ON SUCCESSIVE RADII OF p-SUMS OF CONVEX BODIES

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ABSTRACT. We study the behavior of the so called successive inner and outer radii with respect to the p-sums of convex bodies, $p \geq 1$, which were introduced by Firey in 1962. For all $p \geq 1$ we get the upper and lower bounds for the radii of the p-sum of two convex bodies in terms of the radii of the sets. These results generalize the corresponding relations for the classical Minkowski addition.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ and $| \cdot |_2$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively, and denote by e_i the i-th canonical unit vector.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$, and for $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement. For $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$, the orthogonal projection of K onto L is denoted by $K|L$. With $\mathrm{lin}\{u_1, \ldots, u_m\}$ we represent the linear hull of the vectors $u_1, \ldots, u_m$ and with $[u_1, u_2]$ the line segment with end-points u_1, u_2. Finally, for $S \subset \mathbb{R}^n$ we denote by $\mathrm{conv}\, S$ the convex hull of S and by $\mathrm{bd}\, S$ its boundary. Moreover, we write $\mathrm{relbd}\, S$ to denote the relative boundary of S, i.e., the boundary of S relative to its affine hull $\mathrm{aff}\, S$.

The diameter, minimal width, circumradius and inradius of a convex body K are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [4, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine space A, then we write $f(K; A)$ to stress that f has to be evaluated with respect to the space A. Successive outer and inner radii are defined in the following way.

Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$ let*

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad and \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big).$$

Notice that $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with i-dimensional spherical cross section containing K, and $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. It is clear that the outer radii are increasing in i, whereas the inner radii are decreasing in i, and we obviously have

$$\mathrm{R}_n(K) = \mathrm{R}(K), \ \mathrm{R}_1(K) = \frac{\omega(K)}{2}, \ \mathrm{r}_n(K) = \mathrm{r}(K) \ \text{ and } \ \mathrm{r}_1(K) = \frac{\mathrm{D}(K)}{2}.$$

The first systematic study of the successive radii was developed in [1]. For more information on these radii and their relation with other measures, we refer, for instance, to [1, 2, 5, 8, 11, 13, 14, 15, 19, 20, 21, 22, 24].

From now on we write $\mathcal{K}_0^n$ to denote the class of convex bodies containing the origin. In [7] Firey introduced the following generalization of the classical Minkowski addition (i.e., vectorial addition). Let $p \geq 1$ be given. Then, for $K, K' \in \mathcal{K}_0^n$, the p-*sum* of K and K' is the unique convex body $K +_p K'$ for which the support function

$$(1) \qquad\qquad h(K +_p K', \cdot)^p = h(K, \cdot)^p + h(K', \cdot)^p.$$

We recall that $h(K, u) = \max\{\langle x, u \rangle : x \in K\}$, $u \in \mathbb{S}^{n-1}$; as usual, $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$ (see e.g. [23, s. 1.7]). Clearly, when $p = 1$, formula (1) defines the usual Minkowski sum $K + K'$, and for $p = \infty$ it holds $h(K +_\infty K', u) = \max\{h(K, u), h(K', u)\}$, i.e., $K +_\infty K' = \mathrm{conv}(K \cup K')$. Moreover, in [7, Theorem 1] it is shown that

$$(2) \qquad\qquad K +_q K' \subseteq K +_p K'$$

for all $1 \leq p \leq q$. Observe that for the p-sums of sets, except in the case $p = 1$, the translation invariance is lost.

In [16, 17] Lutwak studied p-sums of convex bodies systematically, and developed a theory nowadays known as Brunn-Minkowski-Firey theory. In the last years many important developments of this theory have come out; we mention e.g. [3, 6, 18] and the references inside.

In this point we would like to notice that usually p-sums are defined for convex bodies containing the origin as a relative interior point, since this condition is needed in some aspects of the Brunn-Minkowski-Firey theory; however, regarding the functionals we are working on, this condition can be withdrawn, and thus we allow the origin to lie on the boundary of the convex bodies.

The behavior of the diameter, minimal width, circumradius and inradius with respect to the Minkowski sum is well known (see e.g. [23, p. 42]), namely,

$$(3) \quad \begin{array}{ll} \mathrm{D}(K + K') \leq \mathrm{D}(K) + \mathrm{D}(K'), & \omega(K + K') \geq \omega(K) + \omega(K'), \\ \mathrm{R}(K + K') \leq \mathrm{R}(K) + \mathrm{R}(K'), & \mathrm{r}(K + K') \geq \mathrm{r}(K) + \mathrm{r}(K'), \end{array}$$

and in [9] we obtained the corresponding relations for the more general successive inner and outer radii. Here we are interested in generalizing those results to the p-sums of convex bodies. We prove the following theorems.

Theorem 1.1. *Let $K, K' \in \mathcal{K}_0^n$ and $p \geq 1$. Then*

(4) $\quad 2^{\frac{p-1}{p}} \, \mathrm{R}_1(K +_p K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K') \quad$ *for all $p \geq 1$,*

(5) $\quad 2^{\frac{3p-2}{2p}} \, \mathrm{R}_i(K +_p K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K') \quad$ *for $1 \leq p \leq 2$, $i = 2, \ldots, n$,*

(6) $\qquad \mathrm{R}_i(K +_p K') \geq \max\{\mathrm{R}_i(K), \mathrm{R}_i(K')\} \quad$ *for $p \geq 2$, $i = 2, \ldots, n$.*

All inequalities are best possible.

Moreover, a reverse inequality can be obtained only in the case of the circumradius:

Proposition 1.1. *Let $K, K' \in \mathcal{K}_0^n$ and $p \geq 1$. Then*

$$\mathrm{R}_n(K +_p K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K'),$$

which is tight, and for any $i = 1, \ldots, n-1$, there exists no constant $c > 0$ such that $c \, \mathrm{R}_i(K +_p K') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$.

Theorem 1.1 and Proposition 1.1 become [9, Theorem 1.1 and Remark 3.1] when $p = 1$.

Theorem 1.2. *Let $K, K' \in \mathcal{K}_0^n$ and $p \geq 1$. Then*

(7) $\quad 2^{\frac{p-1}{p}} \, \mathrm{r}_n(K +_p K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K') \quad$ *for all $p \geq 1$,*

(8) $\quad 2^{\frac{3p-2}{2p}} \, \mathrm{r}_i(K +_p K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K') \quad$ *for $1 \leq p \leq 2$, $i = 1, \ldots, n-1$,*

(9) $\qquad \mathrm{r}_i(K +_p K') \geq \max\{\mathrm{r}_i(K), \mathrm{r}_i(K')\} \quad$ *for $p \geq 2$, $i = 1, \ldots, n-1$.*

All inequalities are best possible.

A reverse inequality can be obtained only in the case of the diameter:

Proposition 1.2. *Let $K, K' \in \mathcal{K}_0^n$ and $p \geq 1$. Then*

$$\mathrm{r}_1(K +_p K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K'),$$

which is tight, and for any $i = 2, \ldots, n$, there exists no constant $c > 0$ such that $c \, \mathrm{r}_i(K +_p K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.

Theorem 1.2 and Proposition 1.2 become [9, Theorem 1.2 and Remark 3.2] when $p = 1$.

We notice that in both Theorems 1.1 and 1.2, the last two inequalities are valid for all $p \geq 1$. We point out that the distinction depending on the range of p is needed for the sharpness.

In Section 2 we present the proofs of the main results, as well as some remarks. Finally, Section 3 is devoted to study a particular case for which the bounds in Theorems 1.1 and 1.2 can be improved, the so called p-difference body of a convex set K, i.e., $K +_p (-K)$.

2. Proofs of the main results

For $p \geq 1$ let B_n^p be the unit p-ball associated to the p-norm $|\cdot|_p$, i.e.,

$$B_n^p = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \leq 1 \right\},$$

with $|x|_\infty = \max\{|x_i| : i = 1, \dots, n\}$. For the sake of brevity, we will write $B_n = B_n^2$ to denote the n-dimensional Euclidean unit ball. Moreover, for $L \in \mathcal{L}_i^n$, we will write $B_{i,L} = B_n \cap L$ and $B_{i,L}^p = B_n^p \cap L$.

On the one hand, for any $1 \leq p \leq q$, it is a direct consequence of Hölder's inequality for q/p (see e.g. [12, p. 15]) that $|\cdot|_q \leq |\cdot|_p \leq n^{1/p-1/q} |\cdot|_q$, which is equivalent to the inclusions $B_n^p \subseteq B_n^q \subseteq n^{1/p-1/q} B_n^p$. On the other hand, it is known (see e.g. [10]) that $[-e_1, e_1] +_p \cdots +_p [-e_n, e_n] = B_n^q$ for $q \geq 1$ such that $1/p + 1/q = 1$. Therefore we get, in particular, that

$$(10) \qquad [-e_1, e_1] +_p \cdots +_p [-e_n, e_n] \subseteq \begin{cases} n^{1/p-1/2} B_n & \text{for } 1 \leq p \leq 2, \\ B_n & \text{for } p \geq 2. \end{cases}$$

We start proving the lower bound for the outer radii $R_i(K +_p K')$ of the p-sum of two convex bodies in terms of the corresponding radii.

Proof of Theorem 1.1. In [7, Theorem 1] it is shown that

$$(11) \qquad \frac{1}{2^{(p-1)/p}} (K + K') \subseteq K +_p K' \subseteq K + K'.$$

Therefore $2^{(p-1)/p} R_i(K +_p K') \geq R_i(K + K')$ for all $i = 1, \dots, n$, and by [9, Theorem 1.1] we get

$$2^{(p-1)/p} R_1(K +_p K') \geq R_1(K + K') \geq R_1(K) + R_1(K'),$$

$$2^{(p-1)/p} R_i(K +_p K') \geq R_i(K + K') \geq \frac{1}{\sqrt{2}} (R_i(K) + R_i(K')),$$

$i = 2, \dots, n$, which shows inequalities (4) and (5). Notice also that it always holds $K, K' \subseteq K +_p K'$, which leads to

$$R_i(K +_p K') \geq \max\{R_i(K), R_i(K')\}.$$

Since for any non-negative real numbers $a, b \geq 0$ it holds that if $p \geq 2$ then $\max\{a, b\} \geq 1/2^{(3p-2)/(2p)}(a + b)$, inequality (6) is obtained.

So it remains to be shown that the three inequalities are best possible. For the first one, let $K = K'$. Then $K +_p K = 2^{1/p} K$ and thus

$$2^{(p-1)/p} R_1(K +_p K) = 2^{(p-1)/p} 2^{1/p} R_1(K) = R_1(K) + R_1(K).$$

Next, for $i \in \{2, \dots, n-1\}$ we consider the convex bodies

$$K = [-e_1, e_1] + \sum_{k=i+1}^n [-e_k, e_k], \quad K' = [-e_2, e_2] + \sum_{k=i+1}^n [-e_k, e_k],$$

i.e., the 0-symmetric $(n-i+1)$-cubes with edges parallel to the coordinate axes and length 2, of the subspaces $L_j = \lin\{e_j, e_{i+1}, \ldots, e_n\} \in \mathcal{L}_{n-i+1}^n$, $j = 1, 2$, and for $i = n$ we take $K = [-e_1, e_1]$, $K' = [-e_2, e_2]$. Clearly $\mathrm{R}(K|L), \mathrm{R}(K'|L) \geq 1$ for all $L \in \mathcal{L}_i^n$. Moreover, $\mathrm{R}\big(K|\lin\{e_1, \ldots, e_i\}\big) = \mathrm{R}\big(K'|\lin\{e_1, \ldots, e_i\}\big) = 1$, which shows that $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 1$.

Let $1 \leq p \leq 2$, and we compute $\mathrm{R}_i(K +_p K')$. Let $L \in \mathcal{L}_i^n$. On the one hand, since $\dim L_j \cap L \geq (n-i+1) + i - n = 1$, there exist $x \in K \cap L$ and $x' \in K' \cap L$ with $|x|_2, |x'|_2 \geq 1$, and because of the central symmetry, we may assume that $\langle x, x' \rangle \geq 0$. Then

$$\left| \frac{(x + x')}{2^{(p-1)/p}} \right|_2 \geq \frac{\big(|x|_2^2 + |x'|_2^2\big)^{1/2}}{2^{(p-1)/p}} \geq \frac{2^{1/2}}{2^{(p-1)/p}} = 2^{(2-p)/(2p)}$$

and thus, since $(x + x')/2^{(p-1)/p} \in (K +_p K') \cap L$ (see (11)), we get

$$(12) \qquad \mathrm{R}\big((K +_p K')|L\big) \geq \mathrm{R}\big((K +_p K') \cap L\big) \geq 2^{(2-p)/(2p)}$$

for all $L \in \mathcal{L}_i^n$. On the other hand, since the orthogonal projection of the p-sum of two convex bodies onto any lower dimensional linear subspace is the p-sum of the projections (see [7, pp. 21–22]), and using (10), we get

$$(K +_p K')|\lin\{e_1, \ldots, e_i\} = K|\lin\{e_1, \ldots, e_i\} +_p K'|\lin\{e_1, \ldots, e_i\}$$

$$= K|\lin\{e_1\} +_p K'|\lin\{e_2\} = [-e_1, e_1] +_p [-e_2, e_2]$$

$$\subseteq 2^{(2-p)/(2p)} B_{2, \lin\{e_1, e_2\}},$$

which gives $\mathrm{R}\big((K +_p K')|\lin\{e_1, \ldots, e_i\}\big) \leq 2^{(2-p)/(2p)}$. Then, together with (12) we get $\mathrm{R}\big((K +_p K')|\lin\{e_1, \ldots, e_i\}\big) = 2^{(2-p)/(2p)}$, and moreover,

$$\mathrm{R}_i(K +_p K') = \min_{L \in \mathcal{L}_i^n} \mathrm{R}\big((K +_p K')|L\big) = 2^{(2-p)/(2p)} = \frac{1}{2^{(3p-2)/(2p)}}(1 + 1)$$

$$= \frac{1}{2^{(3p-2)/(2p)}}\big(\mathrm{R}_i(K) + \mathrm{R}_i(K')\big),$$

as required.

Now let $p \geq 2$ and K, K' as before. First notice that $\mathrm{R}_i(K +_p K') \geq \mathrm{R}_i(K) = \mathrm{R}_i(K') = 1$. With an analogous argument to the previous one, but using (10) when $p \geq 2$, we get that

$$\mathrm{R}\big((K +_p K')|\lin\{e_1, \ldots, e_i\}\big) \leq \mathrm{R}\big(B_{2, \lin\{e_1, e_2\}}\big) = 1.$$

Both inequalities give $\mathrm{R}_i(K +_p K') = 1 = \mathrm{R}_i(K) = \mathrm{R}_i(K')$. $\qquad \square$

However, as we show next, there is no chance to get a reverse inequality for all outer radii.

Proof of Proposition 1.1. In the case of the circumradius we easily get, using (11) and (3), that

$$\mathrm{R}_n(K +_p K') \leq \mathrm{R}_n(K + K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K').$$

Notice that the inequality is tight, because the circumradius is a continuous functional and equality is attained if $K' = \{0\}$.

In order to show the non-existence of a reverse inequality, $i = 1, \ldots, n-1$, we take the convex bodies

$$K = [-\mathrm{e}_{n-i+1}, \mathrm{e}_{n-i+1}] \quad \text{and} \quad K' = \sum_{k=1}^{n-i}[-\mathrm{e}_k, \mathrm{e}_k].$$

On the one hand, since

$$K|\lin\{\mathrm{e}_{n-i}, \mathrm{e}_{n-i+2}, \ldots, \mathrm{e}_n\} = K'|\lin\{\mathrm{e}_{n-i+1}, \ldots, \mathrm{e}_n\} = \{0\},$$

then $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 0$, i.e., $\mathrm{R}_i(K) + \mathrm{R}_i(K') = 0$. On the other hand,

$$K +_p K' \supseteq \frac{1}{2^{(p-1)/p}}(K + K') = \frac{1}{2^{(p-1)/p}}\sum_{j=1}^{n-i+1}[-\mathrm{e}_j, \mathrm{e}_j],$$

i.e., $K +_p K'$ contains an $(n-i+1)$-dimensional convex body, which implies that $\dim\big((K +_p K')|L\big) \geq 1$ for all $L \in \mathcal{L}_i^n$. Then, $\mathrm{R}_i(K +_p K') > 0$, which shows the result. $\qquad\square$

Remark 2.1. *We notice that the inequality* $\mathrm{R}(K +_p K') \leq \mathrm{R}(K) + \mathrm{R}(K')$, $p \geq 1$, *can be strengthened in the particular case when the circumcenter of both* $K, K' \in \mathcal{K}_0^n$ *lies at the origin: since*

$$h(K +_p K', u) = \big(h(K, u)^p + h(K', u)^p\big)^{1/p} \leq \big(\mathrm{R}(K)^p + \mathrm{R}(K')^p\big)^{1/p}$$

for all $u \in \mathbb{S}^{n-1}$, *then we get*

$$\mathrm{R}(K +_p K') \leq \big(\mathrm{R}(K)^p + \mathrm{R}(K')^p\big)^{1/p}.$$

If $K' = K$ *equality holds.*

Now we deal with the inner radii $\mathrm{r}_i(K +_p K')$ and prove Theorem 1.2.

Proof of Theorem 1.2. By (11) we have $2^{(p-1)/p}\,\mathrm{r}_i(K +_p K') \geq \mathrm{r}_i(K + K')$ for all $i = 1, \ldots, n$, and applying [9, Theorem 1.2] we get

$$2^{(p-1)/p}\,\mathrm{r}_n(K +_p K') \geq \mathrm{r}_n(K + K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K'),$$

$$2^{(p-1)/p}\,\mathrm{r}_i(K +_p K') \geq \mathrm{r}_i(K + K') \geq \frac{1}{\sqrt{2}}\big(\mathrm{r}_i(K) + \mathrm{r}_i(K')\big),$$

$i = 1, \ldots, n-1$, which shows inequalities (7) and (8). Again, since $K, K' \subseteq K +_p K'$, then

$$\mathrm{r}_i(K +_p K') \geq \max\big\{\mathrm{r}_i(K), \mathrm{r}_i(K')\big\},$$

which, together with the fact that $\max\{a, b\} \geq 1/2^{(3p-2)/(2p)}(a+b)$ for $p \geq 2$ and $a, b \geq 0$, leads to inequality (9).

So, we have to show that these inequalities are best possible. For the first one, with $K = K'$ we get

$$2^{(p-1)/p}\,\mathrm{r}_n(K +_p K) = 2^{(p-1)/p}\,2^{1/p}\,\mathrm{r}_n(K) = \mathrm{r}_n(K) + \mathrm{r}_n(K).$$

Next we fix $i \in \{1, \ldots, n-1\}$ and consider the i-dimensional unit balls $B_{i,L}$ and $B_{i,L'}$ of the following i-dimensional linear subspaces: if $2i > n$, let

$$L = \lin\{\mathrm{e}_1, \ldots, \mathrm{e}_{2i-n}, \mathrm{e}_{2i-n+1}, \ldots, \mathrm{e}_i\}, \quad L' = \lin\{\mathrm{e}_1, \ldots, \mathrm{e}_{2i-n}, \mathrm{e}_{i+1}, \ldots, \mathrm{e}_n\};$$

if $2i \leq n$, let

$$L = \mathrm{lin}\{e_1, \ldots, e_i\}, \quad L' = \mathrm{lin}\{e_{i+1}, \ldots, e_{2i}\}.$$

Clearly, $\mathrm{r}_i(B_{i,L}) = \mathrm{r}_i(B_{i,L'}) = 1$. We notice also that, since $B_{i,L}, B_{i,L'}$ are 0-symmetric, then $B_{i,L} +_p B_{i,L'}$ is also 0-symmetric, and then

$$\mathrm{r}_i(B_{i,L} +_p B_{i,L'}) = \max_{\bar{L} \in \mathcal{L}_i^n} \mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big).$$

Let $1 \leq p \leq 2$. We are going to show that

(13) $$\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 2^{(2-p)/(2p)}$$

for all $\bar{L} \in \mathcal{L}_i^n$, which will imply that

$$2^{(2-p)/(2p)} = \frac{1}{2^{(3p-2)/(2p)}}(1+1) = \frac{1}{2^{(3p-2)/(2p)}}\big(\mathrm{r}_i(B_{i,L}) + \mathrm{r}_i(B_{i,L'})\big)$$

$$\leq \mathrm{r}_i(B_{i,L} +_p B_{i,L'}) \leq 2^{\frac{2-p}{2p}},$$

i.e., $\mathrm{r}_i(B_{i,L} +_p B_{i,L'}) = 1/2^{(3p-2)/(2p)}\big(\mathrm{r}_i(B_{i,L}) + \mathrm{r}_i(B_{i,L'})\big)$, as required.

Let $L'' = \mathrm{lin}\{e_{j+1}, \ldots, e_n\}$, where $j = 2i - n$ if $2i > n$ and $j = 0$ if $2i \leq n$. In both cases,

$$\dim(\bar{L} \cap L'') = \dim \bar{L} + \dim L'' - \dim(\bar{L} + L'') = i + n - j - \dim(\bar{L} + L'')$$

$$\geq i + n - j - n = i - j \geq 1.$$

Therefore, since $\dim(B_{i,L} +_p B_{i,L'}) = n$, then, for all $\bar{L} \in \mathcal{L}_i^n$, we can always find $z \in \mathrm{relbd}(B_{i,L} +_p B_{i,L'}) \cap \bar{L} \cap L''$, $z \neq 0$. Moreover, notice that, in particular, $z \in \mathrm{lin}\{e_{j+1}, \ldots, e_{2i-j}\}$, and so it can be expressed in the form $z = x + x' \in \mathrm{lin}\{e_{j+1}, \ldots, e_i\} + \mathrm{lin}\{e_{i+1}, \ldots, e_{2i-j}\} = (L \cap L'') + (L' \cap L'')$; observe that x, x' lie in orthogonal subspaces. Writing $u = z/|z|_2$, we have

$$|z|_2 = \langle z, u \rangle \leq h\big(B_{i,L} +_p B_{i,L'}, u\big) = \Big(h\big(B_{i,L}, u\big)^p + h\big(B_{i,L'}, u\big)^p\Big)^{1/p},$$

and since

$$h(B_{i,L}, u) = \sup_{y \in B_{i,L}} \langle y, u \rangle = \frac{1}{|z|_2} \sup_{y \in B_{i,L}} \langle y, x \rangle = \frac{1}{|z|_2}\left\langle \frac{x}{|x|_2}, x \right\rangle$$

$$= h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right], u\right)$$

and analogously $h(B_{i,L'}, u) = h\big([-x'/|x'|_2, x'/|x'|_2], u\big)$, we obtain that

$$|z|_2 \leq \left(h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right], u\right)^p + h\left(\left[-\frac{x'}{|x'|_2}, \frac{x'}{|x'|_2}\right], u\right)^p\right)^{1/p}$$

$$= h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right] +_p \left[-\frac{x'}{|x'|_2}, \frac{x'}{|x'|_2}\right], u\right) = h\left(B_{2,\mathrm{lin}\{x,x'\}}^{p/(p-1)}, u\right)$$

$$\leq \mathrm{R}\left(B_{2,\mathrm{lin}\{x,x'\}}^{p/(p-1)}\right) \leq 2^{(2-p)/(2p)}\mathrm{R}\big(B_{2,\mathrm{lin}\{x,x'\}}\big) = 2^{(2-p)/(2p)}$$

by (10). This implies $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 2^{(2-p)/(2p)}$, showing (13) and concluding the proof of the case $1 \leq p \leq 2$.

Now let $p \geq 2$. Notice that $\mathrm{r}_i\left(B_{i,L} +_p B_{i,L'}\right) \geq \mathrm{r}_i\big(B_{i,L}\big) = \mathrm{r}_i\big(B_{i,L'}\big) = 1$. So, it suffices to show that $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 1$ for all $\bar{L} \in \mathcal{L}_i^n$. With an analogous argument as before, but using (10) when $p \geq 2$, we get that there exists $z \in \mathrm{relbd}(B_{i,L} +_p B_{i,L'}) \cap \bar{L} \cap L''$ such that

$$|z|_2 \leq \mathrm{R}\left(B_{2,\mathrm{lin}\{x,x'\}}^{p/(p-1)}\right) \leq \mathrm{R}\left(B_{2,\mathrm{lin}\{x,x'\}}\right) = 1.$$

It shows that $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 1$ and concludes the proof.   □

We deal again with the possible existence of a reverse inequality, showing Proposition 1.2.

Proof of Proposition 1.2. In the case of the diameter we easily get, using (11) and (3), that

$$\mathrm{r}_1(K +_p K') \leq \mathrm{r}_1(K + K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K').$$

The inequality is tight, since the diameter is a continuous functional and equality is attained if $K' = \{0\}$.

In order to show the non-existence of a reverse inequality, $i = 2, \ldots, n$, we take the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{k=2}^{i} [-\mathrm{e}_k, \mathrm{e}_k].$$

Clearly $\mathrm{r}_i(K) = \mathrm{r}_i(K') = 0$, because they have dimensions $\dim K = 1$ and $\dim K' = i - 1$. However,

$$\mathrm{r}_i(K +_p K') \geq 2^{-(p-1)/p} \mathrm{r}_i(K + K') = 2^{-(p-1)/p} \mathrm{r}_i \left(\sum_{k=1}^{i} [-\mathrm{e}_k, \mathrm{e}_k]\right) = 2^{-(p-1)/p},$$

which shows the result.   □

3. On the p-difference body of a convex set

The difference body of a convex body $K \in \mathcal{K}^n$ is defined as the Minkowski addition $K - K := K + (-K)$ which, in particular, is a 0-symmetric body. Thus, this operation defines the so called central symmetral of a convex body K, just taking $K^0 = (K - K)/2$ (see e.g. [4, p. 79] for properties of the central symmetrization).

Similarly, the p-difference body can be defined: for $K \in \mathcal{K}_0^n$, we take the p-sum $K -_p K := K +_p (-K)$. We observe that $K -_p K$ is also a 0-symmetric convex body: in fact,

$$h(K -_p K, -u)^p = h(K, -u)^p + h(-K, -u)^p = h(-K, u)^p + h(K, u)^p$$
$$= h(K -_p K, u)^p.$$

In [3], a sharp Rogers-Shephard inequality for the p-difference body of a planar convex body was obtained, i.e., the best (upper) bound for the volume

of the set $K -_p K$ in terms of the volume of the original body K. Here we are interested in obtaining upper and lower bounds for the in- and outer radii of the p-difference body $K -_p K$ in terms of the ones of K. In [9, Proposition 4.2] we already studied the behavior of the radii regarding the usual difference body: for all $i = 1, \ldots, n$,

(14)
$$\sqrt{2}\sqrt{\frac{i+1}{i}} \mathrm{R}_i(K) \leq \mathrm{R}_i(K - K) \leq 2\mathrm{R}_i(K),$$
$$2\mathrm{r}_i(K) \leq \mathrm{r}_i(K - K) < 2(i+1)\mathrm{r}_i(K).$$

Next proposition extends the above results to the p-difference body, $p \geq 1$, showing moreover that the bounds in Theorems 1.1 and 1.2 can be improved and that, in this particular case, there are non-trivial reverse inequalities (cf. Propositions 1.1 and 1.2).

Proposition 3.1. *Let* $K \in \mathcal{K}_0^n$. *Then for all* $i = 1, \ldots, n$ *and all* $p \geq 1$,

(15)
$$\mathrm{R}_i(K -_p K) \leq 2\mathrm{R}_i(K),$$

(16)
$$2^{1/p-1/2}\sqrt{\tfrac{i+1}{i}}\mathrm{R}_i(K) \leq \mathrm{R}_i(K -_p K) \qquad \text{if } 1 \leq p \leq 2,$$

(17)
$$\max\left\{2^{1/p-1/2}\sqrt{\tfrac{i+1}{i}}, 1\right\}\mathrm{R}_i(K) \leq \mathrm{R}_i(K -_p K) \qquad \text{if } p \geq 2.$$

(18)
$$2^{1/p}\mathrm{r}_i(K) \leq \mathrm{r}_i(K -_p K) < 2(i+1)\mathrm{r}_i(K).$$

Inequalities (15), (16) *and the lower bound in* (18) *are best possible.*

Proof. By (11) we have, for all $i = 1, \ldots, n$, that

$$\frac{1}{2^{(p-1)/p}}\mathrm{R}_i(K - K) \leq \mathrm{R}_i(K -_p K) \leq \mathrm{R}_i(K - K),$$

and analogously for the inner radii. Then applying (14), together with the fact that $K \subseteq K -_p K'$, we directly get the required inequalities.

We observe that if $p \geq 2$, then $\max\{2^{1/p-1/2}\sqrt{(i+1)/i}, 1\} = 1$ for all $i \geq 2$ and all $p \geq (2\log 2)/\log(2i/(i+1)) \in (2, 4.81\ldots)$.

So we deal with the sharpness of the inequalities, starting with the left hand side in (18). In this case, just notice that if K is a 0-symmetric convex body then $K = -K$ and hence $\mathrm{r}_i(K -_p K) = \mathrm{r}_i(K +_p K) = 2^{1/p}\mathrm{r}_i(K)$.

Next we study (15). We fix $i \in \{1, \ldots, n\}$ and consider the convex body $K = [0, \mathrm{e}_1] + \sum_{j=i+1}^n [-\mathrm{e}_j, \mathrm{e}_j]$, for which it clearly holds

$$\mathrm{R}_i(K) = \mathrm{R}\big(K | \lin\{\mathrm{e}_1, \ldots, \mathrm{e}_i\}\big) = \mathrm{R}\big([0, \mathrm{e}_1]\big) = \frac{1}{2};$$

here, if $i = n$ we are taking $K = [0, \mathrm{e}_1]$. Now, on the one hand we notice that $(K -_p K) | \lin\{\mathrm{e}_1, \ldots, \mathrm{e}_i\} = [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0]$ and that, by (2),

$$[-\mathrm{e}_1, \mathrm{e}_1] = \conv\big([0, \mathrm{e}_1] \cup [-\mathrm{e}_1, 0]\big) \subseteq [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0] \subseteq [-\mathrm{e}_1, \mathrm{e}_1],$$

i.e., $(K -_p K) | \lin\{\mathrm{e}_1, \ldots, \mathrm{e}_i\} = [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0] = [-\mathrm{e}_1, \mathrm{e}_1]$. On the other hand we observe that $\conv\big(K \cup (-K)\big) = C_{n-i+1}$ is the $(n-i+1)$-dimensional

cube of edge-length 2 contained in $\mathrm{lin}\{e_1, e_{i+1}, \ldots, e_n\}$ and thus, by (2), we get that for all $L \in \mathcal{L}_i^n$

$$
\begin{aligned}
\mathrm{R}\big((K -_p K)|L\big) &\geq \mathrm{R}\big(\mathrm{conv}\big(K \cup (-K)\big)|L\big) = \mathrm{R}(C_{n-i+1}|L) \\
&\geq \mathrm{R}\big(C_{n-i+1}|\mathrm{lin}\{e_1, \ldots, e_i\}\big) = \mathrm{R}\big([-e_1, e_1]\big) \\
&= \mathrm{R}\big((K -_p K)|\mathrm{lin}\{e_1, \ldots, e_i\}\big).
\end{aligned}
$$

Therefore,

$$
\mathrm{R}_i(K -_p K) = \mathrm{R}\big((K -_p K)|\mathrm{lin}\{e_1, \ldots, e_i\}\big) = 1 = 2\mathrm{R}_i(K).
$$

Finally we consider the equality case in (17).

If $i = n$, let S_n be the n-dimensional simplex, embedded in $\mathbb{R}^{n+1}$, lying in the hyperplane $\big\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j = 0\big\}$, given by

$$
S_n = \mathrm{conv}\left\{p_k : p_{kk} = \frac{n}{n+1}, \ p_{kj} = \frac{-1}{n+1} \text{ for } j \neq k, \ k = 1, \ldots, n+1\right\}.
$$

Since $S_n -_p S_n$ is a 0-symmetric n-dimensional convex body, then

$$
\mathrm{R}_n(S_n -_p S_n) = \max\left\{h(S_n -_p S_n, u) : |u|_2 = 1 \text{ and } \sum_{j=1}^{n+1} u_j = 0\right\}.
$$

Let $u \in \mathbb{R}^{n+1}$ with $|u|_2 = 1$ and $\sum_{j=1}^{n+1} u_j = 0$. Recall that the value of the support function of a convex body at any vector is attained in an extreme point (cf. e.g. [12, Theorem 5.6]), so, in order to compute $h(S_n, u)$ it suffices to consider the vertices of S_n. Since

$$
\langle p_k, u \rangle = \frac{n}{n+1}u_k - \frac{1}{n+1}\sum_{j \neq k} u_j = u_k,
$$

then $h(S_n, u) = \max\{\langle p_k, u \rangle : k = 1, \ldots, n+1\} = \max\{u_1, \ldots, u_{n+1}\}$. Without loss of generality we may assume that $u_1 \geq \cdots \geq u_{n+1}$. Hence

$$
\begin{aligned}
h(S_n -_p S_n, u)^p &= h(S_n, u)^p + h(-S_n, u)^p = h(S_n, u)^p + h(S_n, -u)^p \\
&= u_1^p + (-u_{n+1})^p.
\end{aligned}
$$

We observe that $u_1 \geq 0$ and $u_{n+1} \leq 0$. Then, by elementary calculations it can be shown that the maximum of the function $u_1^p + (-u_{n+1})^p$, $1 \leq p \leq 2$, under the conditions $|u|_2 = 1$ and $\sum_{j=1}^{n+1} u_j = 0$, is attained in the point $\big(1/\sqrt{2}, 0, \ldots, 0, -1/\sqrt{2}\big)$. Therefore,

$$
\mathrm{R}_n(S_n -_p S_n) = \left(\frac{1}{2^{p/2}} + \frac{1}{2^{p/2}}\right)^{1/p} = 2^{1/p-1/2}.
$$

Since $\mathrm{R}_n(S_n) = \sqrt{n/(n+1)}$, then we get the required equality:

$$
\mathrm{R}_n(S_n -_p S_n) = 2^{1/p-1/2} = 2^{1/p-1/2}\sqrt{\frac{n+1}{n}}\,\mathrm{R}_n(S_n).
$$

If $i < n$, we take the i-dimensional simplex S_i and consider the convex body $K = S_i + MC_{n-i}$, where $C_{n-i} \subset (\text{aff } S_i)^{\perp}$ represents the $(n-i)$-dimensional unit cube and $M > 0$ is sufficiently large such that $\mathrm{R}_i(K -_p K) = \mathrm{R}(S_i -_p S_i)$ and $\mathrm{R}_i(K) = \mathrm{R}(S_i)$. The above argument gives the result. $\qquad\square$

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