Lawson-type problems in non-standard 3-spheres

Manuel Barros, Angel Ferrández and Pascual Lucas
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Abstract

We show that there exist infinitely many metrics on $\mathbb{S}^3$ which provide a discrete family of non congruent embedded minimal tori in $\mathbb{S}^3$. In particular, we obtain a metric which gives a foliation in the once punctured $\mathbb{R}P^3$ whose leaves are pairwise non congruent embedded minimal tori. This contrasts with the recent solution of a well known conjecture of H.B. Lawson.

1. Introduction

It is well known that there exist no compact minimal surfaces (not even immersed) in the Euclidean space. In a recent paper, P. Tomter, [12], has shown that there exist no compact $\mathbb{S}^1$-invariant surfaces, with nonzero genus and constant mean curvature (in particular, minimal tori) in the classic Heisenberg group.

Three main facts should be pointed out regarding the standard 3-sphere. First, an already classical result of H.B. Lawson (see [8]) which in turn implies that the only flat, immersed, minimal torus in $\mathbb{S}^3$ is the Clifford one. Second, the popular conjecture of Lawson [9] and its recent solution [1], which states that *the only torus minimally embedded in* $\mathbb{S}^3$ *is the Clifford torus*. Third, a result of H. Mori [11] which provides a one-parameter family of tori minimally *immersed* (of course, non congruent each other) in $\mathbb{S}^3$.

The main purpose of this note concerns to study of Lawson-type problems in $\mathbb{S}^3$ relative to a class of metrics less symmetric than the standard one. We regard $\mathbb{S}^3$ as a Lie group and consider metrics on $\mathbb{S}^3$ which are $\mathbb{S}^1$-invariant ($\mathbb{S}^1$ is viewed as a subgroup of $\mathbb{S}^3$) and giving $\mathbb{S}^2$ as space of orbits, (see [6] for left $\mathbb{S}^1$-invariant metrics on $\mathbb{S}^3$). Then, we give a large subclass in this class of metrics, providing (as many as we wish but in a discrete way) flat embedded minimal tori, non congruent and $\mathbb{S}^1$-invariant. We also show the existence of at least one metric in this family which admits a foliation (with a pair of antipodal singularities) by embedded, flat minimal tori.

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2. Flat Hopf tori

Let $\mathbb{S}^3 \subset \mathbb{C}^2$ be the 3-sphere endowed with its standard contact structure. If $N = -z$ is an outward normal, then $T\mathbb{S}^3 \equiv \pi^* T\mathbb{S}^2 \oplus iN$, where $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is the usual Hopf map. Let ω be the canonical 1-form connection on this principal $\mathbb{S}^1$-bundle and define on $\mathbb{S}^3$ the class of metrics $\bar{h}_u = \pi^*(h) + (u \circ \pi)^2 \omega^*(dt^2)$, h being a Riemannian metric and u a positive smooth function both on $\mathbb{S}^2$, which makes orthonormal the above splitting. It is easy to see that $\pi : (\mathbb{S}^3, \bar{h}_u) \to (\mathbb{S}^2, h)$ is a Riemannian submersion.

If γ is an immersed closed curve in $\mathbb{S}^2$, then $M_\gamma = \pi^{-1}(\gamma)$ is an immersed torus in $\mathbb{S}^3$, which is embedded if γ is simple in $\mathbb{S}^2$. The universal covering of this torus is $\Psi : \mathbb{R}^2 \to M_\gamma$, given by $\Psi(s,t) = e^{it} \cdot \bar{\gamma}(s)$, where $\bar{\gamma}$ is a lift of γ in $\mathbb{S}^3$.

From now on $\mathbb{S}^2$ by itself will denote the 2-sphere with its standard metric.

A direct computation allows us to find that the Gaussian curvature K of the $\bar{h}_u$-induced metric on M_γ is

$$K = -\frac{u_{ss}}{u} = -\frac{1}{u}\frac{d^2}{ds^2}(u(\gamma(s))). \tag{1}$$

If u is chosen to be a constant u_0 along γ, then M_γ is flat relative to the $\bar{h}_u$-induced metric. Therefore, based on the computation of the curvature and the holonomy of ω, (see [2], [3] and [5]) we obtain the isometry type of M_γ.

Proposition 2.1 *Let γ be an immersed closed curve with length $L > 0$ in $(\mathbb{S}^2, h)$. Let A be the enclosed area by γ in $\mathbb{S}^2$ and let u be a positive smooth function on $\mathbb{S}^2$ which is a constant u_0 along γ. Then the corresponding Hopf torus $M_\gamma = \pi^{-1}(\gamma)$ in $(\mathbb{S}^3, \bar{h}_u)$ is isometric to the flat torus $\mathbb{R}^2/\Gamma$, Γ being the lattice generated by $(0, 2\pi u_0)$ and $(L, 2A)$.*

The shape operator A^u of M_γ relative to the orthonormal basis $\{\Psi_s, \frac{1}{u}\Psi_t\}$ has a matrix of form

$$A^u = \begin{pmatrix} \kappa & \tau \\ \tau & -\xi(\log u) \end{pmatrix},$$

where $\tau = ug(\gamma', \gamma')$ is the torsion of any horizontal lift of γ to $(\mathbb{S}^3, \bar{h}_u)$. Then we have

Proposition 2.2 *Let γ be an immersed curve in $\mathbb{S}^2$ and M_γ its Hopf tube. Then the mean curvature function α_u of M_γ in $(\mathbb{S}^3, \bar{h}_u)$ is given by*

$$\alpha_u = \frac{1}{2}u\tilde{\kappa},$$

where $\tilde{\kappa}$ is the curvature function of γ in $(\mathbb{S}^2, \tilde{h})$ and $\tilde{h} = u^2 h$.

We combine the last proposition with a classical result of L. Lusternik and L. Schnirelmann, [10], to obtain the following.

Corollary 2.3 *For any metric $\bar{h}_u$ on $\mathbb{S}^3$ there exist, at least, three embedded minimal tori in $(\mathbb{S}^3, \bar{h}_u)$.*

3. Main results

If we choose $(\mathbb{S}^2, \tilde{h})$ to be an ellipsoid with three different axes, all having approximately the same length, then it has exactly three closed embedded geodesics. Consequently $(\mathbb{S}^3, \bar{h}_u)$ has three embedded minimal tori. The Lawson conjecture, [9], states that the Clifford torus is the only one minimally embedded in the standard 3-sphere $\mathbb{S}^3$. This conjecture has been proved to be true in [1]. The existence of a one-parameter family of immersed minimal tori in $\mathbb{S}^3$, whose Gaussian curvature takes values in a neighborhood of zero, was showed in [11]. It was proved in [4] that the space of compact embedded minimal surfaces of a fixed genus in a 3-dimensional Riemannian manifold of positive Ricci curvature is compact. This result is false if we relax the assumption of positive Ricci curvature. Indeed, in [7], a sequence of embedded minimal tori in $\mathbb{S}^2 \times \mathbb{S}^1$ (with the standard Riemannian product structure) having no convergent subsequence is given. The same problem can be considered for the metrics $\bar{h}_u$ on $\mathbb{S}^3$.

We would like to point out that the Ricci curvature of the Riemannian metric $\bar{h}_u$ is not signed. In fact, the Ricci curvature r on horizontal vectors is given by (see [3])

$$r(X,X) = 4 - 2|A_X|^2 - (X(\ln u))^2 - XX(\ln u).$$

In spite of Corollary 2.3, we cannot give a negative answer to the above stated Lawson-type conjecture relative to $\bar{h}_u$. In fact, the embedded minimal tori obtained there could be pairwise congruent in $(\mathbb{S}^3, \bar{h}_u)$. However, we can construct large classes of Riemannian metrics on $\mathbb{S}^3$ which admit non congruent embedded minimal tori.

Theorem 3.1 *Let G be a crystallographic subgroup of order m in $SO(3)$. Let β be a closed simple curve in $\mathbb{S}^2$ such that $f(\beta) \cap \beta = \emptyset$, for every $f \in G$. Then there exist infinitely many metrics $\bar{h}_u$ on $\mathbb{S}^3$ such that $\{M_{f(\beta)} \ : \ f \in G\}$ are embedded minimal tori in $(\mathbb{S}^3, \bar{h}_u)$ which are pairwise non congruent.*

Proof. Set $G = \{f_1 = I, f_2, \ldots, f_m\}$. Let $\{u_1, \ldots, u_m\}$ be pairwise distinct real numbers and choose a positive smooth funcion u on $\mathbb{S}^2$ such that $u|_{f_j(\beta)} = u_j$, $j = 1, \ldots, m$. Then $M_{f_j(\beta)}$, equipped with the $\bar{h}_u$-induced metric, is a flat torus. Furthermore, its isometry type is $\mathbb{R}^2/\Gamma_j$, where $\Gamma_j = \operatorname{span}\{(0, 2\pi u_j), (L, 2\pi A)\}$, L being the length of β and A the area enclosed by β in $\mathbb{S}^2$. Since $\{u_j\}$ are pairwise distinct, $\{M_{f_j(\beta)}\}$ are also pairwise non congruent. Next we choose the function u such that $\xi(\log u) = \kappa$ along β, where κ stands for the curvature function of β in $\mathbb{S}^2$ and ξ denotes its unit normal vector field. Said otherwise, we take u in such a way that β is a geodesic of $(\mathbb{S}^2, u^2 h)$, where h is the standard metric on $\mathbb{S}^2$. Therefore $\{M_{f_j(\beta)}\}$ are minimally embedded in $(\mathbb{S}^3, \bar{h}_u)$.

The following result should be compared with the above theorem, as well as with the solution of Lawson's conjecture [1] and the results contained in [4] and [7].

Theorem 3.2 *There exists a metric $\bar{h}_u$ on $\mathbb{S}^3$ such that $(\mathbb{S}^3, \bar{h}_u)$ admits a foliation, with a pair of singularities, whose leaves are flat embedded minimal tori.*

Proof. Let $B(p_0, \delta)$ be a small geodesic ball centered at $p_0 = (0,0,1)$ in $\mathbb{S}^2$ and take a positive smooth function u on $\mathbb{S}^2$ such that u restricted to $\mathcal{W} = \mathbb{S}^2 \setminus \{B(p_0,\delta), B(-p_0,\delta)\}$ is $u(x) = x_1^2 + x_2^2$. For any $p \in \mathcal{W}$, let γ_p be the parallel through p and let κ_p be the curvature of γ_p in $\mathbb{S}^2$. It is clear that u is constant along any parallel γ_p. Furthermore

$$\xi_p(\log u) = \kappa_p.$$

As a consequence, M_{γ_p} is a flat torus which is minimal in $(\mathbb{S}^3, \bar{h}_u)$. Finally, notice that δ can be chosen to be as small as we wish.

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