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A very tasty menu. Starters: Curves. Main Course: Surfaces. Dessert: Manifolds

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Abstract

This is an intuitive and accelerate introduction for beginners to approach to the realm of Differential Geometry.

1 Starters: space curves

Definition 1.1. *A curve in* $\mathbb{R}^3$ *is a map*

$$\alpha : I \to \mathbb{R}^3, \quad \alpha(t) = (\alpha_1(t), a_2(t), \alpha_3(t))\,,$$

where $I \subset \mathbb{R}$ *is open and the functions* $\alpha_i(t)$ *admit continuous derivatives of any order.*

Definition 1.2. *The velocity vector or tangent vector of* $\alpha(t)$ *is the map*

$$\alpha' : I \to \mathbb{R}^3, \quad \alpha'(t) = \left(\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)\right).$$

Remark 1.1. *If* α *takes values in* $\mathbb{R}^2$*, then* α *will be called a plane curve.*

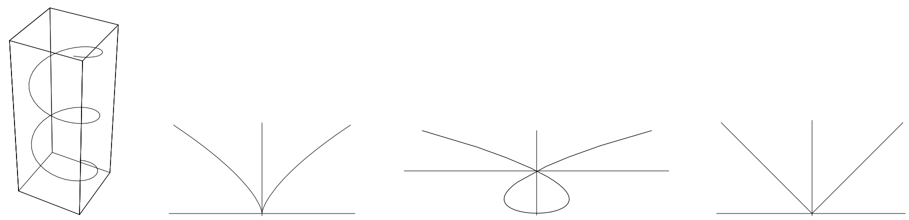
Example 1.1. *(1) The curve* $\alpha(t) = (a\cos t, a\,\mathrm{sen}\, t, b\,t)$*, with* $a, b > 0$*, is called cylindrical helix.*

(2) The curve $\alpha(t) = (t^3, t^2)$ *is differentiable, however there exists the velocity vector* $\alpha'(0) = (0, 0)$ *at* $t = 0$*, though it has no direction. We will not consider such a situation.*

(3) It is clear that $\alpha(t) = (t^3 - 4t, t^2 - 4)$ *is nos an injective map (observe that* $\alpha(2) = \alpha(-2) = (0, 0)$*). Then we will say that* $\alpha(t)$ *is not simple.*

(4) The map $\alpha(t) = (t, |t|)$ *is not differentiable, so it will be dropped in our study.*

I wish to thank the organizers for the opportunity to disseminate the nice world of Differential Geometry and for their hospitality during the meeting.



Why the curve should be differentiable? Why we wish to avoid the vanishing of the tangent vector at a point? The answer will implicitly be given in the following definition.

Definition 1.3. *A curve $\alpha : I \to \mathbb{R}^3$ will be called regular when $\alpha'(t) \neq 0$, for all $t \in I$.*

When $\alpha'(t) \neq 0$, we can consider the tangente line of α at the point $\alpha(t)$, i. e., the straightline through $\alpha(t)$ and direction given by $\alpha'(t)$ (so the curve must be regular).

Otherwise, if $t_0 \in I$ is such that $\alpha'(t_0) = 0$, we will say that α has a singular point at t_0.

We only consider simple (without self-intersections) and regular curves.

1.1 The arclength function

Definition 1.4. *Let $\alpha : I \to \mathbb{R}^3$ be a curve. A parameter change is any differentiable map*

$$h : J \to I,$$

$J \subset \mathbb{R}$ open, whose inverse function h^{-1} is also differentiable.

The curve $\beta = \alpha \circ h : J \to I \to \mathbb{R}^3$ will be called a reparametrization of α.

Example 1.2. *Take the circle $\alpha(t) = (\cos t, \operatorname{sen} t)$. The map $h(s) = 2s$ preserves the orientation, while $h(s) = -s$ reverses it.*

Definition 1.5. *The arclength between the points $\alpha(a)$ and $\alpha(b)$ is given by*

$$L_a^b(\alpha) = \int_a^b |\alpha'(t)|\, dt.$$

A curve $\alpha : I \to \mathbb{R}^3$ is said to be arclength parametrized (a.l.p.) provided $|\alpha'(t)| = 1$, for all $t \in I$.

Remark 1.2. *In that case we have*

$$L_0^t(\alpha) = \int_0^t |\alpha'(u)|\, du = \int_0^t du = t.$$

Proposition 1.1. *Any regular curve can always be arclength parametrized.*

Example 1.3. *The catenary is the curve defined by $\alpha(t) = (t, \cosh t)$. Its arclength parameter is*

$$s = \int_0^t |\alpha'(u)|\, du = \int_0^t \cosh u\, du = \operatorname{senh} u.$$

1.2 Curvature and torsion functions. The Frenet trihedron

Let $\alpha : I \to \mathbb{R}^3$ an a.l.p. regular curve and let s be the arclength parameter. It then has unit tangent vector $\mathbf{T}(s) = \alpha'(s)$, i. e., $\langle \mathbf{T}(s), \mathbf{T}(s)\rangle = 1$. Then $\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle = 0$ and $\mathbf{T}'(s) \perp \alpha'(s)$.

Definition 1.6. *The function*

$$k(s) = |\mathbf{T}'(s)| = |\alpha''(s)|$$

will be called the curvature of α at the point $\alpha(s)$.

Remark 1.3. *We have that $k(s) \geq 0$. Furthermore, α is a straightline if, and only if, $k(s) \equiv 0$.*

Definition 1.7. *Let α be an a.l.p. regular curve. For any $s \in I$, such that $k(s) \neq 0$, the vector*

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{k(s)} = \frac{\alpha''(s)}{|\alpha''(s)|}$$

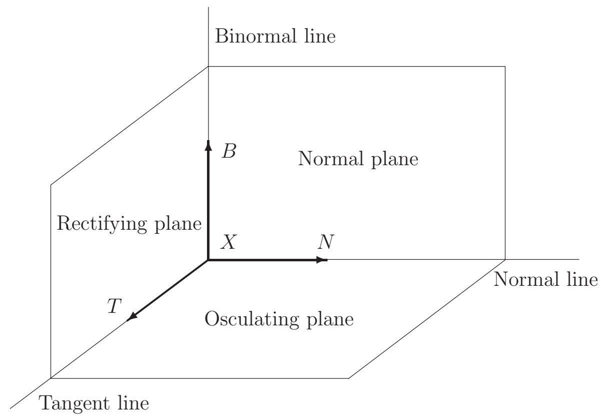
will be called the normal vector of α at the point $\alpha(s)$.

Definition 1.8. *Let α be an a.l.p. regular curve such that $k(s) \neq 0$ anywhere. The plane $\{\mathbf{T}(s), \mathbf{N}(s)\}$ will be called the osculating plane of α at the point $\alpha(s)$.*

Henceforth we will assume that $k(s) \neq 0$ anywhere.

Definition 1.9. $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ *will be called the binormal vector.*

It is easy to see that $|\mathbf{B}(s)| = 1$ and $\mathbf{B}(s)$ is orthogonal to the osculating plane.



Definition 1.10. *For any $s \in I$, where $k(s) > 0$, the set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is an orthonormal basis of $\mathbb{R}^3$, usually called the Frenet trihedron or Frenet frame along $\alpha(s)$.*

Remark 1.4. *How vectors $\mathbf{T}(\mathbf{s})$, $\mathbf{N}(\mathbf{s})$ and $\mathbf{B}(\mathbf{s})$ evolve along $\alpha(s)$?*

To see that, write

$\mathbf{T}'(s) = k(s)\mathbf{N}(s)$,
$\mathbf{B}'(s) = \mathbf{T}(s) \times \mathbf{N}'(s) \Rightarrow \mathbf{B}'(s) \perp \mathbf{T}(s)$,
$|\mathbf{B}(s)| = 1$, so $\mathbf{B}'(s) \perp \mathbf{B}(s)$.

Then $\mathbf{B}'(s)$ and $\mathbf{N}(s)$ should be collinear, so that

$$\mathbf{B}'(s) = \langle \mathbf{B}'(s), \mathbf{N}(s) \rangle \mathbf{N}(s) = \langle \mathbf{T}(s) \times \mathbf{N}'(s), \mathbf{N}(s) \rangle \mathbf{N}(s).$$

It seems natural the following

Definition 1.11. *The torsion of α is the function $\tau : I \longrightarrow \mathbb{R}$ defined by*

$$\tau(s) := \langle \mathbf{T}(s) \times \mathbf{N}'(s), \mathbf{N}(s) \rangle = \langle \mathbf{B}'(s), \mathbf{N}(s) \rangle.$$

Frenet formulas.

$\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$.
$\mathbf{N}'(s) = \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle \mathbf{T}(s) + \langle \mathbf{N}'(s), \mathbf{N}(s) \rangle \mathbf{N}(s) + \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle \mathbf{B}(s)$.

- $\langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$, because $\langle \mathbf{N}(s), \mathbf{N}(s) \rangle = 1$);

- $\langle \mathbf{N}'(s), \mathbf{T}(s)\rangle = -\langle \mathbf{N}(s), \mathbf{T}'(s)\rangle = -k(s)$, because $\langle \mathbf{N}(s), \mathbf{T}(s)\rangle = 0$; and
- $\langle \mathbf{N}'(s), \mathbf{B}(s)\rangle = -\langle \mathbf{N}(s), \mathbf{B}'(s) = -\tau(s)$, because $\langle \mathbf{N}(s), \mathbf{B}(s)\rangle = 0$.

We finally get $\mathbf{N}'(s) = -k(s)\mathbf{T}(s) - \tau(s)\mathbf{B}(s)$. Summarizing, we have obtained the so called Frenet formulas:

$$\left\{ \begin{array}{rcccc} \mathbf{T}'(s) & = & & k(s)\mathbf{N}(s), & \\ \mathbf{N}'(s) & = & -k(s)\mathbf{T}(s) & & -\tau(s)\mathbf{B}(s), \\ \mathbf{B}'(s) & = & & \tau(s)\mathbf{N}(s). & \end{array} \right.$$

Proposition 1.2. *A curve $\alpha(s)$ with curvature function $k(s) \neq 0$ is a plane curve if, and only if, its torsion function vanishes everywhere.*

Remark 1.5. *If $k(s) \equiv 0$, then α is a straightline, and then a plane curve. When $k(s) = 0$ the torsion can not be defined, so this case is not included in the above proposition.*

Example 1.4. *Let α be the cylindrical helix $\alpha(s) = (a\cos s, asins, bs)$, $a, b > 0$, which we assume a.l.p. (so $|\alpha'(s)|^2 = |(-asins, a\cos s, b)|^2 = a^2+b^2 = 1$).*

The Frenet trihedron of the helix:

$$\begin{array}{rcl} \mathbf{T}(s) & = & (-asins, a\cos s, b), \\ \mathbf{N}(s) & = & (-\cos s, -sins, 0), \\ \mathbf{B}(s) & = & (bsins, -b\cos s, a). \end{array}$$

The curvature of the helix: $k(s) = |\alpha''(s)| = |(-a\cos s, -asins, 0)| = a$.
The torsion of the helix: $\tau(s) = \langle \mathbf{N}(s), \mathbf{B}'(s)\rangle = -b$.

Summarizing.

$$\begin{array}{rcl} k(s) & : & \text{measures to what extent } \alpha \text{ is far from a straightline} \\ \tau(s) & : & \text{measures to what extent } \alpha \text{ is far from a plane curve} \end{array}$$

A special case: a plane curve.

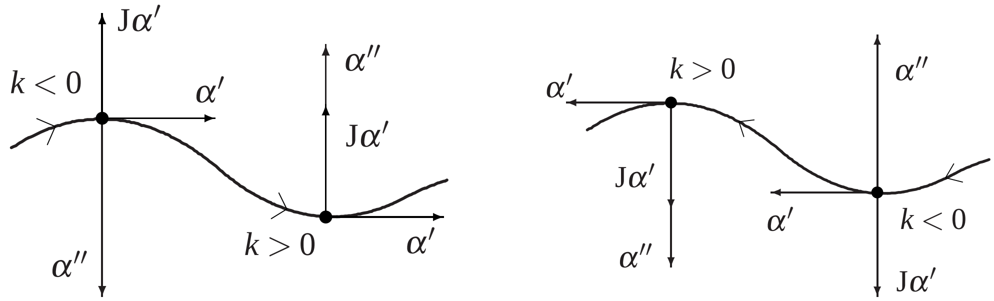
The curvature function is now defined by

$$k(s) = \big\langle \alpha''(s), \mathbf{N}(s) \big\rangle$$

and the Frenet formulas are

$$\left\{ \begin{array}{rcr} \mathbf{T}'(s) & = & k(s)\,\mathbf{N}(s), \\ \mathbf{N}'(s) & = & -k(s)\,\mathbf{T}(s). \end{array} \right.$$

Remark 1.6. *If $\alpha(s)$ is a plane curve, its curvature $k(s)$ is signed.*



2 The main course: surfaces in $\mathbb{R}^3$

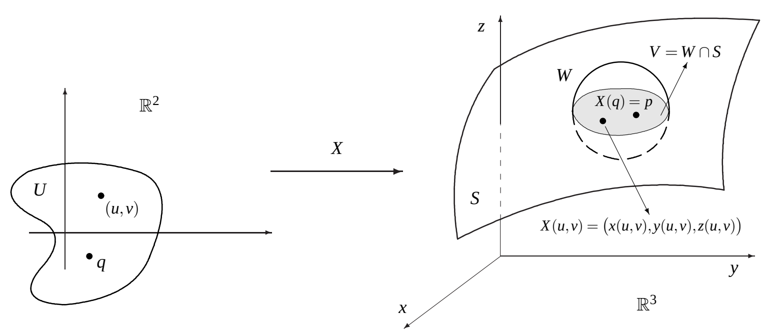
Definition 2.1. *A non-empty set $S \subset \mathbb{R}^3$ is a regular surface if, for any point $p \in S$, there exist an open $U \subset \mathbb{R}^2$, a neighborhood V of p in S (with the relative topology of $S \subset \mathbb{R}^3$) and a map $X : U \longrightarrow \mathbb{R}^3$, such that*

(S1) *$X(U) = V$ and $X : U \longrightarrow \mathbb{R}^3$ is differentiable;*

(S2) *$X : U \longrightarrow V$ is a homeomorphism (i. e., the inverse map $X^{-1} : V \longrightarrow U$ is continuous); and*

(S3) *for any $q \in U$, the differential $dX_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective.*

The map X will be called a parametrization, chart or coordinate system. The neighborhood V is called coordinate neighborhood.



As for the meaning of the above conditions we have:

(i) The differentiability of $X : U \longrightarrow V$ means that if we write $X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$, the functions $x, y, z : U \longrightarrow \mathbb{R}$ are differentiable having continuous derivatives of any order.

(ii) The condition **(S1)** allows us to ensure that the surface S is smooth in the sense that it has neither edges nor vertices.

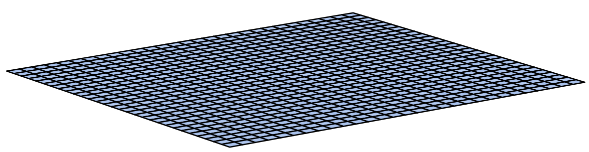
(iii) The condition **(S2)** avoids that S has self-intersections. This is quite important in order to achieve uniqueness when we want to define the tangent plane to the surface at a point.

(iv) The condition **(S3)** will become crucial to ensure the existence of tangent planes at any point of S. This is a similar hypothesis to that of $\alpha'(t) \neq 0$ for regular curves.

An informal way to get a surface. A surface is obtained when we take pieces of a sheet, distort and fold them, then fit them together in such a way that the resulting figure has neither vertices, nor edges and nor self-intersections.

Example 2.1. Regular surfaces.

Plane. *Take the set* $\Pi = \big\{(x,y,z) \in \mathbb{R}^3 : ax + by + cz = d\big\}$,



where a, b, c *do not vanish simultaneously. Assume, without loss of generality, that* $c \neq 0$. *Then we can write* $z = (d - ax - by)/c$. *Take now* $U = \mathbb{R}^2$, $V = \Pi$ *(the coordinate neighborhood will be the whole surface), and let* $X : U \longrightarrow V$ *be the map given by*

$$X(u,v) = \left(u, v, \frac{d - au - bv}{c}\right).$$

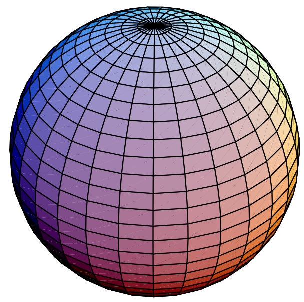
Then we see that:

(i) X *is differentiable, because is linear.*

(ii) $X^{-1} : \Pi \longrightarrow U$ *is the orthogonal projection over the plane* $z = 0$, *which is obviously continuous. Therefore,* X *is a homeomorphism.*

(iii) Vectors $X_u = (1, 0, -a/c)$ *and* $X_v = (0, 1, -b/c)$ *are linearly independent, so that the map* dX_q *is injective for all* q.

Sphere. *Take the set* $\mathbb{S}^2 = \big\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\}$.



Define $X_1 : U \longrightarrow \mathbb{R}^3$ *by*

$$X_1(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right),$$

where $U = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$.

It is clear that X_1 *is differentiable in* U*, then we get* **(S1)**. *Furthermore,* X_1^{-1} *is the orthogonal projection over the plane* $z = 0$*, which is also continuous. Hence,* X_1 *is a homeomorphism on the image* $X_1(U)$*, so it holds* **(S2)**. *Finally, we have that*

$$(X_1)_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \quad (X_1)_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$$

are linearly independent, and so $(dX_1)_q$ *is injective and* **(S3)** *holds. As a consequence,* X_1 *is a parametrization. To show that* $\mathbb{S}^2$ *is a regular surface we have to find a parametrization at any point of* $\mathbb{S}^2$. *We see that* X_1 *covers all points such that* $z > 0$*, but... how many parametrizations we need? The answer is easily obtained, because you have to cover now (i) all points such that* $z < 0$*; (ii) all points such that* $x > 0$*; (iii) all points such that* $x < 0$*; (iv) all points such that* $y > 0$*; and (v) all points such that* $y < 0$. *Summing up, six parametrizations are needed.*

It seems then not easy to find all parametrizations of any surface, even if the surface is as well known as the sphere. Actually, any surface can be parametrized in many different ways, and the experience will show us that some parametrizations will be more suitable than others according to the aim we wish to achieve.

2.1 Practical criteria for determining surfaces

Criterion 1: Graphs. Let $f : U \longrightarrow \mathbb{R}$ be a differentiable function, $U \subset \mathbb{R}^2$ being open. Then the set

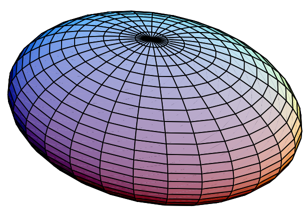
$$G(f) = \left\{\left(u, v, f(u,v)\right) : (u,v) \in U\right\}$$

is a regular surface in $\mathbb{R}^3$. Actually, any graph of a differentiable function is a regular surface.

Criterion 2: Regular values. Let $f : V \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a differentiable function and let a be a regular value of f (i. e., df_p is surjective for all $p \in f^{-1}(a)$). Then $S = f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$, which will be called *level surface*.

Ellipsoid. Take the set

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}.$$

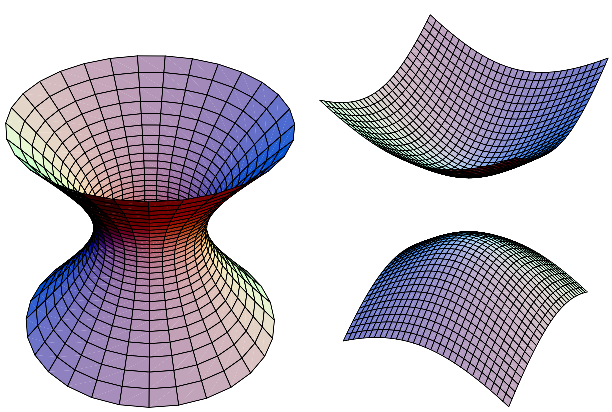


To show that E is a regular surface we define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ as

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

and look for its critical points. It is easy to see that the ellipsoid $E = f^{-1}(1)$ is a regular surface.

Hyperboloids.



The above criterion can be used to show that the following sets

$$H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\} \qquad \text{(One-sheet hyperboloid)}$$
$$H' = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1\} \qquad \text{(Two-sheets hyperboloid)}$$

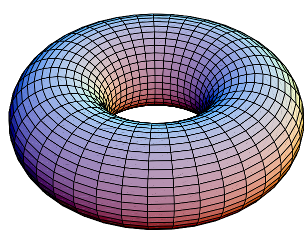
are regular surfaces by using the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R} \quad \text{dada por} \quad f(x,y,z) = x^2 + y^2 - z^2.$$

The only critical value of f is 0, so $H = f^{-1}(1)$ and $H' = f^{-1}(-1)$ are regular surfaces.

The regular value criterion applies not only to show that quadrics are regular surfaces, but also for another kind of surfaces as we exhibit in the following example.

Torus of revolution. Take the circle $\mathbb{S}^1(r)$ in the plane $x = 0$ whose centre is the point $(0, a, 0)$, with $a > r > 0$. A generic point of that circle is given by $(0, y, z)$, so that $r^2 = z^2 + d^2$. By rotating the circle $\mathbb{S}^1(r)$ around the z-axis, we obtain a surface of revolution.



Then we see that

$$\mathbb{T}^2 = \left\{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2\right\}$$

and $\mathbb{T}^2 = f^{-1}(r^2)$, where $f : V \longrightarrow \mathbb{R}$ is given by

$$f(x,y,z) = \left(\sqrt{x^2+y^2} - a\right)^2 + z^2,$$

and $V = \mathbb{R}^3 \backslash \{(0,0,z) : z \in \mathbb{R}\}$.

2.2 The tangent plane

Definition 2.2. *Let $S \subset \mathbb{R}^3$ be a regular surface and take a point $p \in S$. We will say that $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector to S at p if there exist a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.*

Definition 2.3. *The set T_pS of all tangent vectors to S at p, i. e.,*

$$T_pS = \big\{\mathbf{v} \in \mathbb{R}^3 : \exists\alpha : (-\varepsilon, \varepsilon) \longrightarrow S \text{ differentiable} : \alpha(0) = p,\, \alpha'(0) = \mathbf{v}\big\},$$

said otherwise,

$$T_pS = dX_q(\mathbb{R}^2),$$

is a vectorial plane in $\mathbb{R}^3$ which will be called the tangent plane to S at p.

Basis of T_pS. In order to find a basis of T_pS, let $\big\{(1,0),(0,1)\big\}$ be a basis of $\mathbb{R}^2$. As dX_q is injective, we have that $\big\{dX_q(1,0), dX_q(0,1)\big\} = \big\{X_u(q), X_v(q)\big\}$ is a basis of T_pS.

We can consider now the orthogonal complement $(T_pS)^\perp$ of T_pS in $\mathbb{R}^3$, as a Euclidean vector space, to get a vector line that we will call the *normal straightline* to S at p. We then write $\mathbb{R}^3 = T_pS \oplus (T_pS)^\perp$.

Definition 2.4. *For any $p \in S$, we can find a unit vector $N(p)$ generating the normal straightline to S at p, so that we will write $(T_pS)^\perp = \big\{N(p)\big\}$. The vector $N(p)$ is uniquely determined (up to the sign) and will be called the normal vector to S at p.*

As $\big\{X_u(q), X_v(q)\big\}$ is a basis of $T_{X(q)}S$, then we get an explicit expression of the normal vector

$$N\big(X(q)\big) = \pm\frac{X_u(q) \times X_v(q)}{\big|X_u(q) \times X_v(q)\big|}.$$

2.3 Intrinsic geometry: the first fundamental form

Definition 2.5. *The map*

$$\mathtt{I}_p : T_pS \longrightarrow \mathbb{R}, \quad \mathtt{I}_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle_p$$

will be called the first fundamental form (1st f.f.) of S.

In what follows, and unless we want to pinpoint the point on which we are working, we simply write $\langle\,\cdot\,,\cdot\,\rangle$ instead of $\langle\,\cdot\,,\cdot\,\rangle_p$.

Local expression of $\mathtt{I}_p$. Set $\mathbf{v} \in T_pS$ and $\alpha : I \to S$ with initial conditions p and $\mathbf{v}$ (that is, $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$). Let (U, X) be a parametrization of S and let $\widetilde{\alpha}(t) = \big(u(t), v(t)\big)$ the coordinate expression of α. Then $\mathbf{v} = \alpha'(0) = u'(0)X_u(q) + v'(0)X_v(q) = aX_u(q) + bX_v(q)$, where a, b are real numbers and $X(q) = p$. Now we compute $\mathtt{I}_p(\mathbf{v})$ to find

$$\mathtt{I}_p(\mathbf{v}) = |aX_u + bX_v|^2 = a^2 \langle X_u, X_u \rangle + 2ab \langle X_u, X_v \rangle + b^2 \langle X_v, X_v \rangle\,.$$

Definition 2.6. *Set* $E = \langle X_u, X_u \rangle$, $F = \langle X_u, X_v \rangle$ *and* $G = \langle X_v, X_v \rangle$. *These functions (with values in* U*) are clearly differentiable and will be called the coefficients of the 1st f.f.*

Example 2.2. Coefficients of the 1st f.f.

Plane. $\Pi = \big\{ \mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \big\}$, *through the point* $p = (p_1, p_2, p_3)$ *can be parametrized by*

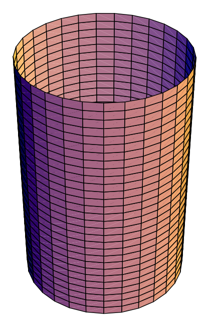
$$X(u, v) = (p_1 + uv_1 + vw_1, p_2 + uv_2 + vw_2, p_3 + uv_3 + vw_3).$$

As $X_u = (v_1, v_2, v_3) = \mathbf{v}$ *and* $X_v = (w_1, w_2, w_3) = \mathbf{w}$, *we get* $E = \langle \mathbf{v}, \mathbf{v} \rangle$, $F = \langle \mathbf{v}, \mathbf{w} \rangle$ *and* $G = \langle \mathbf{w}, \mathbf{w} \rangle$. *Observe that when the basis is orthonormal then* $E = G = 1$ *and* $F = 0$.

Cylinder. *Let* $C = \big\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2 \big\}$ *be the cylinder parametrized by*

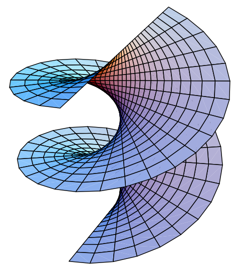
$$X(u, v) = (r \cos u, r sinu, v).$$

As $X_u = (-r sinu, r \cos u, 0)$ *and* $X_v = (0, 0, 1)$, *then* $E = r^2$, $F = 0$ *and* $G = 1$ *are the coefficients of the 1st f.f. with respect to* X.



Helicoid. *Take a parametrized helix* $\alpha(u) = (\cos u, sinu, au)$, *with* $a > 0$. *Observe that this curve is screwing around the* z*-axis, so for each point* $\alpha(u)$ *we take the straightline joining* $\alpha(u)$ *and the point* $(0, 0, au)$ *on the* z*-axis. All those straightlines generate a regular surface which will be called helicoid. A parametrization is given by*

$$X(u, v) = (v \cos u, v sinu, au), \qquad con \;\; u, v \in \mathbb{R}.$$



As $X_u = (-v sinu, v\cos u, a)$ *and* $X_v = (\cos u, sinu, 0)$*, we find that* $E = a^2 + v^2$*,* $F = 0$ *and* $G = 1$ *are the coefficients of the 1st f.f. with respect to* X*.*

Elementary properties of the coefficients of the 1st f.f.

(i) $E, G > 0$,

(ii) $EG - F^2 > 0$.

2.4 Aplications of the 1st f.f.

2.4.1 Measure of lengths

Let $\alpha : I \longrightarrow S$ be a parametrized curve. Its arclength is given by

$$s(t) = \int_0^t \left|\alpha'(r)\right| dr = \int_0^t \sqrt{\langle \alpha'(r), \alpha'(r)\rangle}\, dr = \int_0^t \sqrt{\mathtt{I}_{\alpha(r)}\big(\alpha'(r)\big)}\, dr.$$

In particular, when $\alpha(t) = X\big(u(t), v(t)\big) = X\big(\widetilde{\alpha}(t)\big)$, (U, X) being a parametrization of S, the arclength can be expressed as

$$s(t) = \int_0^t \sqrt{E\big(\widetilde{\alpha}(r)\big)u'(r)^2 + 2F\big(\widetilde{\alpha}(r)\big)u'(r)v'(r) + G\big(\widetilde{\alpha}(r)\big)v'(r)^2}\, dr.$$

Therefore,

$$s'(t) = \sqrt{E\big(\widetilde{\alpha}(t)\big)u'(t)^2 + 2F\big(\widetilde{\alpha}(t)\big)u'(t)v'(t) + G\big(\widetilde{\alpha}(t)\big)v'(t)^2},$$

that is,

$$\left(\frac{ds}{dt}(t)\right)^2 = E\big(\widetilde{\alpha}(t)\big)\left(\frac{du}{dt}(t)\right)^2 + 2F\big(\widetilde{\alpha}(t)\big)\frac{du}{dt}(t)\frac{dv}{dt}(t) + G\big(\widetilde{\alpha}(t)\big)\left(\frac{dv}{dt}(t)\right)^2,$$

which is usually written as $(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$, and it is said that ds is the *arc element* or *line element* of S.

2.4.2 Measure of angles

Let $\alpha : I \longrightarrow S$ and $\beta : I \longrightarrow S$ be two regular parametrized curves crossing each other at a point $\alpha(t_0) = \beta(t_0)$, $t_0 \in I$. The angle θ between them, i. e., the angle of their tangent vectors at that point, is given by

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0)\rangle}{\left|\alpha'(t_0)\right|\left|\beta'(t_0)\right|}.$$

In particular, given a parametrization (U, X), by taking its coordinate curves (when v_0 and u_0 are fixed, that is, $\alpha(u) = X(u, v_0)$, $\beta(v) = X(u_0, v)$), then the angle they form at $X(u_0, v_0)$ is

$$\theta = \arccos\frac{\langle X_u, X_v\rangle}{|X_u||X_v|}(u_0, v_0) = \arccos\frac{F}{\sqrt{EG}}(u_0, v_0).$$

Observe that $\theta \geq 0$ and the coordinate curves of X are orthogonal if, and only if, $F \equiv 0$. If this is the case, we will say that X is an *orthogonal parametrization.*

2.4.3 Measure of areas

Definition 2.7. *Let $R \subset S$ be a region of a regular surface S, such that there exists a parametrization (U, X) with $R \subset X(U)$. The area of R is defined as*

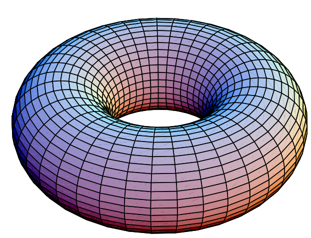
$$A(R) = \int_{X^{-1}(R)} |X_u \times X_v|\, dudv.$$

The following technical result shows that this is a good definition.

Lemma 2.1. *The number $A(R)$ does not depend on the chosen parametrization. Furthermore,*

$$A(R) = \iint_{X^{-1}(R)} \sqrt{EG - F^2}\, dudv.$$

Example 2.3. *As we saw before, the torus of revolution $\mathbb{T}^2$ is a regular surface*



that can be parametrized by

$$X(u,v) = \big((r\cos u + a)\cos v, (r\cos u + a)sinv, r\,sinu\big),$$

where $(u,v) \in U = (0,2\pi)\times(0,2\pi)$. *The coefficients of the 1st f.f. regarding that parametrization are* $E = r^2$, $F = 0$, $G = (a + r\cos u)^2$. *Then* $\sqrt{EG - F^2} = r(a{+}r\cos u)$ *and, for any region* R *contained in the coordinate neighborhood* $X(U)$, *we will have*

$$A(R) = \iint_{X^{-1}(R)} \sqrt{EG - F^2}\,dudv = \iint_{X^{-1}(R)} r(a + r\cos u)\,dudv = 4ar\pi^2.$$

2.5 Orientability of surfaces

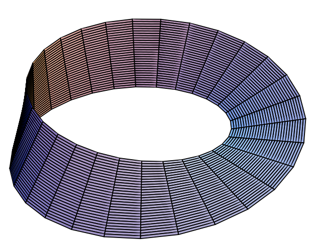
Let us start with two important examples.

Möbius strip. The Möbius strip was independently discovered by the German mathematicians August Ferdinand Möbius and Johann Benedict Listing in 1858. It is a regular surface which can be obtained as follows: take a rectangle made of paper sheet with length, say 4π and width, say 2. The Möbius strip is obtained when gluing the shortest sides but reversing the ends. A parametrization of this surface is given by

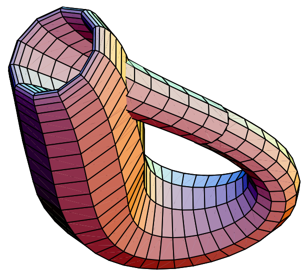
$$X(u,v) = \Big(\big(2 - v\sin(u/2)\big)\mathrm{sin}u, \big(2 - v\sin(u/2)\big)\cos u, v\cos(u/2)\Big),$$

where $0 < u < 2\pi$ and $-1 < v < 1$. As we need another parametrization to cover all points, it is enough to let the parameter u move in the interval $(-\pi,\pi)$.

Later, by using the theory we are going to exhibit, we will see that the Möbius strip is not orientable.



Klein bottle. The Klein bottle was first described by the German mathematician Felix Klein, in 1882, and was originally dubbed Klein surface. A mistranslation from the German language (Flasche = bottle by Fläche = surface) led to the name as today is known.

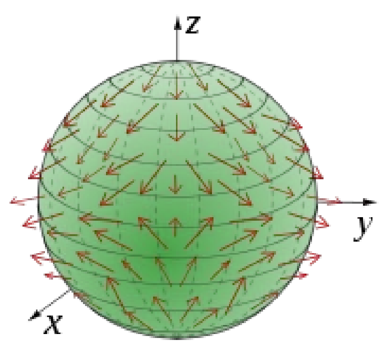


Theorem 2.1. Theorem of Brouwer-Samelson. *Any compact regular surface in $\mathbb{R}^3$ is orientable.*

As the Klein bottle is compact and non orientable, it can not be a regular surface in the sense of our stated definition.

To precise the concept of orientability we need some definitions.

Definition 2.8. *Let S be a regular surface. A vector field ξ on S is a vectorial function $\xi : S \longrightarrow \mathbb{R}^3$, where $\xi(p)$ is a vector of $\mathbb{R}^3$ for any $p \in S$. We will say that ξ is differentiable when it is differentiable as a function from S to $\mathbb{R}^3$.*



Definition 2.9. *A vector field ξ on S will be called a tangent vector field if $\xi(p)$ is tangent to S at p, for any $p \in S$. We will say that ξ is normal to S when $\xi(p)$ is normal to S at p, for any $p \in S$. Finally, remember that ξ is called unit when $\big|\xi(p)\big| = 1$, for any $p \in S$.*

As for differentiable vector fields, $\mathfrak{X}(S)$ will denote those tangent to S and $\mathfrak{X}(S)^{\perp}$ those normal to S.

Definition 2.10. *A regular surface S will be called orientable if there exists a globally defined differentiable unit vector field N normal to S.*

If S is orientable, each vector field N as above will be called an orientation of S. An orientable surface will be say oriented provided an orientation has been chosen.

This definition requires some comments.

Remark 2.1. *(1) Say that S has a normal vector field $N(p)$ at any point $p \in S$ is the same as saying that the tangent plane T_pS is oriented and a direction of rotation is well defined through the vector cross product in $\mathbb{R}^3$. In other words, the existence of a globally defined normal vector field on the surface determines a direction of rotation at any point of S.*

(2) If S is orientable, we can distinguish two sides of the surface: the side to which N is pointing and the opposite one. Then, orientable surfaces have been also called two-sided surfaces.

Otherwise, non orientable surfaces have an only side. This is the case of the Möbius strip: someone walking on the surface would come back to the starting point but in the bottom side.

(3) We can not build a globally defined normal vector field on the Möbius strip. However, we can do that on each coordinate neighborhood, but in the intersection of two of them (which has two connected components) both normal vector fields do not agree whatever the choice of each.

2.6 Curvatures of a surface

Definition 2.11. *A normal section $C_{\mathbf{v}}$ is the plane curve obtained when intersecting the surface S with the plane $\Pi_{\mathbf{v}} = \{\mathbf{v}, N(p)\}$.*

Then $C_{\mathbf{v}} = \Pi_{\mathbf{v}} \cap S$, and we can find an a.l.p. parametrization of $C_{\mathbf{v}}$ given by $\alpha : I \longrightarrow C_{\mathbf{v}} \subset S$, so that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. As $\mathbf{v} \in T_pS$, then $\mathbf{v} \perp N(p)$ and the curve α has normal vector $\mathbf{n} = \mathbf{J}_{\Pi_{\mathbf{v}}}\mathbf{v} = N(p)$, where $\mathbf{J}_{\Pi_{\mathbf{v}}}$ is the $\frac{\pi}{2}$-rotation in the plane $\Pi_{\mathbf{v}}$. Therefore

$$\big\langle \alpha''(0), N(p) \big\rangle = \langle k(0)\mathbf{n}(0), N(p)\rangle = k(0),$$

k being the curvature of α as a plane curve lying in the vectorial plane $\Pi_{\mathbf{v}}$, which is oriented by the positively oriented orthonormal basis $\big\{\mathbf{v}, N(p)\big\}$.

Example 2.4. Normal sections.

Plane. *It is easy to see that they are straightlines.*

Sphere. *Let $\mathbb{S}^2(r)$ be the sphere of radius r, which we suppose oriented by the normal $N(p) = (1/r)p$ pointing outside. If $\mathbf{v} \in T_p\mathbb{S}^2(r)$, then the* *normal sections are great circles in the sphere, i. e., circles of radius* r *whose curvature, as a plane curve, is* $-1/r$.

Observe that the minus sign comes from the chosen orientation: the acceleration vector of the normal sections is always pointing inside, while the orientation of the surface we have chosen is that given by the normal pointing outside.

Cylinder. *Let* C *be the cylinder given by the equation* $x^2 + z^2 = r^2$ *(cylinder whose generatrices are parallel to the* y*-axis) and take the point* $p = (r, 0, 0)$. *The tangent plane* T_pC *is generated by the vectors* $\mathbf{v}_1 = (0, 1, 0)$ *and* $\mathbf{v}_2 = (0, 0, 1)$, *while the normal is* $N(p) = (1, 0, 0)$, *where we are choosing the orientation pointing outside.*

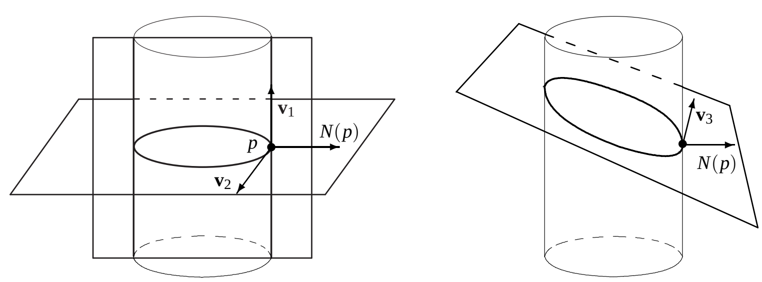
The plane $\Pi_{\mathbf{v}_1}$ *is nothing but* $z = 0$, *so that*

$$\Pi_{\mathbf{v}_1} \cap C = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r^2,\ z = 0\right\} = \left\{(x, y, z) \in \mathbb{R}^3 : x = \pm r,\ z = 0\right\},$$

which are two parallel straightlines. The only one through p *is* $\{x = r, z = 0\}$, *whose curvature is 0.*

The plane $\Pi_{\mathbf{v}_2}$ *is now* $y = 0$, *so that*

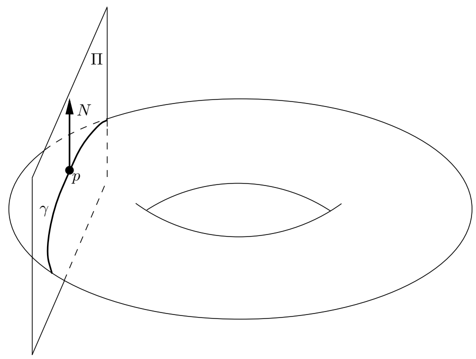
$$\Pi_{\mathbf{v}_2} \cap C = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r^2,\ y = 0\right\},$$



which is a circle of radius r *in the plane* $y = 0$, *whose curvature is* $-1/r$ *(the minus sign comes from the normal to the cylinder pointing in the opposite direction to that of the acceleration of the circle).*

Finally, by taking $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$, *with* $a, b \neq 0$, *an easy computation shows that* $\Pi_{\mathbf{v}_3} \cap C$ *is an ellipse, whose curvature falls within* $(-1/r, 0)$.

The curvatures of a surface can be described by two numbers. To see that, let S be a regular surface oriented by N and let Π be a plane through p containing N. Then $\Pi \cap S$ is a plane curve $\gamma \subset \Pi$ trough p. We compute the curvature $k(p)$ of γ with respect to N and do the same for any plane containing N.



Definition 2.12. *The principal curvatures* $k_1(p)$ *and* $k_2(p)$ *of* S *at* p *are the minimum and maximum of the curvatures of the plane curves obtained as above.*

Definition 2.13. *The Gaussian curvature of* S *at* $p \in S$ *is*

$$K(p) = k_1(p)k_2(p).$$

Definition 2.14. *The mean curvature of* S *at* $p \in S$ *is*

$$H(p) = \frac{k_1(p) + k_2(p)}{2}.$$

Remark 2.2. *It is quite important to note that the principal curvatures are not intrinsic quantities.*

Proposition 2.1. (1) *The Gaussian curvature does not depend on the chosen orientation of the surface.*

(2) *The sign of the mean curvature depends on the chosen orientation of the surface.*

Classifying the points of a surface. Depending on the sign of the Gaussian curvature, the points of a surface can be classified as follows.

Let S be an oriented regular surface and let $p \in S$. Then we will say that

(i) $p \in S$ is elliptic provided $K(p) > 0$;

(ii) $p \in S$ is hyperbolic provided $K(p) < 0$;

(iii) $p \in S$ is parabolic provided $K(p) = 0$, but at least one of the principal curvatures at p does not vanish;

(iv) $p \in S$ is flat provided $k_1(p) = k_2(p) = 0$.

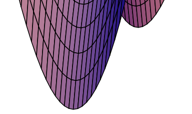
Example 2.5. *We will see how we can distinguish points of some surfaces:*

(i) *Any point of a sphere is elliptic, indeed* $K \equiv 1/r^2 > 0$.

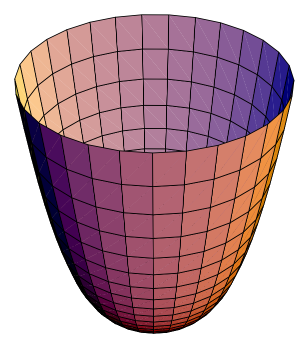
(ii) *In the monkey saddle, parametrized by* $X(u,v) = (u, v, v^2 - u^2)$, *the point* $p = (0,0,0)$ *is hyperbolic, because* $k_1(p) = -2$ *and* $k_2(p) = 2$, *so that* $K(p) = -4 < 0$.

(iii) *Any point of a cylinder is parabolic, because* $k_1 \equiv -1/r$ *and* $k_2 \equiv 0$, *and then* $K \equiv 0$ *for any* p.

(iv) *Any point of a plane is a flat point. They are not the only examples of flat points. In fact, by taking the revolution surface generated by* $z = y^4$ *(revolving around the* z*-axis), then the origin is a flat point of a surface which is not a plane.*

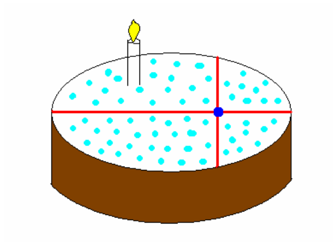


Monkey saddle

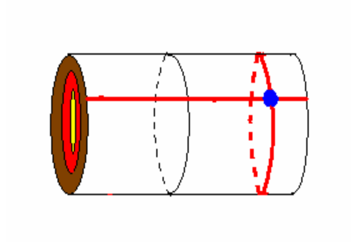


$z = y^4$

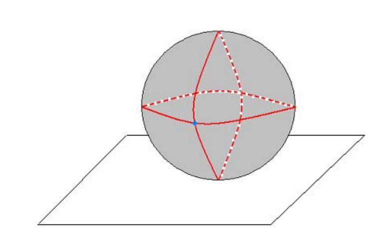
To spread the beauty of Geometry, in recent public exhibition of Science, we have given -to current people- some examples showing some different points on well known surfaces.



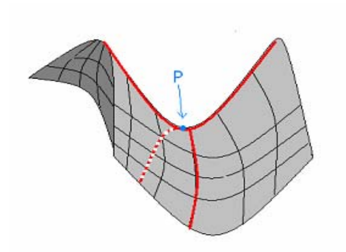
Flat points



Parabolic points



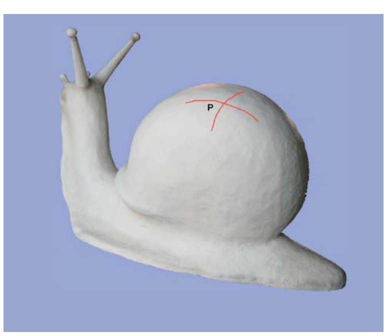
Elliptic points



P hyperbolic point

P elliptic point

Q hyperbolic point



P elliptic point

Q hyperbolic point

2.7 Isometries

Definition 2.15. *A local isometry between two regular surfaces S_1 and S_2 is a differentiable map $\varphi : S_1 \longrightarrow S_2$ preserving the 1st f.f., i. e., preserving the scalar product: for any $p \in S_1$ and any $\mathbf{v}, \mathbf{w} \in T_pS_1$,*

$$\langle d\varphi_p(\mathbf{v}), d\varphi_p(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \,.$$

Definition 2.16. *A global isometry between two regular surfaces S_1 and S_2 is a local isometry which is also a global diffeomorphism. Then we will say that S_1 and S_2 are (globally) isometric. Furthermore, S_1 and S_2 will be locally isometric if, for any $p \in S_1$, there exist a neighborhood $V \subset S_1$ of p and a global isometry $\varphi : V \longrightarrow \varphi(V) \subset S_2$; and similarly for S_2.*

Therefore, if two regular surfaces are (globally) isometric, then they are exactly alike from topological, differentiable and metric points of view. There exist, of course, surfaces which are locally isometric but not globally isometric. We will see the example we proposed at the beginning of this section.

Example 2.6. *The plane and the cylinder are locally isometric, but not globally isometric.*

Theorem 2.2. *Let $\varphi : S_1 \longrightarrow S_2$ be a local isometry between two regular surfaces. Then, for any $p \in S_1$, there exist parametrizations $X : U \longrightarrow S_1$, $\overline{X} : U \longrightarrow S_2$ around $p \in S_1$ and $\varphi(p) \in S_2$, respectively, such that $E = \overline{E}$, $F = \overline{F}$ and $G = \overline{G}$.*

A sort of converse to this result is as follows.

Theorem 2.3. *Let S_1, S_2 be regular surfaces and $X : U \longrightarrow S_1$, $\overline{X} : U \longrightarrow S_2$ parametrizations of S_1 and S_2, respectively, such that $E = \overline{E}$, $F = \overline{F}$ and $G = \overline{G}$. Then, the map $\varphi = \overline{X} \circ X^{-1} : X(U) \subset S_1 \longrightarrow \overline{X}(U) \subset S_2$ is a (global) isometry between the open sets $X(U)$ and $\overline{X}(U)$ of the surfaces S_1 and S_2.*

2.8 Christoffel symbols

Let S be a regular surface oriented by N and let $X : U \longrightarrow V$ a positively oriented parametrization of S, i. e., such that $\{X_u, X_v, N\}$ is a basis of $\mathbb{R}^3$ positively oriented. We can express the derivatives of those vectors in that basis $\{X_u, X_v, N\}$ in terms of some suitable functions Γ_{ij}^k, which we will call Christoffel symbols.

To do that we write the derivatives as follows:

$$\begin{cases} X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L_1 N, \\ X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + L_2 N, \\ X_{vu} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + L_3 N, \\ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + L_4 N, \end{cases}$$

for suitable coefficients Γ_{ij}^k and L_m, where $i, j, k \in \{1, 2\}$ and $m \in \{1, 2, 3, 4\}$.

A key point. It is easy to see that they only depend on the 1st f.f. of the surface. Namely,

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} \dfrac{E_u}{2} & \dfrac{E_v}{2} & F_v - \dfrac{G_u}{2} \\ F_u - \dfrac{E_v}{2} & \dfrac{G_u}{2} & \dfrac{G_v}{2} \end{pmatrix}.$$

2.9 The Gauss Egregium Theorem

A straightforward computation allows us to deduce the so called *Gauss equation* of a surface:

$$\Gamma_{11}^1 \Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - (\Gamma_{12}^2)_u - \Gamma_{12}^2 \Gamma_{12}^2 = EK.$$

This equation represented one of the great advances in Differential Geometry of surfaces, because it is implicitly saying that the Gaussian curvature only depends on Christoffel symbols and, therefore only depends on the 1st f.f. In other words, K is an intrinsic concept, which is quite surprising such as already mentioned, if we consider that the Gaussian curvature is defined from the Gauss map, that is, from the normal to the surface.

Gauss's equation allows to show a major result in Differential Geometry, the so called Gauss Egregium Theorem. This result was proved by Gauss in 1828, and published for the first time in his great work *Disquisitiones circa general curved surfaces.*

Theorem 2.4. *(The Gauss Egregium Theorem) The Gaussian curvature of a regular surface is invariant by local isometries. Said otherwise, if* $\varphi : S_1 \longrightarrow S_2$ *is a local isometry, then* $K_1(p) = K_2\big(\varphi(p)\big)$*, for any* $p \in S_1$*, where* K_1 *and* K_2 *are, respectively, the Gaussian curvatures of* S_1 *and* S_2*.*

Remark 2.3. *The converse is not true, in general.*

2.10 Geodesics

Given a curve α in a surface S, we can split the vector space $\mathbb{R}^3$ as $\mathbb{R}^3 = T_{\alpha(t)}S \oplus \{N(\alpha(t))\}$. Then, $\alpha''(t) \in \mathbb{R}^3$ is written in a unique way as

$$\alpha''(t) = \alpha''(t)^\top + \alpha''(t)^\perp.$$

The vector field $\alpha''(t)^\top$ will be called the tangent acceleration or intrinsic acceleration of α. It represents the acceleration of a particle whose two-dimensional trajectory is described by α.

The vector field $\alpha''(t)^\perp$ is called the normal acceleration or extrinsic acceleration. In particular we have $\alpha''(t)^\perp = \lambda(t)N(\alpha(t))$, for a suitable differentiable function $\lambda(t)$. To compute it we have $\lambda(t) = \langle \alpha''(t), N(\alpha(t)) \rangle$, so we can write $\alpha''(t)^\perp = \langle \alpha''(t), N(\alpha(t)) \rangle N(\alpha(t))$. The following notation is usual

$$\alpha''(t)^\top = \frac{D\alpha'}{dt}(t),$$

so we get

$$\alpha''(t) = \frac{D\alpha'}{dt}(t) + \langle \alpha''(t), N(\alpha(t)) \rangle N(\alpha(t)).$$

If we work in a plane (or in general, in any surface containing a straightline) we know that straightlines are a very special kind of curves. For instance,

(i) they minimizes the length between two points;

(ii) they have zero constant curvature.

Then, working on a surface S, can we always find curves having similar characteristics to those of straightlines in a plane? We will shortly give an affirmative answer, and such curves will be called the geodesic of S.

Definition 2.17. *Let $\gamma : I \to S$ a parametrized curve. We will say that γ is a geodesic of S if its acceleration γ'' is normal to S.*

Properties of geodesics. Let γS be a geodesic of a regular surface S.

(i) $|\gamma'(t)|$ is constant.

(ii) Geodesics are preserved by local isometries, because they only depend on the covariant derivative, and then on Christoffel symbols. Geodesic is an intrinsic property.

(iii) Assume that γ is an a.l.p. curve and $\gamma''(s) \neq \mathbf{0}$ for any s. We know that γ is geodesic if, and only if, $(D\gamma'/ds)(s) = \big(\gamma''(s)\big)^{\top} = \mathbf{0}$, i. e., $\gamma''(s) = k_\gamma(s)\mathbf{n}_\gamma(s)$ points towards the normal vector of the surface at the point $\gamma(s)$. Then γ is a geodesic if, and only if, $\mathbf{n}_\gamma(s) = \pm N(s)$.

(iv) For any couple $(p, \mathbf{v})$, with $p \in S$ and $\mathbf{v} \in T_pS$, there exists a unique geodesic $\gamma \subset S$ such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$.

Example 2.7. Geodesics

Plane. *Given a unit vector* $\mathbf{a}$, *consider the plane* $\Pi = \big\{p \in \mathbb{R}^3 : \langle \mathbf{a}, p\rangle = c\big\}$, *and take* $N(p) = \mathbf{a}$ *as normal vector, for any* $p \in \Pi$. *A curve* γ *in the plane is geodesic if, and only if,* $\gamma''(t) = \mathbf{0}$, *i. e.,* $\gamma(t) = p + t\mathbf{v}$. *The unique geodesics of a plane are, as predictably, straightlines.*

Sphere. *Set* $p \in \mathbb{S}^2(r)$ *and* $\mathbf{v} \in T_p\mathbb{S}^2(r)$. *Assume that the normal to the sphere is* $N(p) = (1/r)p$. *Take the plane generated by* $\mathbf{v}$ *and* $N(p)$, *whose intersection with the sphere is a great circle, which is also a normal section. Therefore, the normal vector of this curve is* $\mathbf{n} = \pm N(p)$, *which shows that the curve is a geodesic. All great circles of a sphere are geodesics, and they are the only geodesics.*

Cylinder. *There exist three kind of geodesics: helices, circles (parallel of the cylinder) and straightlines (meridians).*

3 Dessert: manifolds

Preparing the ground:

- The normal space to a curve in $\mathbb{R}^2$ is 1-dimensional;
- The normal space to a curve in $\mathbb{R}^3$ is 2-dimensional;
- The normal space to a surface in $\mathbb{R}^3$ is 1-dimensional;
- The normal space to a surface in $\mathbb{R}^4$ should be 2-dimensional;
- The normal space to a curve in $\mathbb{R}^n$ is $(n-1)$-dimensional;
- The normal space to a surface in $\mathbb{R}^n$ should be $(n-2)$-dimensional;
- A curve is “something” 1-dimensional; a surface is “something” 2-dimensional; but $\mathbb{R}^3$ es 3-dimensional.
- Our world is 4-dimensional, is not it?

- And that world ... is it lying in any bigger space?

We have to do a geometry as independent of the ambient space as we can.

We have to define something new to generalize curves and surfaces. That will be the notion of Differentiable Manifold.

Definition 3.1. *Let M a set. An n-dimensional chart on M is a bijective map $\varphi : U \subset M \to \mathbb{R}^n$ whose image $V = \varphi(U)$ is an open set in the Euclidean space.*

As in surfaces, we will find sets which can not be covered by a single chart, so that we will need some collections of charts satisfying some compatibility conditions in a sense that we have to precise.

Definition 3.2. *Two charts n-dimensional charts (U, φ) and (V, ψ) on a set M will be said compatible when either $U \cap V = \emptyset$ or $U \cap V \neq \emptyset$, the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^n$ and the maps $\psi \circ \varphi^{-1}$ are diffeomorphisms.*

We are now able to define one of the key concepts.

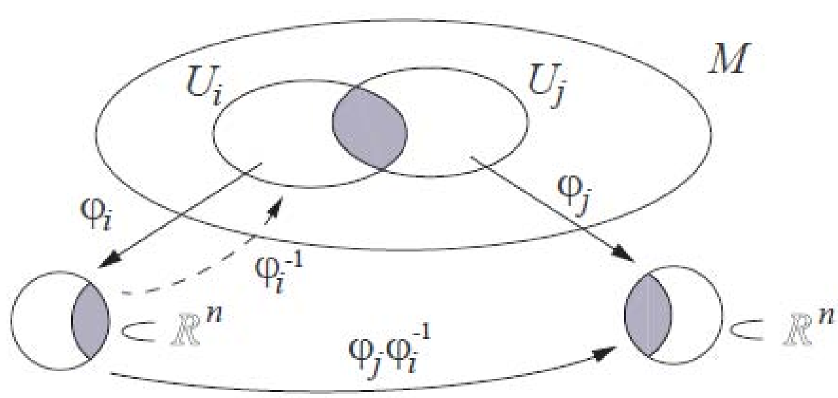
Definition 3.3. *An n-dimensional differentiable atlas on a set M is a family of charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in A}$ satisfying the following conditions:*

(1) $\cup_{i \in A} U_i = M$.

(2) For any pair of indices i and j, the charts (U_i, φ_i) and (U_j, φ_j) are compatible.

Then we will say that the atlas $\mathcal{A}$ determines a differentiable structure on M.

We will always consider atlas of class $\mathcal{C}^\infty$, that is, all maps $\psi \circ \varphi^{-1}$ are differentiable of class $\mathcal{C}^\infty$.



Definition 3.4. *An n-dimensional differentiable manifold is a pair $(M, \mathcal{A})$ of a set M and an n-dimensional atlas $\mathcal{A}$ on M.*

To stand out the dimension n, sometimes we will write M^n instead of M, and when the differentiable structure is known we will omit any reference to it.

Example 3.1. Manifolds

(1) In the Euclidean space $\mathbb{R}^n$ we can define a differentiable structure by taking the identity map as a global chart. We will refer to it as the standard differentiable structure of $\mathbb{R}^n$.

(2) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the map defined by $\varphi(s) = s^3$. Then φ provides a differentiable structure on $\mathbb{R}$ different from the standard one. This is so because the map $\varphi^{-1}(s) = \sqrt[3]{s}$ is not differentiable on the whole $\mathbb{R}$.

(3) A curve, wherever it is, is a 1-dimensional differentiable manifold.

(4) A surface, wherever it is, is a 2-dimensional differentiable manifold.

Definition 3.5. *Let M be a differentiable manifold. A submanifold of M is a pair (N, j) of a manifold N and an injective map $j : N \to M$ such that, at any point $p \in N$, T_pN is a subspace of $T_{j(p)}M$.*

3.1 Tangent space

Definition 3.6. *Sea M una diferenciable manifold. A curve in M is a differentiable map $\alpha : I \to M$ of an open set $I \subset \mathbb{R}$ in M. Without loss of generality, we can assume that $0 \in I$. Given a point $p \in M$ and a curve α in M, we will say that α pass through p if there exists a value $t_0 \in \mathbb{R}$ such that $\alpha(t_0) = p$. When a curve α pass through a point p, we can always reparametrize the curve in such a way that $\alpha(0) = p$.*

Definition 3.7. *A tangent vector to M at p is a tangent vector of a curve α at the point p. The set of all tangent vectors to M at p is called the tangent space to M at p and it will be denoted by T_pM.*

Proposition 3.1. *Let $p \in M^n$. The tangent space T_pM is an n-dimensional (real) vector space.*

3.2 Metric on a manifold

A metric on a manifold M is a 1st f.f. $\mathtt{I}_p \equiv \langle\, , \,\rangle_p$ defined on each tangent space T_pM, which changes smoothly, that is,

$$\mathtt{I}_p : T_pM \times T_pM \longrightarrow \mathbb{R}, \quad \mathtt{I}_p(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle_p .$$

3.3 Curvatures of a manifold $(M^n, \langle\, , \,\rangle)$

When studying surfaces S we defined two curvatures:

- The Gaussian curvature K, which is intrinsic, i. e., K does not depend on the space where the surface S is lying;
- The mean curvature H, which is extrinsic, i. e., H depends on the space where the surface S is lying.

Let $(M^n, \langle\, , \,\rangle)$ be a manifold equipped with a metric. As we are now free of an ambient space, it only makes sense to define an intrinsic curvature of M^n to generalize the Gaussian curvature in surfaces.

To do that, we proceed as follows at a point $p \in M^n$.

The sectional curvature. Set a plane $\Pi \subset T_pM$. We look for all geodesics determined by p and vectors in Π. They generate a surface $S_\Pi \subset M^n$, which inherits the metric of $(M^n, \langle\, , \,\rangle)$.

Now we compute the Gaussian curvature of S_Π at the point p, which will be denoted by $K(\Pi)$ and called sectional curvature of M^n at p with respect to the plane Π.

From the sectional curvature K of M^n, we build two new curvature functions:

The Ricci curvature. Let $v \in T_pM$ be a unit tangent vector, we construct a basis of T_pM of the form $\{v, e_2, \ldots, e_n\}$. The Ricci curvature in the direction of v is defined by

$$Ric(v) = \sum_{i=2}^{n} K(\Pi_i),$$

donde $\Pi_i = \{v, e_i\}$.

The scalar curvature. Given $p \in M^n$ and an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM, the scalar curvature is defined as the map

$$Scal : M^n \to \mathbb{R}, \qquad Scal = 2 \sum_{i<j} K(\Pi_{ij}),$$

where $\Pi_{ij} = \{e_i, e_j\}$.

Remark 3.1. *Let M^2 be a 2-dimensional differentiable manifold, i. e., a surface. Then*

(1) The Gaussian and Ricci curvatures provide the same information about the surface;

(2) $Scal(p) = 2K(p)$.

References

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