Almost isometries of non-reversible metrics with applications to stationary spacetimes

Miguel Ángel Javaloyes (Universidad de Murcia)
(joint work with L. Lichtenfelz and P. Piccione)

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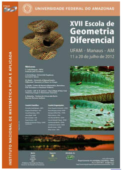

My collaborators

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NOTRE DAME UNIVERSITY (USA)

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Outline

The talk will consist in three parts:

1) Almost isometries of quasi-metrics (abstract setting)

2) Almost isometries of Finsler metrics

3) Applications to stationary spacetimes (Fermat metrics)

First part: Almost isometries of quasi-metrics

Definition

Given a set X, we say that a function $d : X \times X \to \mathbb{R}$ is a *quasi-metric* if

(i) $d(x, y) \geq 0$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

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As a consequence of the lack of symmetry, there are two kinds of balls:

- $B_d^+(x, r) = \{y \in X : d(x, y) < r\}$ (forward balls)
- $B_d^-(x, r) = \{y \in X : d(y, x) < r\}$ (backward balls)

respectively, for $x \in X$ and $r > 0$.

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Definition

A pair (X, d) will be called a *quasi-metric space* endowed with the topology induced by the family $B_d^+(x, r) \cap B_d^-(x, r)$, $x \in M$ and $r > 0$.

Let us observe that this topology coincides with the topology generated by (the balls of) the symmetrized metric $\widetilde{d}(x, y) = \frac{1}{2}\big(d(x, y) + d(y, x)\big)$.

Quasi-metrics

Quasi-metrics spaces have been studied by many mathematicians:

- Fréchet 1909, Hausdorff 1914, Mazurkiewicz 1930, Wilson 1931, Busemann 1944
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Out seminar in the university of Murcia is called “Rey Pastor” after him

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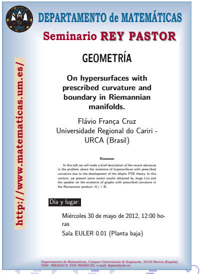

Quasi-metrics and the triangular function

In a quasi-metric space we can define the length of a continuous curve $\alpha : [a, b] \subseteq \mathbb{R} \to X$ as

$$\ell(\alpha) = \sup_{\mathcal{P}} \sum_{1=1}^{r} d(\alpha(s_i), \alpha(s_{i+1})),$$

where $\mathcal{P}$ is the set of partitions $a = s_1 < s_2 < \ldots < s_{r+1} = b,\ r \in \mathbb{N}$.

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- We say that α is *rectifiable* when $\ell(\alpha)$ is finite.
- Moreover, we say that a curve γ in X from p to q is a *minimizing geodesic* if $\ell(\gamma) = d(p, q)$.

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Definition

Let us define the *triangular function* $T : X \times X \times X \to [0, +\infty[$ of a quasi-metric space (X, d) as $T(x, y, z) = d(x, y) + d(y, z) - d(x, z)$ for every $x, y, z \in X$.

Evidently, T is continuous.

Almost isometries

Proposition

A curve $\alpha : [a, b] \subseteq \mathbb{R} \to X$ *is a minimizing geodesic of a quasi-metric space* (X, d) *iff* $T(\alpha(s_1), \alpha(s_2), \alpha(s_3)) = 0$ *for every* $a \leq s_1 < s_2 < s_3 \leq b$.

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Definition

Let (X_1, d_1) and (X_2, d_2) be two quasi-metric spaces. A bijection $\varphi : X_1 \to X_2$ is an *almost isometry* if it preserves the triangular function, that is,

$$T_2(\varphi(x), \varphi(y), \varphi(z)) = T_1(x, y, z)$$

for every $x, y, z \in X_1$, where T_1 and T_2 are the triangular functions associated respectively to (X_1, d_1) and (X_2, d_2).

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Corollary

Almost isometries preserve minimizing geodesics.

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Proposition

Given quasi-metric spaces (X_1, d_1) *and* (X_2, d_2)*, a bijection* $\varphi : X_1 \to X_2$ *is an almost isometry iff* $\exists\ f : X_2 \to \mathbb{R}$ *such that for every* $x, y \in X_1$*:*

$$d_2\big(\varphi(x), \varphi(y)\big) = d_1(x, y) + f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \qquad (1)$$

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Proof.

$\Rightarrow$ (the converse is straightforward)

- Fix a point $x_0 \in X_1$ and define $f : X_2 \to \mathbb{R}$ as $f(z) = d_2(z, \varphi(x_0)) - d_1(\varphi^{-1}(z), x_0)$ for every $z \in X_2$.

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- Given $x, y \in X_1$, as φ preserves the triangular function, we have

$$\begin{aligned} &d_1(x, y) + d_1(y, x_0) - d_1(x, x_0) \\ &\qquad = d_2(\varphi(x), \varphi(y)) + d_2(\varphi(y), \varphi(x_0)) - d_2(\varphi(x), \varphi(x_0)), \end{aligned}$$

which is equivalent to (1).

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  is an isometry, where

$$\widetilde{d}_1(x, y) = \tfrac{1}{2}\big(d_1(x, y) + d_1(y, x)\big),$$
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- Moreover, φ is a homeomorphism and the functions $f : X_2 \to \mathbb{R}$ are continuous

Almost isometries

Notation:

- $\mathrm{Iso}(X, d)$ is the group of isometries of (X, d)
- $\widetilde{\mathrm{Iso}}(X, d)$ is the group of almost isometries of (X, d). It will be called the *extended isometry group* of (X, d).

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Proof.

The proof follows from the inclusions:

$$\mathrm{Iso}(X, d) \subseteq \widetilde{\mathrm{Iso}}(X, d) \subseteq \mathrm{Iso}(X, \tilde{d}).$$

$\square$

Local almost isometries

Definition

Let (X_1, d_1) and (X_2, d_2) be two quasi-metric spaces. A map $\varphi : X_1 \to X_2$ is a *local almost isometry* if $\forall x \in X_1$, $\exists\ U \subseteq X_1$, $V \subseteq X_2$ open subsets, with $x \in U$, such that $\varphi|_U : U \to V$ is an almost isometry.

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- define d_l as the infimum of the lengths of curves between two points. We say that (X, d) is a *length space* when $d_l = d$.
- We say that a quasi-metric space is weakly finitely compact if $B^+(x, r) \cap B^-(x, r)$ are precompact $\forall x \in X$ and $r > 0$.

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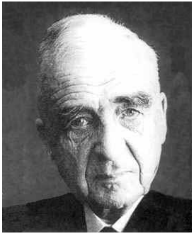
Theorem

Let $\varphi : (X_1, d_1) \to (X_2, d_2)$ be a local almost isometry. Assume that (X_1, d_1) and (X_2, d_2) are length spaces, d_1 is weakly finitely compact and X_2 is locally arc-connected and simply connected. Then φ is an almost isometry.

Second Part: Almost isometries of Finsler metrics

DEFINITION: $F : TM \to [0, +\infty)$ continuous and



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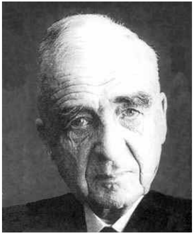

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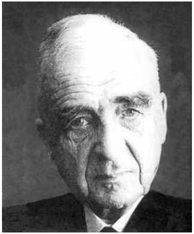

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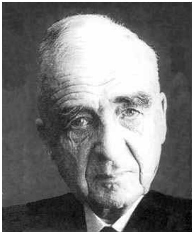

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$$g_v(w,z) = \frac{\partial^2}{\partial t \partial s} F(v + tw + sz)^2|_{t=s=0} = \mathrm{Hess}(F^2)_v(w,z)$$

for every $w, z \in T_{\pi(v)}M$. Then $g_v(w,w) > 0$ for every $0 \neq w \in T_{\pi(v)}M$.

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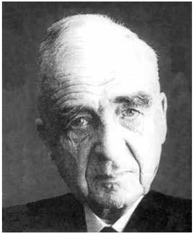

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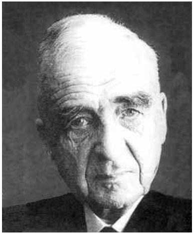

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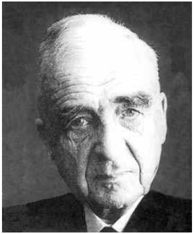

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- Triangle inequality holds in the fibers

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- Randers metrics in a manifold M is a function $R : TM \to \mathbb{R}$ defined as:

$$R(v) = \sqrt{h(v,v)} + \omega(v)$$

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- are basic examples of non-reversible Finsler metrics: $R(-v) \neq R(v)$.

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where h is Riemannian and ω a 1-form with $\|\omega\|_h < 1 \ \forall x \in M$,

- are basic examples of non-reversible Finsler metrics: $R(-v) \neq R(v)$.
- Named after the norwegian physicist Gunnar Randers (1914-1992):

Randers, G.: On an asymmetrical metric in the fourspace of General Relativity. Phys. Rev. (2) **59**, 195–199 (1941)

G. Randers and A. Einstein

Zermelo metrics

Given a Riemannian metric g,
Zermelo metric:

$$Z(v) = \sqrt{\frac{1}{\alpha}g(v,v) + \frac{1}{\alpha^2}g(W,v)^2} - \frac{1}{\alpha}g(W,v),$$

where $\alpha = 1 - g(W,W)$.

Meeting of waters

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Geodesics minimize time in the presence of a wind or current W.

MEETING OF WATERS

Matsumoto metrics

Given a Riemannian metric g, and a one-form β

$$M(v) = \frac{g(v,v)}{\sqrt{g(v,v)} - \beta(v)}$$

Sierra Nevada (near Granada)

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Let us define the symmetrized Finsler metric of F as

$$\hat{F}(v) = \frac{1}{2}\left[F(v) + F(-v)\right]$$

for every $v \in TM$. The sum of Finsler metrics is a Finsler metric:

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Lemma

If $\varphi : (M_1, F_1) \to (M_2, F_2)$ is an almost isometry then

$$\varphi : (M_1, \hat{F}_1) \to (M_2, \hat{F}_2)$$

is an isometry and φ is smooth.

Proof.

- To see that φ is an isometry prove that preserves the length of curves
- φ is smooth because it is an isometry of a Riemannian average metric

Finsler metrics

Proposition

- *If* $\exists$ *an almost isometry* $\varphi : (M_1, F_1) \to (M_2, F_2)$, *then there exists a smooth* $f : M_2 \to \mathbb{R}$ *such that* $\varphi^*(F_1) = F_2 + \mathrm{d}f$.
- *Conversely, if* $\varphi^*(F_1) = F_2 + \mathrm{d}f$, *the map* φ *is an almost isometry.*

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Proposition

Let (M, F) *be a Finsler manifold. Then the extended isometry group* $\widetilde{\mathrm{Iso}}(M, F)$ *is a closed subgroup of* $\mathrm{Iso}(M, \hat{F})$*. In particular,* $\widetilde{\mathrm{Iso}}(M, F)$ *is a Lie group.*

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Proof.

Use that $\widetilde{\mathrm{Iso}}(M, F) \subset \mathrm{Iso}(M, \hat{F})$ $\square$

Randers metrics

Corollary

Let (M, R) be a Randers manifold and $\varphi : M \to M$ an almost isometry for R. Then φ is an isometry for h.

Randers metrics

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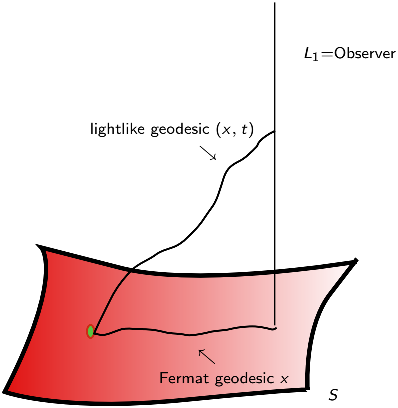
Let (M, R) be a Randers manifold and $\varphi : M \to M$ an almost isometry for R. Then φ is an isometry for h.

Proof.

Just observe that the symmetrized Finsler metric of R is given by $\hat{R}(v) = \sqrt{h(v, v)}$ for $v \in TM$. □

Third part: applications to stationary spacetimes



$(S \times \mathbb{R}, l)$ is a standard stationary spacetime

S is naturally endowed with a Randers metric F called the Fermat metric

Conformally Standard Stationary Spacetimes

- A spacetime (M, g) is Conformastationary if it admits a timelike Conformal field K, that is, a timelike vector field satisfying

$$\mathcal{L}_K g = \lambda g,$$

for some function $\lambda : M \to \mathbb{R}$

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- Standard Conformastationary means that $M = S \times \mathbb{R}$ and

$$g((v,\tau),(v,\tau)) = \varphi(g_0(v,v) + 2\omega(v)\tau - \tau^2),$$

in $(x,t) \in S \times \mathbb{R}$, where $(v,\tau) \in T_x S \times \mathbb{R}$, $\varphi : S \times \mathbb{R} \to (0,+\infty)$

- and g_0 is a Riemannian metric on S and ω a 1-form on S.

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A conformastationary spacetime is standard whenever it is distinguishing and the timelike conformal vector field is complete:

M. A. J. and M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*, Classical Quantum Gravity, 25 (2008), pp. 168001, 7.

Miguel Sánchez

Fermat principle in General Relativity

- First established by Herman Weyl in 1917 for static spacetimes

H. WEYL
→ (1885-1955)

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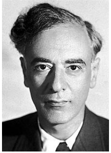


H. WEYL → (1885-1955) | T. LEVI-CIVITA → (1873-1941) | LEV LANDAU → (1908-1968) | E. LIFSHITZ → (1915-1985)

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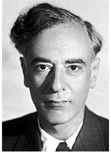

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- Volker Perlick gave a rigorous proof of this general principle in the same year (1990)

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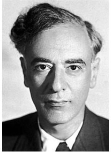

→ Lev Landau (1908-1968)

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→ V. Perlick (Born in 1956)

Fermat principle in standard stationary spacetimes

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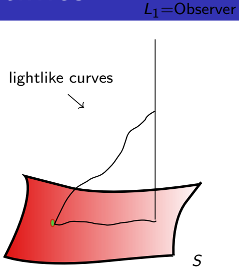

- Relativistic Fermat Principle: lightlike pregeodesics are critical points of the arrival time function corresponding to an *observer* in a suitable class of lightlike curves
- If you consider as observer $s \to L_1(s) = (x_1, s)$ in $(S \times \mathbb{R}, g)$, given a lightlike curve $\gamma = (x, t)$, the arrival time $\mathrm{AT}(\gamma)$ is

$$t(b){=}t(a){+}\int_a^b\left(\omega(\dot{x}){+}\sqrt{g_0(\dot{x},\dot{x}){+}\omega(\dot{x})^2}\right)\mathrm{d}s.$$

Pierre de Fermat (1601-1665)

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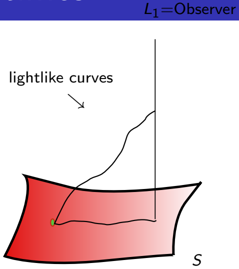

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- because $g_0(\dot{x}, \dot{x}) + 2\omega(\dot{x})\dot{t} - \dot{t}^2 = 0$ $(g(\dot{\gamma}, \dot{\gamma}) = 0)$

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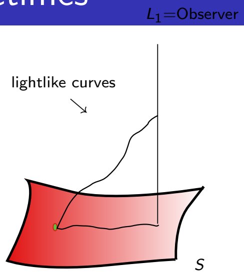

$$t(b)=t(a)+\int_a^b\left(\omega(\dot{x})+\sqrt{g_0(\dot{x},\dot{x})+\omega(\dot{x})^2}\right)\mathrm{d}s.$$

Pierre de Fermat (1601-1665)

- because $g_0(\dot{x},\dot{x}) + 2\omega(\dot{x})\dot{t} - \dot{t}^2 = 0$ $(g(\dot\gamma,\dot\gamma) = 0)$
- Let us define the Fermat (Finslerian) metric in S as

$$F(v)=\omega(v)+\sqrt{g_0(v,v)+\omega(v)^2},$$

References

E. Caponio, M. A. J., and A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, Math. Ann., 351 (2011), pp. 365–392.

E. Caponio, M. A. J., and M. Sánchez, *On the interplay between Lorentzian Causality and Finsler metrics of Randers type*, Rev. Mat. Iberoamericana, 27 (2011), pp. 919–952.

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For a review see:

M. A. J., *Conformally standard stationary spacetimes and Fermat metrics*, arXiv:1201.1841v1 [math.DG], to appear in Proceedings of GeLoGra 2011.

K-conformal maps

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- Now consider a normalized standard stationary spacetime $(S \times \mathbb{R}, g)$ with

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Theorem

If $\psi : (S \times \mathbb{R}, g) \to (S \times \mathbb{R}, g)$ is a K-conformal map, then
$$\psi(x,t) = (\varphi(x), t + f(x)),$$
and $\varphi_*(F) = F + df$ and $\varphi : (S, h) \to (S, h)$ is an isometry, where
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Proof.

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- K-conformal implies that ψ maps orbits of ∂_t to orbits of ∂_t, that is, $\psi(x,t) = (\varphi(x), t + f(x))$

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- K-conformal implies that ψ maps orbits of ∂_t to orbits of ∂_t, that is, $\psi(x,t) = (\varphi(x), t + f(x))$
- As ψ is conformal, maps lightlike pregeodesics to lightlike pregeodesics

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- As ψ is conformal, maps lightlike pregeodesics to lightlike pregeodesics
- Then Fermat metric maps Fermat pregeodesics to Fermat pregeodesics and $\ell_{\varphi_*(F)}(\gamma) = \ell_F(\gamma) + f(\gamma(1)) - f(\gamma(0))$

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- This means that $\varphi_*(F)$ and $F + df$ have the same geodesics and therefore they are equal

$\square$

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Lemma

$\mathrm{Conf}_K(M,g)$ (here $M=\mathbb{R}\times S$) is a closed subgroup of $\mathrm{Conf}(M,g)$. Moreover the one-parameter subgroup $\mathcal{K}$ generated by K is closed and normal in $\mathrm{Conf}(M,g)$.

Proof.

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- First part is obvious in the C^1 topology.

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Proof.

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- If $\psi \in \mathrm{Conf}_K(M, g)$ then $\psi(x, t) = (\varphi(x), t + f(x))$ with $\varphi \in \widetilde{\mathrm{Iso}}(S, F)$

$\square$

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- If $\psi \in \mathrm{Conf}_K(M, g)$ then $\psi(x, t) = (\varphi(x), t + f(x))$ with $\varphi \in \widetilde{\mathrm{Iso}}(S, F)$
- Moreover, $\psi^{-1}(x, t) = (\varphi^{-1}(x), t - f(\varphi^{-1}(x)))$

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- Moreover, $\psi^{-1}(x,t)=(\varphi^{-1}(x),t-f(\varphi^{-1}(x)))$
- Then if $K^T:M\to M$ is given by $K^T(x,t)=(x,t+T)$, it follows that $\psi\circ K^T\circ\psi^{-1}=K^T$ ($\mathcal{K}$ is normal)

$\square$

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Proposition

The map $\pi : \mathrm{Conf}_K(M,g) \to \widetilde{\mathrm{Iso}}(S,F)$ *defined as* $\pi(\psi) = \varphi$ *is a Lie group homomorphism and* $\bar{\pi} : \mathrm{Conf}_K(M,g)/\mathcal{K} \to \widetilde{\mathrm{Iso}}(S,F)$ *is an isomorphism.*

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- Injective: if ψ_1 and ψ_2 project on the same almost isometry map φ, then by last Prop. $\psi_1(x, t) = (\varphi(x), t + f(x) + c_1)$ and $\psi_2(x, t) = (\varphi(x), t + f(x) + c_2)$, $\psi_2 \circ \psi_1^{-1} = K^{c_2 - c_1}$ and $[\psi_1] = [\psi_2]$

K-conformal maps

Proposition

The map $\pi : \mathrm{Conf}_K(M,g) \to \widetilde{\mathrm{Iso}}(S,F)$ *defined as* $\pi(\psi) = \varphi$ *is a Lie group homomorphism and* $\bar{\pi} : \mathrm{Conf}_K(M,g)/\mathcal{K} \to \widetilde{\mathrm{Iso}}(S,F)$ *is an isomorphism.*

Proof.

- We just have to prove that $\bar{\pi}$ is one-to-one.
- Injective: if ψ_1 and ψ_2 project on the same almost isometry map φ, then by last Prop. $\psi_1(x,t) = (\varphi(x), t + f(x) + c_1)$ and $\psi_2(x,t) = (\varphi(x), t + f(x) + c_2)$, $\psi_2 \circ \psi_1^{-1} = K^{c_2 - c_1}$ and $[\psi_1] = [\psi_2]$
- Surjective: given an almost isometry φ, we construct the map

$$\psi(x,t) = (\varphi(x), t + f(x))$$

Clearly, it preserves ∂_t. By Fermat principle, it maps lightlike pregeodesics to lightlike pregeodesics, then it preserves the lightcone and it must be conformal (by Dajcker-Nomizu [83]).

Applications

Corollary

Given a manifold S, for a generic set of data (g_0, ω), the stationary metric $g = g(g_0, \omega)$ on $M = S \times \mathbb{R}$ has discrete K-conformal group $\mathrm{Conf}_K(M, g)/\mathcal{K}$.

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Corollary

If S is compact, then $\mathrm{Conf}_K(S \times \mathbb{R}, g)/\mathcal{K}$ and $\widetilde{\mathrm{Iso}}(S, F)$ are compact Lie groups.

Open problems

- Compute explicitly some extended isometry group
- Which are the Finsler metrics with extended isometry group of maximal dimension?

You can find this talk in http://webs.um.es/majava

Thanks a lot for this wonderful conference