because the set of all Σ-simple functions is norm dense in $L_1(\nu)$.

(iv) It is easy to check that $|\nu|(A) = \sum_{n=1}^{\infty} 2^n \lambda(A \cap I_n) \|x_n\|$ for every $A \in \Sigma$.

(v) This follows from [12, Proposition 2] and (iv).

(vi) We already know that the Schur property of X implies that $L_1(\nu)$ has the positive Schur property, [13, proof of Theorem 4]. Now, (iv) and Corollary 4.5 ensure that $L_1(\nu)$ has the Dunford-Pettis property.

(vii) Suppose, by contradiction, that there exist a σ-algebra $\tilde{\Sigma}$ and $\tilde{\nu} \in \mathrm{ca}(\tilde{\Sigma}, c_0)$ such that $\mathcal{R}(\tilde{\nu})$ is relatively norm compact and $L_1(\nu)$ is lattice-isomorphic to $L_1(\tilde{\nu})$. Then $L_1(\tilde{\nu})$ has the positive Schur property (by (vi)) and we can apply Proposition 2.6 to infer that the integration operator $I_{\tilde{\nu}} : L_1(\tilde{\nu}) \to c_0$ is Dunford-Pettis. Now, Proposition 3.1 and Theorem 3.6 (the usual basis of c_0 is shrinking) imply that $L_1(\tilde{\nu})$ is lattice-isomorphic to an AL-space, which contradicts (v). □

Remark 4.7. Part (vii) of Example 4.6 should be compared with [12, Theorem 1]. That result states that if X is a Banach space, (Ω, Σ) is a measurable space, the vector measure $\nu \in \mathrm{ca}(\Sigma, X)$ is atomless and $L_1(\nu)$ is separable, then there is $\tilde{\nu} \in \mathrm{ca}(\Sigma, c_0)$ such that $L_1(\nu)$ and $L_1(\tilde{\nu})$ are lattice-isometric (cf. [24, Theorem 5] for another proof). For variants in the non-separable setting, see [36] and [37]. In [24, Theorem 5] it was claimed that if, in addition, $\mathcal{R}(\nu)$ is relatively norm compact, then $\tilde{\nu}$ can be chosen so that $\mathcal{R}(\tilde{\nu})$ is relatively norm compact as well. Unfortunately, this turns out to be false in general, as shown in Example 4.6(vii).

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