

Uniqueness of measure extensions in Banach spaces

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Abstract

Let X be a Banach space, $B \subset B_{X^*}$ a norming set and $\sigma(X, B)$ the topology on X of pointwise convergence on B . We study the following question:

Given two (non negative, countably additive and finite) measures μ_1 and μ_2 on $\text{Baire}(X, w)$,

$$\mu_1|_{\text{Baire}(X, \sigma(X, B))} = \mu_2|_{\text{Baire}(X, \sigma(X, B))} \implies \mu_1 = \mu_2 \text{ ?}$$

While this is not true in general, the answer is affirmative provided that both μ_1 and μ_2 are convexly τ -additive (and this condition holds automatically whenever X has the Pettis Integral Property). For a Banach space Y not containing isomorphic copies of ℓ^1 , we show that Y^* has the Pettis Integral Property if and only if every measure on $\text{Baire}(Y^*, w^*)$ admits a unique extension to $\text{Baire}(Y^*, w)$. We also discuss the coincidence of the two σ -algebras involved in such results.

1 Introduction

All the **measures** considered in this poster are non negative, countably additive and finite.

From now on X is a real Banach space with topological dual X^* . We write w (resp. w^*) to denote the weak topology on X (resp. the weak* topology on X^*). As usual, we denote by B_X the closed unit ball of X .

According to a result of G. A. Edgar [3], the Baire σ -algebra of a locally convex space endowed with its weak topology is exactly the σ -algebra generated by the elements of the topological dual. Thus a function $f : \Omega \rightarrow X$, defined on a probability space $(\Omega, \Sigma, \lambda)$, is scalarly measurable if and only if it is Σ - $\text{Baire}(X, w)$ -measurable. Assume further that f is scalarly bounded, i.e. the family of compositions $\{x^* \circ f : x^* \in B_{X^*}\}$ is a bounded subset of $\mathcal{L}^\infty(\lambda)$. It was shown by M. Talagrand [14] that f is Pettis integrable if and only if the image measure λf^{-1} on $\text{Baire}(X, w)$ is convexly τ -additive, in the following sense.

Definition 1 A measure μ on $\text{Baire}(X, w)$ is **convexly τ -additive** if for each decreasing net (C_α) of convex closed elements of $\text{Baire}(X, w)$ with $\bigcap_\alpha C_\alpha = \emptyset$, we have $\lim_\alpha \mu(C_\alpha) = 0$.

Recall that X has the **Pettis Integral Property** (PIP) if every scalarly bounded X -valued function is Pettis integrable; see for instance [9] and the references therein. As a consequence of Talagrand's result, one can deduce that

$$X \text{ has the PIP} \iff \text{every measure on } \text{Baire}(X, w) \text{ is convexly } \tau\text{-smooth.}$$

In view of the characterization above, it is clear that every Banach space with the so-called property (C) of Corson [1] (e.g. a weakly compactly generated space or, more generally, a weakly Lindelöf one) has the PIP.

Now let $B \subset B_{X^*}$ be a norming set, i.e. $\|x\| = \sup\{|x^*| : x^* \in B\}$ for every $x \in X$. The topology $\sigma(X, B)$ on X is the coarsest one for which each element of B is continuous. In view of the aforementioned theorem of G. A. Edgar, $\text{Baire}(X, \sigma(X, B))$ is just the σ -algebra on X generated by B . Clearly, we have $\text{Baire}(X, \sigma(X, B)) \subset \text{Baire}(X, w)$.

The purpose of this work is to discuss the following general question, which turns out to be closely related to the Pettis integral theory:

Given two measures μ_1 and μ_2 on $\text{Baire}(X, w)$ which coincide on $\text{Baire}(X, \sigma(X, B))$, does it follow that $\mu_1 = \mu_2$?

2 Uniqueness of convexly τ -additive extensions

It is convenient to introduce a definition.

Definition 2 Let $\Sigma' \subset \Sigma$ be two σ -algebras on a set Ω and let \mathcal{M} be a family of measures on Σ . We say that Σ' has the **uniqueness property** with respect to \mathcal{M} if for every pair $\mu_1, \mu_2 \in \mathcal{M}$ we have

$$\mu_1|_{\Sigma'} = \mu_2|_{\Sigma'} \implies \mu_1 = \mu_2.$$

As a consequence of the results of R. G. Douglas [2] we can characterize the uniqueness property as follows.

Proposition 1 Let $\Sigma' \subset \Sigma$ be two σ -algebras on a set Ω and let \mathcal{M} be a family of measures on Σ such that:

- $\mu_1 + \mu_2 \in \mathcal{M}$ for every $\mu_1, \mu_2 \in \mathcal{M}$.
- If $\nu \in \mathcal{M}$ and μ is a measure on Σ such that $\mu \leq \nu$, then $\mu \in \mathcal{M}$.

Then the following statements are equivalent:

- Σ' has the uniqueness property with respect to \mathcal{M} .
- For every $\mu \in \mathcal{M}$ and every $E \in \Sigma$ there exists $E' \in \Sigma'$ such that $\mu(E \triangle E') = 0$.

Our approach to the uniqueness problem relies on G. A. Edgar's work [5] (going back to A. Ionescu-Tulcea [8]) about the continuity of the mapping

$$I : \mathcal{R} \rightarrow \mathbb{R}, \quad I(f) = \int_{\Omega} f \, d\lambda,$$

where $(\Omega, \Sigma, \lambda)$ is a probability space and $\mathcal{R} \subset \mathcal{L}^1(\lambda) \subset \mathbb{R}^{\Omega}$ is endowed with the pointwise convergence topology. For a thorough study on this subject we refer the reader to [15].

We can now state a general result on the uniqueness of convexly τ -additive extensions.

Theorem 2 Let $B \subset B_{X^*}$ be a norming set. Then $\text{Baire}(X, \sigma(X, B))$ has the uniqueness property with respect to the family of all convexly τ -additive measures on $\text{Baire}(X, w)$.

Corollary 3 Suppose that X has the PIP and let $B \subset B_{X^*}$ be a norming set. Then $\text{Baire}(X, \sigma(X, B))$ has the uniqueness property with respect to the family of all measures on $\text{Baire}(X, w)$.

3 Uniqueness in dual Banach spaces

In this section we deal with a particular case of the situation considered above: we assume that $X = Y^*$ for some Banach space Y and we take $B = B_Y$, so that $\sigma(X, B)$ is just the weak* topology.

Definition 3 We say that Y^* has **property (U)** if $\text{Baire}(Y^*, w^*)$ has the uniqueness property with respect to the family of all measures on $\text{Baire}(Y^*, w)$.

From Corollary 3 it follows that

$$Y^* \text{ has the PIP} \implies Y^* \text{ has property (U).}$$

We do not know whether the reverse implication holds in general, but this is always the case provided that Y contains no subspace isomorphic to ℓ^1 (we write $Y \not\supset \ell^1$ for short), as we point out below.

When $Y \not\supset \ell^1$, a result of R. Haydon [7] ensures that the identity mapping $I : B_{Y^*} \rightarrow Y^*$ is Pettis integrable with respect to the completion of each Radon measure on $\text{Borel}(B_{Y^*}, w^*)$. This fact is a basic tool to prove the following

Theorem 4 Suppose that $Y \not\supset \ell^1$. Then every measure on $\text{Baire}(Y^*, w^*)$ can be extended in a unique way to a convexly τ -additive measure on $\text{Baire}(Y^*, w)$.

Corollary 5 Suppose that $Y \not\supset \ell^1$. The following statements are equivalent:

- Y^* has the PIP.
- Every measure on $\text{Baire}(Y^*, w^*)$ can be extended in a unique way to a measure on $\text{Baire}(Y^*, w)$.
- Y^* has property (U).

Recall that a topological space is **realcompact** if it is homeomorphic to a closed subset of \mathbb{R}^I for some set I . G. A. Edgar [4] showed that every Banach space with the PIP is realcompact for its weak topology. We can now obtain the same conclusion for any dual Banach space with property (U).

Proposition 6 If Y^* has property (U), then (Y^*, w) is realcompact.

The converse of Proposition 6 does not hold in general. Indeed, the space ℓ_∞ is weakly realcompact [1], whereas it fails property (U), as D. H. Fremlin pointed out to us. His example involves the so-called Talagrand's measure [13] already used in [6] when proving that ℓ_∞ does not enjoy the PIP.

4 Coincidence of $\text{Baire}(X, \sigma(X, B))$ and $\text{Baire}(X, w)$

Of course, our results on the uniqueness of measure extensions are not of interest when the norming set $B \subset B_{X^*}$ satisfies

$$\text{Baire}(X, \sigma(X, B)) = \text{Baire}(X, w).$$

For instance, this equality holds true whenever (B_{X^*}, w^*) is angelic (e.g. for a weakly compactly generated X).

In this section we discuss the coincidence of both σ -algebras in two particular cases: (i) Dual Banach spaces and the weak* topology. (ii) Banach spaces of real-valued continuous functions on a compact Hausdorff topological space and the pointwise convergence topology.

The Odell-Rosenthal theorem [11] states that a separable Banach space Y is w^* -sequentially dense in Y^{**} if and only if $Y \not\supset \ell^1$. For non necessarily separable spaces we have the following characterization due to M. Raja.

Proposition 7 For a Banach space Y the following statements are equivalent:

- Y is weak*-sequentially dense in Y^{**} .
- $\text{Baire}(Y^*, w^*) = \text{Baire}(Y^*, w)$.
- $Y \not\supset \ell^1$ and for every $y^{**} \in Y^{**}$ there exists a countable set $D \subset Y$ such that $y^{**} \in \overline{D}^{w^*}$.

For instance, $\ell^1(\omega_1)$ is an example of dual Banach space with the PIP for which $\text{Baire}(\ell^1(\omega_1), w^*) \neq \text{Baire}(\ell^1(\omega_1), w)$.

On the other hand, R. Pol's [12] dual characterization of property (C) allows us to deduce the following

Corollary 8 Let Y be a Banach space such that Y^* has property (C). Then $\text{Baire}(Y^*, w^*) = \text{Baire}(Y^*, w)$.

Given a compact Hausdorff topological space K , we write $C(K)$ to denote the Banach space of all real-valued continuous functions on K endowed with the supremum norm. Notice that the set $B = \{\delta_t : t \in K\} \subset B_{C(K)^*}$ of "point masses" is norming and that $\sigma(C(K), B)$ is just the pointwise convergence topology $\mathfrak{T}_p(K)$.

Proposition 9 Let K be a compact Hausdorff topological space and ν a Radon measure on K . If the functional

$$\iota(\nu) : C(K) \rightarrow \mathbb{R}, \quad h \mapsto \int_K h \, d\nu,$$

is $\text{Baire}(C(K), \mathfrak{T}_p(K))$ -measurable, then there exists a closed separable set $F \subset K$ such that $\nu(K \setminus F) = 0$.

Corollary 10 Let K be a compact Hausdorff topological space such that $\text{Baire}(C(K), \mathfrak{T}_p(K)) = \text{Baire}(C(K), w)$. Then for each Radon measure ν on K there exists a closed separable set $F \subset K$ such that $\nu(K \setminus F) = 0$.

Under the continuum hypothesis, there exists a compact Hausdorff topological space K (the Kunen-Haydon-Talagrand space, cf. [10]) such that $C(K)$ has the PIP and $\text{Baire}(C(K), \mathfrak{T}_p(K)) \neq \text{Baire}(C(K), w)$.

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