ISOLATING SUBGAPS OF A MULTIPLE GAP

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Abstract. Given an analytic multiple gap $\Gamma = \{\Gamma_i : i < n\}$, we study the family $\mathfrak{B}(\Gamma)$ of the sets $A \subset n$ for which there is a restriction $\{\Gamma_i|_a : i \in A\}$ which is still a multiple gap, while $a \in \Gamma_i^\perp$ for $i \not\in A$. This family always contains at least two sets of cardinality 2, and every set of cardinality k is contained in a set from $\mathfrak{B}(\Gamma)$ of cardinality $J(k)$, a number that grows as $\frac{3}{8\sqrt{2\pi k}} \cdot 9^k$. All these results can be stated in terms of the topology of the Čech-Stone remainder ω^* and in terms of sequences in Banach spaces. For example, for any finite family of analytic open sets of ω^* with non-disjoint closures there is always a point that lies in exactly two closures. And given a sequence $\{x_n\}_{n<\omega}$ of vectors in a Banach space that contains subsequences equivalent to $\ell_1, \ell_2, \ldots, \ell_n$ in a way that cannot be separated, it always contains a subsequence $\{x_{n_k}\}_{k<\omega}$ where the ℓ_1 and ℓ_2 subsequences cannot be separated, while there are at most 6 (and this is sharp) of the remaining p's for which $\{x_{n_k}\}_{k<\omega}$ contains subsequences equivalent to ℓ_p.

1. Introduction

Let N be a countable infinite set, and $Fin(N)$ the family of its finite subsets. Let us say that a family I of subsets of N is a *preideal* if whenever we have $a \subset b$ with a infinite and $b \in I$, it follows that $a \in I$ as well. For two preideals I and J, it is equivalent to say that $I \cap J \subset Fin(N)$ or that $x \cap y$ is finite for all $x \in I$ and $y \in J$. In such a case we say that I and J are *orthogonal*. A trivial reason for I and J to be orthogonal is that there exist two disjoint sets $a, b \subset N$ such that $x \subset^* a$ and $y \subset^* b$ for all $x \in I$ and $y \in J$ (here, $u \subset^* v$ means that u is almost contained in v in the sense that $u \setminus v$ is finite). When such a and b exist we say that I and J are *separated*. A pair (I, J) of orthogonal but not separated preideals is the classical well-studied notion of a *gap* [6, 7]. This was generalized to higher dimensions [1] in the following way:

- The preideals $\Gamma_0, \ldots, \Gamma_{n-1}$ are *separated* if there exist $a_0, \ldots, a_{n-1} \subset N$ such that $a_0 \cap \cdots \cap a_{n-1} = \emptyset$ and $x_i \subset^* a_i$ for all $i < n$.
- An *n-gap* (or, in this paper, simply a *gap*) is a family $\Gamma = \{\Gamma_0, \ldots, \Gamma_{n-1}\}$ of n-many preideals that are pairwise orthogonal and are not separated.

For a preideal I and $a \subset N$ we can consider *the restriction* $I|_a = \{x \in I : x \subset a\}$. If $\Gamma = \{\Gamma_0, \ldots, \Gamma_{n-1}\}$ is a gap and $a \subset N$, then the restriction $\Gamma|_a = \{\Gamma_0|_a, \ldots, \Gamma_{n-1}|_a\}$ may still be a gap or may become separated. And it may be the case that while $\Gamma|_a$ becomes separated, the subgap $\{\Gamma_i : i \in A\}$ remains a gap after

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restriction to a, for some subsets $A \subset \{0, \ldots, n-1\}$. We look at the family of all subgaps of a given gap which can be isolated from the rest by a restriction:

$$\mathfrak{B}(\Gamma) = \left\{ \begin{array}{c} A \subset \{0, \ldots, n-1\} \\ 1 < |A| < n \end{array} : \exists a \subset N : \begin{array}{c} \{\Gamma_i|_a : i \in A\} \text{ is still a gap,} \\ \text{but } \Gamma_i|_a \subset Fin(N) \text{ if } i \notin A. \end{array} \right\}$$

In the language of [1], Γ is a B-*clover* when $B \notin \mathfrak{B}(\Gamma)$. It is convenient for us to have a name for the opposite property: we say that Γ is B-*broken* if $B \in \mathfrak{B}(\Gamma)$. We constructed n-gaps in [1] such that $\mathfrak{B}(\Gamma) = \emptyset$ but we needed to use Bernstein sets for that. On the other hand, all the nicely definable gaps that we found seemed to have a large family $\mathfrak{B}(\Gamma)$. Let us say that $\Gamma = \{\Gamma_0, \ldots, \Gamma_{n-1}\}$ is *analytic* if each Γ_i is analytic as a subset of 2^N with its natural product topology. So the natural question arises: Can we build analytic gaps with any prescribed family $\mathfrak{B}(\Gamma)$? In particular, can we construct, for $n \geq 3$, analytic n-gaps with $\mathfrak{B}(\Gamma) = \emptyset$? In this paper we show that the answer to these questions is negative. Indeed, the fact that Γ is analytic implies a number of restrictions on the structure of the family $\mathfrak{B}(\Gamma)$. We are not able to give a completely satisfactory characterization of the families of sets that can be found as $\mathfrak{B}(\Gamma)$ for Γ analytic gap. Instead, we give a number of sample results that can be obtained with a general technique, like the following:

Theorem 1.1. *If Γ is an analytic n-gap with $n \geq 3$, then $\mathfrak{B}(\Gamma)$ contains at least two sets of cardinality 2.*

Theorem 1.2. *For every natural number k there is a natural number $J(k) > k$ such that for every analytic gap Γ and every $A \in \mathfrak{B}(\Gamma)$ of cardinality k there exists $B \in \mathfrak{B}(\Gamma)$ of cardinality at most $J(k)$ such that $A \subset B$.*

The optimal value $J(k)$ in the above theorem can be computed as:

$$J(n) = 2^n - 1 + \sum_{i=1}^{n}\sum_{j=1}^{n} \binom{i+j-1}{j} B(i,j,n);$$

$$B(i,j,n) = \binom{n}{i}\binom{n}{j} - \sum_{p=0}^{n-\max(i,j)} \binom{n-p-1}{j-1}\binom{n-p-1}{i-1}.$$

Its first values are $J(2) = 8$, $J(3) = 61$, $J(4) = 480\ldots$ with an asymptotic behavior

$$J(n) \sim \frac{3}{8\sqrt{2\pi n}} \cdot 9^n.$$

Through Stone duality, our results can be restated in terms of the topology of $\omega^* = \beta\omega \setminus \omega$, the Čech-Stone remainder of the natural numbers. Let us say that an open set $U \subset \omega^*$ is *analytic* if $\{a \subset \omega : \overline{a} \setminus a \subset U\}$ is an analytic family of subsets of ω. As an illustration, this is the translation into this language of Theorem 1.2 when $n = 3$:

Theorem 1.3. *If $U_1, \ldots, U_n$ are pairwise disjoint analytic open subsets of ω^* and $\overline{U_1} \cap \overline{U_2} \cap \overline{U_3} \neq \emptyset$, then there exists a point $x \in \overline{U_1} \cap \overline{U_2} \cap \overline{U_3}$ such that $|\{i : x \in \overline{U}_i\}| \leq 61$. The number 61 is optimal in this result.*

Another natural context where our results can be applied is for sequences of vectors in a Banach space, where many classes of subsequences which are usually considered (ℓ_1-sequences, c_0-sequences, etc.) are in fact analytic. Again, for the purpose of illustration, this is a corollary of Theorem 1.2 for $n = 2$:

Theorem 1.4. *Let F be a finite set of reals with $\{1, 2\} \subset F \subset [1, +\infty)$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in a Banach space. Then one of the following holds:*

(1) *either there is a decomposition $\mathbb{N} = a \cup b$ such that $\{x_n\}_{n\in a}$ does not contain any ℓ_1-subsequence and $\{x_n\}_{n\in b}$ does not contain any ℓ_2-subsequence,*
(2) *or there is an infinite subset $c \subset \mathbb{N}$ which does not admit any decomposition like above, and moreover there are at most eight many $p \in F$ for which $\{x_n\}_{n\in c}$ has an ℓ_p-subsequence.*

The number 8 is optimal in this result.

Concerning the proofs, they essentially rely on the machinery developed in our previous works [3] and [4]. The paper is structured as follows: In Section 2 we recall some concepts and facts from [3, 4] that will be used later. In Section 3 we give some examples and we prove a number of facts concerning the family $\mathfrak{B}(\Gamma)$. In Sections 4 and 5 we interpret our results in terms of the topology of ω^* and in terms of sequences in Banach spaces, respectively. Finally, Section 6 is devoted to the study of the function $J(k)$.

It is worth noticing that although we state all our results for analytic gaps, they also hold for more general projective gaps under the axiom of projective determinacy, cf.[3]. Another remark concerns countable separation and strong gaps. The preideals $\{\Gamma_i : i < n\}$ are *countably separated* if there exists a countable family of sets $\mathcal{C}$ such that whenever we pick $x_i \in \Gamma_i$ for $i = 0, \ldots, n-1$, there exist $a_i \in \mathcal{C}$ such that $x_i \subset a_i$ for all $i < n$ and $\bigcap_{i<n} a_i = \emptyset$. When a gap is not countably separated it is called a *strong gap*. In such case, we can consider the family of subgaps

$$\mathfrak{B}^+(\Gamma) = \left\{ \begin{array}{c} A \subset \{0, \ldots, n-1\} \\ 1 < |A| < n \end{array} : \exists a \subset N : \begin{array}{c} \{\Gamma_i|_a : i \in A\} \text{ is a strong gap,} \\ \text{but } \Gamma_i|_a \subset Fin(N) \text{ if } i \notin A. \end{array} \right\}.$$

There is a theory of strong gaps analogous to that of gaps, which is in fact simpler. The role of *types* explained below is played by so-called (i, j)-combs, for each couple $i, j \in k$, cf. [2]. Similar results as the ones proved for $\mathcal{B}(\Gamma)$ hold for $\mathfrak{B}^+(\Gamma)$ when Γ is strong, with analogous but much easier proofs. For example, Theorem 1.2 is true for $\mathfrak{B}^+(\Gamma)$ when Γ is strong but with k^2 instead of $J(k)$. The analogue of Theorem 1.1 holds as well. Both follow from the results in [2].

2. Preliminaries

In this section we introduce basic terminology and we state some results that we need, that are coming from our previous works [3, 4]. Given a positive integer n, the *n-adic tree* is the set $n^{<\omega}$ of all finite sequences of numbers from $\{0, \ldots, n-1\}$. If $t = (t_1, \ldots, t_p)$ and $s = (s_1, \ldots, s_q)$ are elements of $n^{<\omega}$, the *length* of t is $p = |t|$, and the *concatenation* of t and s is $t^\frown s = (t_1, \ldots, t_p, s_1, \ldots, s_q)$. We consider two orders on $n^{<\omega}$.

- The *tree partial order*: $t \leq s$ if and only if there exists $r \in n^{<\omega}$ such that $s = t^\frown r$. We write $t < s$ if $t \leq s$ and $t \neq s$.

- The *total order*: $t \prec s$ if and only if either $|t| < |s|$, or $|t| = |s|$ and $t_i < s_i$ where i is the first index at which t and s differ. We write $t \preceq s$ if either $t \prec s$ or $t = s$.

The *meet* $t \wedge s$ is the element $r \in n^{<\omega}$ of maximal length that satisfies $r \leq t$ and $r \leq s$. When $t \leq r$, the difference $r \setminus t$ is the s such that $r = t^\frown s$. For every $k < n$ we consider

$$W_k = \{(t_1, \ldots, t_p) \in n^{<\omega} : t_1 = k, t_2 \leq k, \ldots, t_p \leq k\}.$$

Definition 2.1. A *type* τ is a triple $(\tau^0, \tau^1, \lhd)$ where τ^0 and τ^1 are subsets of $\{0, \ldots, n-1\}$, and $\lhd$ is a total order relation on $\tau^0 \times \{0\} \cup \tau^1 \times \{1\}$ satisfying the following constrains:

- $\tau^0 \neq \emptyset$.
- If $\tau^1 \neq \emptyset$, then $\min(\tau^0) \neq \min(\tau^1)$.
- $(k, i) \lhd (l, i)$ if $k < l$ and $i \in \{0, 1\}$.
- The maximal element of the order $\lhd$ is of the form $(k, 0)$.

Definition 2.2. Consider a type τ where $\tau^0 = \{k_0 < \cdots < k_p\}$ and $\tau^1 = \{l_0 < \ldots < l_q\}$. We say that a couple (u, v) is a rung of type τ if the following conditions hold:

(1) u can be written as $u_0{}^\frown \cdots {}^\frown u_p$ where $u_i \in W_{k_i}$,
(2) v can be written as $v_0{}^\frown \cdots {}^\frown v_q$ where $v_i \in W_{l_i}$,
(3) $(k_i, 0) \lhd (l_j, 1)$ if and only if $u_0{}^\frown \cdots {}^\frown u_i \prec v_0{}^\frown \cdots {}^\frown v_j$.

When $\tau^1 = \emptyset$, this means that $v = \emptyset$ and u satisfies condition (1) above.

Definition 2.3. Consider a type τ. We say that an infinite set $X \subset m^{<\omega}$ is of type τ if there exists $u \in m^{<\omega}$ and a sequence of rungs $(u_0, v_0), (u_1, v_1), \ldots$ of type τ such that we can write $X = \{x_0, x_1, \ldots\}$ and

$$x_k = u^\frown u_0{}^\frown u_1{}^\frown \cdots {}^\frown u_{k-1}{}^\frown v_k$$

for $k = 0, 1, \ldots$

A direct application of Ramsey's theorem shows that every infinite subset of $n^{<\omega}$ contains an infinite set that is of some type τ [3, Lemma 3.4]. The set of all types made with numbers from $\{0, 1, \ldots, n-1\}$ is denoted by $\mathfrak{T}_n$. This is a finite list of types, whose cardinality is the number that we will denote by $J(n)$. If a subset of $n^{<\omega}$ has type τ according to Definition 2.3, then $\tau \in \mathfrak{T}_n$. We denote each type with two rows of numbers between brackets, the upper row being τ^1 (it is omitted if $\tau^1 = \emptyset$), the lower row being τ^0, and the order $\lhd$ being read from left to right. For example, there are 8 types in $\mathfrak{T}_2$,

$$\mathfrak{T}_2 = \{[0], [1], [01], [{}^1{}_{01}], [{}^0{}_1], [{}^1{}_0], [{}^{01}{}_1], [{}_0{}^1{}_1]\}$$

If $\tau^1 = \emptyset$, then τ is called a *chain type*, and if $\tau^1 \neq \emptyset$, then τ is called a *comb type*. These names correspond to the shapes of the respective sets of type τ.

Now, let us consider again gaps. If I is a preideal of subsets of N, let $I^\perp = \{a \subset N : I|_a \subset Fin(N)\}$. Notice that $I \cap I^\perp$ consists only of finite sets, and from our point of view finite sets will be negligible. If $\Gamma = \{\Gamma_0, \ldots, \Gamma_{n-1}\}$, we write $\Gamma^\perp = \bigcap_{i<n} \Gamma_i^\perp$. If Γ and Δ are two n-gaps on countable sets N and M, we write $\Gamma \leq \Delta$ if there exists a one-to-one function $f : N \longrightarrow M$ such that

(1) If $a \in \Gamma_i$ then $f(a) \in \Delta_i$,

(2) If $a \in \Gamma_i^\perp$, then $f(a) \in \Delta_i^\perp$.

This is relevant for us due to the following observation:

Lemma 2.4. *If $\Gamma \leq \Delta$ then $\mathfrak{B}(\Gamma) \subset \mathfrak{B}(\Delta)$.*

Proof. Let $f : N \longrightarrow M$ be the map that witnesses that $\Gamma \leq \Delta$. If $A \in \mathfrak{B}(\Gamma)$ then we have $a \subset N$ such that $\{\Gamma_i|_a : i \in A\}$ is a gap but $\Gamma_i|_a \subset Fin(N)$ for $i \notin A$. We claim that $A \in \mathfrak{B}(\Delta)$ because $\{\Delta_i|_{f(a)} : i \in A\}$ is a gap but $\Delta_i|_{f(a)} \subset Fin(M)$ for $i \notin A$. The latter fact is just part (2) in the above definition of $\Gamma \leq \Delta$. For the first fact, if $\{\Delta_i|_{f(a)} : i \in A\}$ were separated, we would have sets $b_i \subset M$ such that $\bigcap_i b_i = \emptyset$ and $y \subset^* b_i$ for $y \in \Delta_i$, $i < n$. But then the sets $a_i = f^{-1}(b_i)$ would show that $\{\Gamma_i|_a : i \in A\}$ are separated, because if $x \in \Gamma_i|_a$, then $f(x) \in \Delta_i|_{f(a)}$, hence $f(x) \subset^* b_i$, hence by injectivity of f, $x \subset^* f^{-1}(b_i)$. □

This lemma and the following theorem show that we can restrict our attention to a particular kind of analytic gaps: those made from types. Given a set $S \subset \mathfrak{T}_n$, we denote by Γ_S the preideal consisting of all subsets of $n^{<\omega}$ that are of type τ for some type $\tau \in S$.

Theorem 2.5. *Let $n, m < \omega$ be natural numbers.*

(1) *If $S_0, \ldots, S_{n-1}$ are nonempty pairwise disjoint subsets of $\mathfrak{T}_m$, then the preideals $\{\Gamma_{S_0}, \ldots, \Gamma_{S_{n-1}}\}$ form an analytic n-gap.* This follows from [3, Lemma 4.3] and [4, Theorem 4.5].

(2) *For every analytic n-gap Δ there exist pairwise disjoint subsets $S_0, \ldots, S_{n-1}$ of $\mathfrak{T}_n$, and a permutation $\varepsilon : n \longrightarrow n$ such that $[\varepsilon(i)] \in S_i$ for each $i < n$ and $\{\Gamma_{S_{\varepsilon(0)}}, \ldots, \Gamma_{S_{\varepsilon(n-1)}}\} \leq \Delta$* [3, Theorem 4.5].

In the above theorem, notice that $[\varepsilon(i)]$ is denoting a chain type τ for which τ^0 has a single integer $\varepsilon(i)$, following the bracket notation that we introduced for types. An analytic n-gap of the form $\{\Gamma_{S_{\varepsilon(0)}}, \ldots, \Gamma_{S_{\varepsilon(n-1)}}\}$ as above is called *standard*. Since $\mathfrak{T}_n$ is finite, there are only finitely many standard n-gaps, and what the above theorem says is that any analytic n-gap contains a standard n-gap. For simplicity, if we have a gap of the form $\{\Gamma_{S_i} : i < n\}$ where each S_i is a set of types in $n^{<\omega}$, when we say that a type τ belongs to Γ_{S_i} we mean that $\tau \in S_i$.

In [4] we introduced the notion of *normal embedding*. We do not need to recall the formal definition with all its technicalities, the only thing that we need to know for our purposes is that a normal embedding is an injective map $\phi : m^{<\omega} \longrightarrow n^{<\omega}$ with the property that it induces a function $\bar{\phi} : \mathfrak{T}_m \longrightarrow \mathfrak{T}_n$ such that $\phi(A)$ is a set of type $\bar{\phi}\tau$ whenever A is a set of type τ. For gaps constructed from types, the relation $\leq$ is always witnessed by normal embeddings [4, Corollary 4.4]:

Lemma 2.6. *Let $S_0, \ldots, S_{k-1}$ be disjoint sets of types in $n^{<\omega}$ and $\tilde{S}_0, \ldots, \tilde{S}_{k-1}$ disjoint sets of types in $m^{<\omega}$. The following are equivalent:*

(1) $\{\Gamma_{S_i} : i < k\} \leq \{\Gamma_{\tilde{S}_i} : i < k\}$.

(2) *There exists a normal embedding $\phi : n^{<\omega} \to m^{<\omega}$ such that for every $i < k$ and for every $\tau \in \mathfrak{T}_n$ we have that $\tau \in S_i$ if and only if $\bar{\phi}\tau \in \tilde{S}_i$.*

For a type τ, $\max(\tau)$ is the largest integer that appears in that type. The following are Theorem 4.5 and Corollary 4.6 in [4]:

Theorem 2.7. *For a family $\{\tau_i : i \in n\} \subset \mathfrak{T}_m$ the following are equivalent:*

(1) *There exists a normal embedding* $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ *such that* $\bar{\phi}[i] = \tau_i$ *for* $i = 0, \ldots, n-1$,
(2) $\max(\tau_0) \leq \cdots \leq \max(\tau_{n-1})$.

Corollary 2.8. *If* $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ *is a normal embedding and* $\tau, \sigma \in \mathfrak{T}_n$ *satisfy* $\max(\tau) \leq \max(\sigma)$, *then* $\max(\bar{\phi}\tau) \leq \max(\bar{\phi}\sigma)$.

A type $\tau = (\tau^0, \tau^1, \triangleleft)$ is called a *top-comb type* if it is a comb type and moreover the penultimate position in the order $\triangleleft$ is occupied by an element coming from $\tau^1 \times \{1\}$. We say that the type τ dominates the type σ if τ is a top-comb type and moreover $\max(\tau^1) \geq \max(\sigma)$. By [4, Corollary 6.4], if τ dominates σ and $\phi : m^{<\omega} \longrightarrow n^{<\omega}$ is a normal embedding, then $\bar{\phi}\tau$ either dominates or is equal to $\bar{\phi}\sigma$. We shall frequently use the following fact, which is a particular case of [4, Theorem 6.2]

Theorem 2.9. *If* $\tau_0, \tau_1 \in \mathfrak{T}_m$ *are two types and* τ_1 *dominates* τ_0, *then there exists a normal embedding* $\phi : 2^{<\omega} \longrightarrow m^{<\omega}$ *such that* $\bar{\phi}[0] = \tau_0$ *and* $\bar{\phi}\sigma = \tau_1$ *for all* $\sigma \in \mathfrak{T}_2 \setminus \{[0]\}$.

Finally, the following lemma is part of [4, Lemma 5.2], and it says that normal embeddings send a comb type to a chain type only in trivial cases.

Lemma 2.10. *Let* $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ *be a normal embedding and let* $k \leq n$. *The following are equivalent:*

(1) *There exists a comb type* τ *with* $\max(\tau^1) = k-1$ *and a chain type* σ *such that* $\bar{\phi}(\tau) = \sigma$.
(2) *There exists a chain type* σ *such that* $\bar{\phi}(\tau) = \sigma$ *for all types* τ *with* $max(\tau) < k$.

3. Studying the family $\mathfrak{B}(\Gamma)$

Our main tool to study the family $\mathfrak{B}(\Gamma)$ for analytic gaps is the following lemma:

Lemma 3.1. *Let* $\{S_i : i < n\}$ *be nonempty pairwise disjoint sets of types in* $m^{<\omega}$, *and* $\Gamma = \{\Gamma_{S_i} : i \in n\}$ *the corresponding* n*-gap. For a set* $B \subset n$ *with* $1 < |B| < n$ *the following are equivalent:*

(1) $B \in \mathfrak{B}(\Gamma)$.
(2) *There exists some* k *and a normal embedding* $\phi : k^{<\omega} \longrightarrow m^{<\omega}$ *such that the range of* $\bar{\phi}$ *intersects each* S_i *with* $i \in B$, *but it is disjoint from* S_j *for* $j \notin B$.
(3) *There exists some* k, *a normal embedding* $\phi : k^{<\omega} \longrightarrow m^{<\omega}$ *and a bijection* $u : k \longrightarrow B$ *such that* $\bar{\phi}[i] \in S_{u(i)}$ *for all* $i < k$, *but* $\bar{\phi}\tau \notin S_j$ *whenever* $\tau \in \mathfrak{T}_k$ *and* $j \notin B$.

Proof. The obvious fact is that (3) implies (2). If (2) holds, then the image $a = \phi(k^{<\omega})$ is the set that witnesses that $B \in \mathfrak{B}(\Gamma)$. This is because, on the one hand, $\Gamma_{S_i}|_a = \emptyset$ when $i \notin B$ since S_i is disjoint from $\bar{\phi}(\mathfrak{T}_k)$. And on the other hand, $\{\Gamma_{S_i}|_a : i \in B\}$ are not separated because otherwise $\{\phi^{-1}(\Gamma_{S_i}) : i \in B\}$ would be separated, and $\phi^{-1}(\Gamma_{S_i}) \supset \Gamma_{\bar{\phi}^{-1}(S_i)}$ and these sets are not separated by Theorem 2.5. So this proves that (2) implies (1). Now, suppose that $B \in \mathfrak{B}(\Gamma)$ and $a \subset m^{<\omega}$ is such that $a \in \Gamma_i^{\perp}$ for $i \notin B$ while $\{\Gamma_{S_i}|_a : i \in B\}$ is a gap. Then by Theorem 2.5, we can find a standard gap $\{\Gamma_{\tilde{S}_i} : i \in B\}$ on some $k^{<\omega}$ such that

$$\{\Gamma_{\tilde{S}_i} : i \in B\} \leq \{\Gamma_{S_i}|_a : i \in B\}.$$

By Lemma 2.6, this relation $\leq$ is witnessed by a normal embedding $\phi : k^{<\omega} \longrightarrow a \subset m^{<\omega}$, which satisfies condition (3). $\square$

Lemma 3.2. *Let $\phi : m^{<\omega} \longrightarrow n^{<\omega}$ be a normal embedding. Suppose that we have $j < n$ and a type $\tau \in \mathfrak{T}_m$ such that $\bar{\phi}\tau = [j]$. Then there exists $i < m$ such that $\bar{\phi}[i] = [j]$.*

Proof. If τ is a comb type, it follows from Lemma 2.10 that $\bar{\phi}[0] = [j]$. So we suppose that τ is a chain type, and we write $\tau = [i_0\ i_1\ \cdots\ i_k]$ being $i_0 < i_1 < \cdots < i_k < m$. We are going to prove that the desired i is precisely $i = i_0$, the smallest integer in τ. Let us use the following notation: given s, t two finite sequences of integers and given an integer p, we write $t \sim_p s$ if there exists $r \in W_p$ such that $s = t^\frown r$ or $t = s^\frown r$. Thus, a $[p]$-chain is just an infinite set A such that $s \sim_p t$ for all different $s, t \in A$. It is enough for us to prove that if $x \in m^{<\omega}$ and $r \in W_i$, then $\phi(x) \sim_j \phi(x^\frown r)$. For this, pick $r_1 \in W_{i_1}, \ldots, r_k \in W_{i_k}$ and consider $w = r^\frown i^\frown r_1^\frown \cdots ^\frown r_k$, $x_1 = x^\frown w$, $x_2 = x^\frown w^\frown w$, and in general $x_{\xi+1} = x_\xi{}^\frown w$. Notice that both $\{x, x_1, x_2, x_3 \ldots\}$ and $\{x^\frown r, x_1, x_2, x_3, \ldots\}$ are chains of type τ, hence both $\{\phi(x), \phi(x_1), \phi(x_2), \ldots\}$ and $\{\phi(x^\frown r), \phi(x_1), \phi(x_2), \ldots\}$ are chains of type $[j]$. This easily implies that $\phi(x) \sim_j \phi(x^\frown r)$ as desired. $\square$

Given two chain types $\tau = [\tau_1 < \tau_2 < \cdots < \tau_p]$ and $\sigma = [\sigma_1 < \sigma_2 < \cdots < \sigma_q]$, we define the chain type

$$\tau * \sigma = [\tau_1 < \tau_2 < \cdots < \tau_p < \sigma_\xi < \sigma_{\xi+1} < \cdots < \sigma_q]$$

where ξ is the least integer such that $\sigma_\xi > \tau_p$. If there is no such integer, we declare $\tau * \sigma = \tau$. By [4, Lemma 5.1], if $\phi : m^{<\omega} \longrightarrow n^{<\omega}$ is a normal embedding and σ, τ, $\bar{\phi}\sigma$ and $\bar{\phi}\tau$ are all chain types, then $\bar{\phi}(\tau * \sigma) = \bar{\phi}\tau * \bar{\phi}\sigma$. As a corollary, we get:

Lemma 3.3. *Let $\{S_i : i < n\}$ be pairwise disjoint sets of types in $m^{<\omega}$, and $\Gamma = \{\Gamma_{S_i} : i < n\}$ the corresponding gap. Fix $A \in \mathfrak{B}(\Gamma)$. If $\tau \in S_i$, $\sigma \in S_j$ are chain types and $i, j \in A$, then $\sigma * \tau \notin \bigcup_{k \notin A} S_k$.*

Proof. Find ϕ and u as in Lemma 3.1(3), and then use the property mentioned above. $\square$

Before stating any positive results, we give some examples to set some limits on what we can expect:

Theorem 3.4. *We have the following examples:*

1. *For every n, there is an n-gap with $\mathfrak{B}(\Gamma) = \{A : 1 < |A| < n\}$.*
2. *For every n, there is an n-gap with $\mathfrak{B}(\Gamma) = \{A : 1 < |A| < n,\ n-1 \in A\}$.*
3. *There is a 4-gap with $\mathfrak{B}(\Gamma) = \{\{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{0,1,2\}, \{1,2,3\}\}$.*

Proof. The examples that we are going to provide in order to check the three statements above are all of the form $\{\Gamma_{S_i} : i < n\}$ for some pairwise disjoint sets of types $S_i \subset \mathfrak{T}_m$. For (1), take $m = n$ and $S_i = \{[i]\}$. For $A \subset n$, the set $a = A^{<m}$ of all finite sequences with entries in A, witnesses that $A \in \mathfrak{B}(\Gamma)$. For the second example, take $m = 2n-2$, $S_i = \{[i], [n-1+i]\}$ for $i < n-1$, and $S_{n-1} = \mathfrak{T}_m \setminus \{[i] : i < m\}$. If $n-1 \in A$ and $B = \{i, n-1+i : i \in A \setminus \{n-1\}\}$ then the set $B^{<\omega}$ witnesses that $A \in \mathfrak{B}(\Gamma)$. On the other hand, suppose that $n-1 \notin A \in \mathfrak{B}(\Gamma)$. Then there exists a normal embedding $\phi : k^{<\omega} \longrightarrow m^{<\omega}$ such that the range of $\bar{\phi}$ does not hit S_{n-1} while hitting all S_j for $j \in A$. Lemma 2.10 implies that $\bar{\phi}\left[{}^{k-1}{}_0\right]$ is a comb type, so it belongs to S_{n-1}, a contradiction. For the third example, take $n = 4$, $m = 3$, $S_0 = \{[0]\}$, $S_1 = \{[12]\}$, $S_2 = \{[012]\}$ and $S_3 = \{[1]\}$. The inclusion $[\subset]$ follows from Lemma 3.3, since $[0] * [12] = [012]$ and $[1] * [012] = [12]$. For the other inclusion, for a set $W \subset 3^{<\omega}$, let us denote

$$a(W) = \{w_1^\frown w_2^\frown \cdots {}^\frown w_p : w_1, \ldots, w_p \in W\}$$

the set of all finite concatenations of elements of W. That each of the indicated sets belongs to $\mathfrak{B}(\Gamma)$ is witnessed by the sets $a(0, 012)$, $a(0, 1)$, $a(12, 012)$, $a(12, 1)$, $a(0, 12, 012)$ and $a(12, 012, 1)$ respectively. $\square$

We are now ready to prove Theorem 1.1:

Proof. We can suppose that $\Gamma = \{\Gamma_k : k < n\}$ is a standard gap of the form $\{\Gamma_{S_k} : k < n\}$ for some pairwise disjoint sets $S_k \subset \mathfrak{T}_n$ with $[k] \in S_k$. For each pair $\{i, j\} \subset m$ we can consider the restriction $\{\Gamma_{S_k}|_{\{i,j\}^{<\omega}} : k < n\}$. We notice that if $A(i, j)$ is the set of all k such that S_k contains at least one type made of integers only i and j, then we have that $\{\Gamma_{S_k}|_{\{i,j\}^{<\omega}} : k \in A(i, j)\}$ is a gap, while $\Gamma_{S_k} \in (\{i, j\}^{<\omega})^\perp$ for $k \notin A(i, j)$. Notice that $i, j \in A(i, j)$ because $[k] \in S_k$. If $|A(i, j)| = 2$ for all pairs i, j, then $\mathfrak{B}(\Gamma)$ contains all pairs and we are done. Otherwise, there is a pair $\{i, j\}$ such that $|A(i, j)| > 2$. For simplicity, we suppose that $\{i, j\} = \{0, 1\}$ and $A(i, j) = \{0, 1, 2, \ldots, m\}$ with $2 \leq m < 8$. Remember that there are just eight types in $\mathfrak{T}_2$. The types $[{}^{1}{}_{0}]$, $[{}^{01}{}_{1}]$ and $[{}_{0}{}^{1}{}_{1}]$ are top-comb types with $\max(\tau^1) = 1$, so they dominate any of the other types. If any of these three types τ belongs to some S_k with $k \leq m$, then we will be done. This is because in this case we can choose two other types $\sigma 1$ and $\sigma 2$ in two other different sets S_{k1} and S_{k2}, and then Lemma 2.9 provides, for $u = 0, 1$, embeddings $\phi_u : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $\bar{\phi_u}[0] = \sigma u$ and $\bar{\phi_u}\rho = \tau$ for all $\rho \neq [0]$. This means that $\{k, k1\}, \{k, k2\} \in \mathfrak{B}(\Gamma)$. So, we can suppose that the three types $[{}^{1}{}_{0}]$, $[{}^{01}{}_{1}]$ and $[{}_{0}{}^{1}{}_{1}]$ do not hit any of the sets S_k, and so $2 \leq m < 5$. We can use the following table to visualize the distribution of the types in the gap:

$[0]$	$[1]$	$[01]$	$[{}^{1}{}_{01}]$	$[{}^{0}{}_{1}]$	$[{}^{1}{}_{0}]$	$[{}_{0}{}^{1}{}_{1}]$	$[{}^{01}{}_{1}]$
S_0	S_1	?	?	?	no	no	no

We are going to give normal embeddings $\varphi_1, \varphi_2, \ldots, \varphi_9$ from $2^{<\omega}$ into itself. We summarize in the following non-exhaustive table the facts about these embeddings that will be most relevant for our discussion:

τ	$[0]$	$[1]$	$[01]$	$[{}^{1}{}_{01}]$		$[0]$	$[1]$	$[01]$	$[{}^{1}{}_{01}]$	$[{}^{0}{}_{1}]$
$\overline{\varphi_1}\tau$	$[0]$	$[{}^{0}{}_{1}]$	$[{}^{0}{}_{1}]$	$[{}^{0}{}_{1}]$		X	$-$	$-$	$-$	X
$\overline{\varphi_2}\tau$	$[1]$	$[{}^{0}{}_{1}]$	$[1]$	$[{}_{0}{}^{1}{}_{1}]$		$-$	X	$-$	$-$	X
$\overline{\varphi_3}\tau$	$[01]$	$[{}^{0}{}_{1}]$	$[01]$	$[{}_{0}{}^{1}{}_{1}]$		$-$	$-$	X	$-$	X
$\overline{\varphi_4}\tau$	$[{}^{0}{}_{1}]$	$[{}^{1}{}_{01}]$	$[{}^{0}{}_{1}]$			$-$	$-$	$-$	X	X
$\overline{\varphi_5}\tau$	$[0]$	$[{}^{1}{}_{01}]$	$[{}^{1}{}_{01}]$			X	$-$	$-$	X	
$\overline{\varphi_6}\tau$	$[1]$	$[{}^{1}{}_{01}]$	$[1]$			$-$	X	$-$	X	
$\overline{\varphi_7}\tau$	$[01]$	$[{}^{1}{}_{01}]$				$-$	$-$	X	X	
$\overline{\varphi_8}\tau$	$[0]$	$[01]$				X	$-$	X		
$\overline{\varphi_9}\tau$	$[01]$	$[1]$				$-$	X	X		

In the left half of the table we indicate the value of $\overline{\varphi_i}\tau$ for each of the four types indicated in the top row. In the right half of the table we indicate with the symbol X which of those five types is in the range of $\overline{\varphi_i}$ and with the symbol $-$ those which are not. The purpose is, of course, to apply Lemma 3.1. We do not need to worry about the remaining three types on the right half of the table because we already assumed that they do not belong to any S_k.

Before describing what the embeddings φ_i are, let us note how we can deduce from the information in the table that there are three different i, j, k such that $\{i,j\}, \{i,k\} \in \mathfrak{B}(\Gamma)$. This is easy, by a consideration of cases. In the first case, if $[^0{}_1]$ is in some S_i, then two other types from [0], [1], [01] or $[^1{}_{01}]$ must belong to two other S_j and S_k, and then two of the embeddings φ_1, φ_2, φ_3 or φ_4 give, using Lemma 3.1, that $\{i,j\}, \{i,k\} \in \mathfrak{B}(\Gamma)$. In the second case, we assume that $[^0{}_1]$ is not in any S_i but $[^1{}_{01}]$ is in some of them. Then, similarly as in the first case two of the embeddings φ_5, φ_6 of φ_7 give the desired conclusion. Finally, in the third case if neither $[^0{}_1]$ nor $[^1{}_{01}]$ is in any S_k, then φ_8 and φ_9 show that $\{0,2\}, \{1,2\} \in \mathfrak{B}(\Gamma)$.

Theorem 2.7 ensures that if $\sigma, \sigma' \in \mathfrak{T}_2$ are such that $\max(\sigma) \leq \max(\sigma')$ then there exists a normal embedding such that $\bar{\phi}[0] = \sigma$ and $\bar{\phi}[1] = \sigma'$. Thus, for each row of the table above we can find a normal embedding which takes the prescribed values on the types [0] and[1]. We take φ_5, φ_7, φ_8 and φ_9 in this way, just given by Theorem 2.7. However, the value at [0] and [1] does not determine the action on the rest of types, and we describe more precisely the rest of cases. The embedding $\overline{\varphi_1}$ will be the one provided by Theorem 2.9 for $\tau_0 = [0]$ and $\tau_1 = [^0{}_1]$, so that $\overline{\varphi_1}[0] = [0]$ and $\overline{\varphi_1}\tau = [^0{}_1]$ for any type $\tau \neq [0]$. For the rest, we need some notation. Given $s^0, s^1, s \in 2^{<\omega}$, we define a function $\psi[s^0, s^1, s] : 2^{<\omega} \longrightarrow 2^{<\omega}$ whose value on a given $t \in 2^{<\omega}$ is the sequence obtained by substituting each 0 by the sequence s^0, each 1 by the sequence s^1, and finally adding a single copy of the sequence s at the end. We consider $\psi_2 = \psi[01, 11, 0]$, $\psi_3 = \psi[001, 111, 0]$, $\psi_4 = \psi[1111, 0001, 10]$, $\psi_6 = \psi[1111, 0001, 1]$. It is a straightforward exercise to check that if A is a set of type τ, then $\psi_i(A)$ is a set of the type $\overline{\varphi_i}\tau$ given by the table (in the cases where the table indicates what this type should be). The functions ψ_i are injective but we did not say that they are normal embeddings. But [4, Theorem 4.2] states that for an injective function ψ_i there is always a *nice embedding* $u_i : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $\psi_i \circ u_i$ is a normal embedding. We do not need to recall now the technical definition of a nice embedding; it is enough to know that a nice embedding is a function $m^{<\omega} \longrightarrow m^{<\omega}$ that maps sets of type τ onto sets of type τ for all $\tau \in \mathfrak{T}_m$, cf. Proposition 2.4 and remarks before Lemma 3.4 in [3]. So we can take $\varphi_i = \psi_i \circ u_i$ and in this way we already have all the normal embeddings that satisfy the left half of our table.

Now let us check the right half of the table. The X's are filled just by looking at these values at [0] and [1], so the point is how to make sure that we can put a $-$ sign in the corresponding places.

The values for $\overline{\varphi_1}$ are clear by definition. The three excluded types $[^1{}_0]$, $[^1{}_{01}]$ and $[^{01}{}_1]$ dominate [1], hence their images under a $\overline{\varphi_i}$ dominate (or equal) $\overline{\varphi_i}[1]$ which has maximum 1 for $i \geq 2$. Therefore the image of any of these three types is again one of those three types or equals $\bar{\varphi}_i[i]$, so we do not need to care about their values to fill the right half of the table. Since $[^0{}_1]$ dominates [0], $\overline{\varphi_i}[^0{}_1]$ dominates or equals $\overline{\varphi_i}[0]$. Hence $\overline{\varphi_i}[^0{}_1]$ is either one of the three excluded types or $[^0{}_1]$ itself, or $\bar{\varphi}_i[0]$. Therefore, we do not need to care about the image of $[^0{}_1]$ to fill the right half the table. The values for $\overline{\varphi_5}$ are given by [4, Lemma 8.1]. The $-$ signs for the columns of [0] and [1] follow from Lemma 3.2. The fact that $[^1{}_{01}]$ cannot be sent to a chain type follows from Lemma 2.10. The rest of $-$ signs follow from the observations above and the information on the left side of the table. $\square$

A gap Γ on the set N is called *dense* if every infinite subset of N contains an infinite subset of one the preideals of Γ. In other words, if $\Gamma^\perp = Fin(N)$. By [3, Lemma 3.4] every infinite subset of $m^{<\omega}$ contains an infinite subset of one type. Therefore a gap on $m^{<\omega}$ of the form $\{\Gamma_{S_i} : i < n\}$ is dense if and only if $\bigcup_{i<n} S_i = \mathfrak{T}_m$.

Theorem 3.5. *For every dense analytic n-gap Γ there exists $k \in n$ such that $\{k,i\} \in \mathfrak{B}(\Gamma)$ for all $i \in n \setminus \{k\}$. Moreover, $B \cup \{k\} \in \mathfrak{B}(\Gamma)$ whenever $B \in \mathfrak{B}(\Gamma)$.*

Proof. Suppose that Γ is a standard n-gap in $n^{<\omega}$ and let k be such that Γ_k contains a top-comb type τ with $\max(\tau^1) = n-1$. The theorem follows from application of Theorem 2.9. The last statement follows similarly from the more general [4, Theorem 6.2]. $\square$

The above result is not true for general gaps, as example 3 in Theorem 3.4 shows. On the other hand, by example 2 in Theorem 3.4, Theorem 3.5 is an optimal result concerning the doubletons that we can find in $\mathfrak{B}(\Gamma)$ for Γ dense and analytic.

For a natural number m, let $J(m) = |\mathfrak{T}_m|$ be the number of types in $m^{<\omega}$. Theorem 1.2 follows now from Theorem 3.6 below. The function J will be studied in more detail later in Section 6.

Theorem 3.6. *For every analytic n-gap Γ and for every set $A \subset n$ with $|A| = k$ there exists $B \supset A$ with $|B| \leq J(k)$ such that Γ can be B-broken. The number $J(k)$ is optimal in this result.*

Proof. Let $\phi : k^{<\omega} \longrightarrow m^{<\omega}$ be a normal embedding witnessing that we have a standard k-gap $\Gamma' \leq \{\Gamma_i : i \in A\}$. Since there are only $J(k)$ types in $k^{<\omega}$, at most $J(k)$ ideals from Γ can be present after this reduction, so this proves the statement of the theorem. Concerning optimality, let $\{\tau_i : i \in J(k)\}$ be an enumeration of all types in $k^{<\omega}$ where $\tau_i = [i]$ for $i < k$. Consider the $J(k)$-gap $\Gamma = \{\Gamma_{\{\tau_i\}} : i \in J(k)\}$ in $k^{<\omega}$. We claim that if Γ can be B-broken for some $B \supset A = \{i : i < k\}$, then B is the whole $J(k)$. Suppose that we have a normal embedding $\phi : p^{<\omega} \longrightarrow k^{<\omega}$ witnessing that Γ is B-broken. Then since $A \subset B$, there must exist types σ_i for $i < k$ such that $\bar{\phi}\sigma_i = [i]$. Since $\max[i] < \max[j]$ when $i < j$, it follows from Corollary 2.8 that $\max(\sigma_i) < \max(\sigma_j)$ when $i < j$. By Theorem 2.7, we get a normal embedding $\psi : k^{<\omega} \longrightarrow p^{<\omega}$ such that $\bar{\psi}[i] = \sigma_i$ for $i < k$. Then, $\bar{\phi}\bar{\psi}[i] = [i]$ for $i < k$ and this implies that $\bar{\phi}\bar{\psi}\tau = \tau$ for all types τ in $k^{<\omega}$, hence B must be the whole set $J(k)$. $\square$

4. Interpretation in ω^*

The phenomena studied in this paper have a nice topological interpretation through Stone duality. Let us remind the reader that $\beta\omega$, the Čech-Stone compactification of ω, is a compact Hausdorff topological space that is characterized by the fact that it contains ω as a dense subset of isolated points with the property that $\overline{a} \cap \overline{b} = \emptyset$ whenever a and b are disjoint subsets of ω. The Čech-Stone remainder of ω is the compact space $\omega^* = \beta\omega \setminus \omega$. The clopen subsets of ω^* are the sets of the form $\bar{a} \setminus \omega$ with $a \subset \omega$, and they form a basis of the topology of ω^*.

To each open subset V of $\omega^* = \beta\omega \setminus \omega$ we associate the preideal (in fact the ideal) $I(V) = \{a \subset \omega : \overline{a} \setminus \omega \subset V\}$. We say that V is analytic if $I(V)$ is an analytic subset of 2^ω. The following lemma contains just some elementary manipulations in ω^*. The first four statements are implicit in [1] but we sketch a proof anyway.

Lemma 4.1. *Let $\{U_i : i < n\}$ be open subsets of ω^*.*

1. *The preideals $\{I(U_i) : i < n\}$ are pairwise orthogonal if and only if the open sets $\{U_i : i < n\}$ are pairwise disjoint.*
2. *The preideals $\{I(U_i) : i < n\}$ are separated if and only if $\bigcap_{i\in n} \overline{U_i} = \emptyset$.*
3. *The preideals $\{I(U_i) : i < n\}$ are an n-gap if and only if the open sets $\{U_i : i < n\}$ are pairwise disjoint and $\bigcap_{i\in n} \overline{U_i} \neq \emptyset$.*
4. *$I(U_i)^\perp = \{a \subset \omega : \overline{a} \cap \overline{U_i} = \emptyset\}$.*
5. *For $B \subset n$ the n-gap $\{I(U_i) : i < n\}$ can be B-broken if and only if $\bigcap_{i\in B} \overline{U_i} \setminus \bigcup_{i\not\in B} \overline{U_i} \neq \emptyset$.*

Proof. For (1), if $I(U)$ and $I(V)$ are not orthogonal, then there is an infinite $a \in I(U) \cap I(V)$, and then $\emptyset \neq \overline{a} \setminus \omega \subset U \cap V$. Conversely if $U \cap V \neq \emptyset$, then $U \cap V$ contains a basic open set of the form $\bar{a} \setminus \omega$, and then $a \in I(U) \cap I(V)$. For (2), if the preideals are separated by some $a_0, \ldots, a_{n-1} \subset \omega$, then since $\bigcap_{i<n} a_i = \emptyset$, we get that $\bigcap_{i<n} \overline{a_i} = \emptyset$. Moreover, $U_i \subset \overline{a_i}$ because if $\bar{a} \setminus \omega$ is a basic clopen set inside U_i, then $a \in I(U_i)$ and therefore $a \subset^* a_i$ and $\overline{a} \setminus \omega \subset \overline{a_i}$. This proves that $\bigcap_{i<n} \overline{U_i} = \emptyset$. Conversely, if $\bigcap_{i<n} \overline{U_i} = \emptyset$, then there are clopen sets $C_i \supset U_i$ such that $\bigcap_{i<n} C_i = \emptyset$ (cf. [1, Lemma 9]). The sets C_i can be taken of the form $C_i = \overline{a_i} \setminus \omega$ with $\bigcap_{i<n} a_i = \emptyset$, and the a_i's separate the $I(U_i)$'s. Part (3) just follows from (1) and (2). Concerning (4), first notice that the closure of U_i is superfluous in that expression since $\overline{a}$ is open. The expression in (4) is just saying that, for $a \subset \omega$, $\overline{a}$ is disjoint from U_i if and only if it is disjoint from all basic clopens inside U_i. For part (5), if the gap can be B-broken, then we have a set $a \in \bigcap_{i\not\in B} I(U_i)^\perp$ such that $\{I(U_i)|_a : i \in B\}$ is a gap. But this means that $\overline{U_i} \cap \overline{a} = \emptyset$ for $i \not\in B$ and $\overline{a} \cap \bigcap_{i\in B} \overline{U_i} \neq \emptyset$. Therefore,

$$\bigcap_{i\in B} \overline{U_i} \setminus \bigcup_{i\not\in B} \overline{U_i} \supset \overline{a} \cap \bigcap_{i\in B} \overline{U_i} \neq \emptyset.$$

Conversely, if $\bigcap_{i\in B} \overline{U_i} \setminus \bigcup_{i\not\in B} \overline{U_i} \neq \emptyset$, then we can pick a point x in this set, and a clopen neighborhood V of x which is disjoint from $\bigcup_{i\not\in B} \overline{U_i}$. This clopen set V must be of the form $V = \overline{a} \setminus \omega$ for some $a \subset \omega$, and this set a witnesses that the gap $\{I(U_i) : i < n\}$ can be B-broken. □

After this lemma, we can automatically start getting corollaries from the results in Section 3.

Corollary 4.2. *Let $\mathcal{U}$ be a finite family of pairwise disjoint analytic open subsets of ω^*. Then*

1. *either their closures $\{\overline{U} : U \in \mathcal{U}\}$ are also pairwise disjoint,*
2. *or there exists a point $x \in \omega^*$ such that $|\{U \in \mathcal{U} : x \in \overline{U}\}| = 2$.*

Moreover, in the second case, unless all but two of those closures are disjoint from all others, there are in fact at least two points $x, y \in \omega^*$ for which the sets $\{U \in \mathcal{U} : x \in \overline{U}\}$ and $\{U \in \mathcal{U} : y \in \overline{U}\}$ are different and both of cardinality 2.

Proof. Suppose that (1) does not hold, and consider $\{U_i : i \in n\}$ a maximal subfamily of $\mathcal{U}$ with $\bigcap_{i\in n} \overline{U_i} \neq \emptyset$. We can find a clopen set $V \supset \bigcap_{i\in n} \overline{U_i}$ such that $V \cap \overline{U} = \emptyset$ for all $U \in \mathcal{U}$, $U \neq U_i$, $i < n$. The result follows from Lemma 4.1 and Theorem 1.1 applied to the gap $\{I(U_i \cap V) : i < n\}$. □

Corollary 4.3. *Let $\mathcal{U}$ be a countable family of pairwise disjoint analytic open subsets of ω^*, and let $\{U_i : i \in k\} \subset \mathcal{U}$ be a finite subfamily with $\bigcap_{i<k} \overline{U_i} \neq \emptyset$. Then, there exists a point $x \in \bigcap_{i<k} \overline{U_i}$ such that $|\{U \in \mathcal{U} : x \in \overline{U}\}| \leq J(k)$. Moreover, $J(k)$ is optimal in this result.*

Proof. First, consider the case when $\mathcal{U}$ is finite, and we write it in the form $\mathcal{U} = \{U_i : i \in n\}$, for some $k \leq n < \omega$. Pick $y \in \bigcap_{i<k} \overline{U_i}$, and let us suppose as well that we have $k \leq m \leq n$ such that $y \in \bigcap_{i<m} \overline{U_i}$ but $y \notin \bigcup_{m\leq i<n} \overline{U_i}$. Let C be a clopen subset of ω^* such that $y \in C$ and $C \cap \bigcup_{m\leq i<n} \overline{U_i} = \emptyset$. Since C is a clopen subset of ω^*, it is of the form $C = \bar{c} \setminus \omega$ for some infinite set $c \subset \omega$. By Theorem 3.6, we can find $B \supset A = k$ with $|B| \leq J(k)$ such that the gap $\{I(U_i \cap C) : i < m\}$ can be B-broken. This implies, by Lemma 4.1 that

$$\bigcap_{i\in B} \overline{U_i \cap C} \setminus \bigcup_{i\in m\setminus B} \overline{U_i \cap C} \neq \emptyset.$$

Any point x in the intersection above satisfies that $\{i \in n : x \in \overline{U_i}\} \subset B$, so it is as required.

Now, suppose that $\mathcal{U}$ is infinite, and write it as $\mathcal{U} = \{U_i : i < \omega\}$. We define inductively a decreasing sequence of clopen subsets C_n of ω^* and points $x_n \in C_n \cap \bigcap_{i<k} \overline{U_i}$. We can start with $C_0 = \omega^*$ and x_0 any point from $\bigcap_{i<k} \overline{U_i}$. Using the finite case proved above, we pick $x_{n+1} \in C_n \cap \bigcap_{i<k} \overline{U_i}$ (this set is nonempty, since we had x_n from the previous step) such that $|\{i < n+1 : x_{n+1} \in \overline{U_i}\}| \leq J(k)$. Then, we can choose $C_{n+1} \subset C_n$ so that $x_{n+1} \in C_{n+1}$ and C_{n+1} is disjoint from all $\overline{U_i}$ such that $x_{n+1} \notin \overline{U_i}$ and $i < n+2$. Let x be a cluster point of the sequence $\{x_n : n < \omega\}$. On the one hand, $x \in \bigcap_{i<k} \overline{U_i}$ since all x_n belong to that intersection. Suppose for contradiction that $|\{i < \omega : x \in \overline{U_i}\}| > J(k)$. Find n such that $|\{i < n : x \in \overline{U_i}\}| > J(k)$. But the construction of our sequence was done in such a way that $|\{i < n : y \in \overline{U_i}\}| \leq J(k)$ for all $y \in C_n$, and $x \in C_n$ because $x_m \in C_n$ for all $m \geq n$. This is a contradiction.

For the optimality, consider the gap on ω that witnessed optimality in Theorem 3.6. That is, we have $\Gamma = \{\Gamma_i : i < J(k)\}$ which cannot be B-broken for any $B \supset k$, $B \neq J(k)$. Consider the open sets $U_i = \bigcup_{a\in\Gamma_i} \overline{a} \setminus \omega$. It is easy to see that $\Gamma_i \subset I(U_i)$ and that every element of $I(U_i)$ is contained in a finite union of elements of Γ_i. From these facts we can transfer all properties of $\{\Gamma_i : i < n\}$ to $\{I(U_i) : i < J(k)\}$ which is, in this way, also a gap which cannot be B-broken for any $B \supset k$, $B \neq J(k)$. By Lemma 4.1, this means that the family $\{U_i : i < J(k)\}$ witnesses optimality. □

Finally, the direct translation of Theorem 3.5 is the following fact:

Corollary 4.4. *Whenever $\{U_i : i < n\}$ are pairwise disjoint analytic open subsets of ω^* such that $\bigcap_{i<n} \overline{U_i} \neq \emptyset$ and $\bigcup_{i<n} \overline{U_i} = \omega^*$, then there exists $k < n$ such that* *for all* $B \subset n$

$$\bigcap_{i\in B} \overline{U_i} \setminus \bigcup_{i\notin B} \overline{U_i} \neq \emptyset \quad \Rightarrow \quad \bigcap_{i\in B\cup\{k\}} \overline{U_i} \setminus \bigcup_{i\notin B\cup\{k\}} \overline{U_i} \neq \emptyset.$$

5. Sequences in Banach spaces

If $(x_k)_{k<\omega}$ is a sequence of vectors in a Banach space, having a preideal $I \subset \omega$ is equivalent to having a class of subsequences of $(x_k)_{k<\omega}$ which is hereditary, that is, it is closed under taking further subsequences. Thus, a gap $\{\Gamma_i : i < n\}$ is viewed as finitely many hereditary families of subsequences which are pairwise orthogonal (that is, no subsequence can be in two different classes at the same time) and are not separated. Separation would mean that there exists a decomposition $\omega = \bigcup_{i<n} b_i$ such that $(x_n)_{n\in b_i}$ does not contain any subsequence from Γ_i. This is just a reformulation of separation by looking at the sets $b_i = \omega \setminus a_i$ instead of the a_i's in the original definiton. Classes of sequences considered in Banach space theory are often analytic, sometimes coanalytic and rarely from higher projective classes. In any case, they are typically definable in some sense so the results of this paper apply. Just to give an example, for a real number $p \in [1,\infty)$, a sequence of vectors $x = \{x_k\}_{k<\omega}$ in a Banach space are said to be an ℓ_p-sequence if there exists a constant $L > 0$ such that

$$\frac{1}{L}\left(\sum_{i=1}^{m} a_i^p\right)^{1/p} \leq \left\|\sum_{i=1}^{m} a_i x_i\right\| \leq L\left(\sum_{i=1}^{m} a_i^p\right)^{1/p}$$

for all real numbers $a_1,\dots,a_m$. Clearly, the definition does not change if we take L and $a_1,\dots,a_m$ rational. It is obvious then that the class of ℓ_p-subsequences of a given sequence is Borel. After these explanations, it is clear that Theorem 1.4 is a corollary of Theorem 3.6 for $n = 2$, except for the fact that the constant $J(k)$ is sharp. This requires to check that the corresponding counterexample can be made with this particular kind of preideals, which is the content of the following statement:

Proposition 5.1. *Let $\{S_i : i < n\}$ be pairwise disjoint nonempty sets of types in $m^{<\omega}$, and let $\{p_i : i < n\}$ be numbers with $1 \leq p_i < \infty$. Then there exists a Banach space E and a sequence of vectors $\{x_k : k \in m^{<\omega}\} \subset E$ in a Banach space such that $\{x_k : k \in X\}$ is an ℓ_{p_i}-sequence whenever X is a set of type $\tau \in S_i$.*

Proof. Let $\{x_k : k \in m^{<\omega}\}$ be the canonical basis of the completion of $c_{00}(m^{<\omega})$ (the set of all functions $m^{<\omega} \longrightarrow \mathbb{R}$ which vanish out of a finite set) endowed with the norm

$$\|f\| = \sup\left\{\left(\sum_{k<\omega} |f(s_k)|^{p_i}\right)^{1/p_i} : \{s_0, s_1, \dots\} \text{ is of type } \tau \in S_i\right\}.$$

Just take into account that the intersection of two sets of different types has cardinality at most 2. $\square$

6. Asymptotic behavior of the number of types

The function J can be computed as follows:

$$J(n) = 2^n - 1 + \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\begin{array}{c} i+j-1 \\ j \end{array} \right) B(i,j,n);$$

$$B(i,j,n) = \left(\begin{array}{c} n \\ i \end{array} \right)\left(\begin{array}{c} n \\ j \end{array} \right) - \sum_{p=0}^{n-\max(i,j)} \left(\begin{array}{c} n-p-1 \\ j-1 \end{array} \right)\left(\begin{array}{c} n-p-1 \\ i-1 \end{array} \right).$$

In the first formula above, $2^n - 1$ is the number of chain types, $B(i,j,n)$ is the number of pairs (τ^0, τ^1) corresponding to types where τ^0 has i many numbers and τ^1 has j many numbers, while the combinatorial number that multiplies $B(i,j,n)$ is the number of possible order relations $\triangleleft$ once τ^0 and τ^1 are fixed increasing sequences of i many and j many numbers. In the second formula, the first summand is the total number of pairs of increasing sequences (τ^0, τ^1) while in each summand of the sum on the right we are subtracting those pairs of sequences that begin with the same integer p, that must be excluded. The first values of $J(n)$ are the following:

n	1	2	3	4	5	6	7	$\cdots$
$J(n)$	1	8	61	480	3881	31976	266981	$\cdots$

It is possible to find some recursive formulas as well. But it is perhaps more interesting to analyze the asymptotic behavior, to get an idea of the magnitude of the numbers. This is stated in Proposition 6.3, which requires some previous computations along two lemmas. We are not experts in probability, and for the proof of Lemma 6.1, we must acknowledge the hint provided in the web math.stackexchange.com (cf. question 353748).

Given $n < \omega$, and $p \in \mathbb{Z}$, let $M_n(p)$ be the set of all $2 \times n$ matrices with entries $\{-1, 0, 1\}$ such that the number of -1's in the upper row equals p plus the number of -1's in the lower row.

Given $u, v \in \{1, -1\}$, let $M_n(p)[{}^u_v]$ be the set of all matrices from $M_n(p)$ such that the first (leftmost) nonzero element of the upper row takes value u, and the first nonzero element of the lower row takes value v.

Let $M_n^{=}(p)[{}^u_v]$ be the set of all matrices in $M_n(p)[{}^u_v]$ such that the first nonzero element of the upper row and the first nonzero element of the lower row appear in the same column. Let $M_n^{\neq}(p)[{}^u_v]$ be the set of all matrices in $M_n(p)[{}^u_v]$ such that the first nonzero element of the upper row and the first nonzero element of the lower row appear in different columns.

Lemma 6.1. *For fixed p, as n goes to infinity,*

$$|M_n(p)| \sim \frac{3 \cdot 9^n}{2\sqrt{2\pi n}}.$$

Proof. We can consider our matrices as random matrices in which each entry takes the value 0, 1 or -1 with equal probability. Let X_i be the the random variable that provides the difference between number of -1's in the upper row and number of -1's in the lower row, but looking only at column i. Thus X_i takes value 0 with probability 5/9, value 1 with probability 2/9, and value -1 with probability 2/9.

We have that

$$|M_n(p)| = 9^n \cdot \mathbb{P}\left(\sum_{i=1}^n X_i = p\right).$$

The random variables X_i are independent, equidistributed, have mean $\mu = 0$ and variance $\sigma = 2/3$. Let

$$Y_n = \frac{3}{2\sqrt{n}} \sum_{i=1}^n X_i$$

be their standardized sums, which converge to a normal distribution $N(0,1)$. Let $F_n = \mathbb{P}(Y_n \le x)$ be the distribution function of Y_n. The X_i are concentrated in $\{-1,0,1\} \subset \{x_0 + dz : z \in \mathbb{Z}\}$ for $x_0 = 0$ and $d = 1$, so they are so-called lattice random variables. The way in which the functions $F_n(x)$ converge to the Gaussian $\Phi(x) = \int_{-\infty}^x \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$ is estimated by the following Esseen's formula [5, Theorem 3, p. 56]:

$$F_n(x) - \Phi(x) = \frac{\alpha_3(1-x^2)}{6\sigma^3\sqrt{2\pi n}} e^{-x^2/2} + \frac{d}{\sigma\sqrt{2\pi n}} Q_1\left(\frac{(x-\xi_n)\sigma\sqrt{n}}{d}\right) e^{-x^2/2} + o(n^{-1/2})$$

where $Q_1(t) = [t] - t + 1/2$ is the translation of the decimal part of t to the interval $(-1/2, 1/2]$ and $o(n^{-1/2})$ denotes a function f such that $n^{1/2} f$ converges to 0 uniformly on x as n goes to $+\infty$. The numbers ξ_n are described in [5, (29), p. 55] but in our case $\xi_n = 0$ since $x_0 = 0$. The other constants appearing in the formula take in our case the values $d = 1$, $\sigma = 2/3$ and the third moment [5, p.40] of the distributions X_i is $\alpha_3 = 0$, so we get:

$$F_n(x) - \Phi(x) = \frac{3}{2\sqrt{2\pi n}} Q_1\left(\frac{2x\sqrt{n}}{3}\right) e^{-x^2/2} + o(n^{-1/2}).$$

Thus,

$$\begin{aligned} 9^{-n} \cdot |M_n(p)| &= \mathbb{P}\left(Y_n = \frac{3p}{2\sqrt{n}}\right) \\ &= F_n\left(\frac{3p}{2\sqrt{n}}\right) - F_n\left(\frac{3(p-1)}{2\sqrt{n}}\right) \\ &= \Phi\left(\frac{3p}{2\sqrt{n}}\right) - \Phi\left(\frac{3(p-1)}{2\sqrt{n}}\right) + o(n^{-1/2}) \end{aligned}$$

and taking into account that $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} x + o(x)$, we get

$$9^{-n} \cdot |M_n(p)| = \frac{3}{2\sqrt{2\pi n}} + o(n^{-1/2}) \sim \frac{3}{2\sqrt{2\pi n}}$$

as desired. □

Lemma 6.2. *For fixed p, u, v, as n grows to infinity,*

$$|M_n^{\neq}(p)[{}^u_v]| \sim |M_n^{=}(p)[{}^u_v]| \sim \frac{3 \cdot 9^n}{16\sqrt{2\pi n}}.$$

Proof. We are going to prove that $|M_n^=(p)[^u_v]| \gtrsim \frac{1}{8}M_n(p)$ and $|M_n^{\neq}(p)[^u_v]| \gtrsim \frac{1}{8}M_n(p)$. Since

$$\begin{aligned} |M_n(p)| &= |M_n^=(p)[^1_1]| + |M_n^=(p)[^{+1}_{-1}]| + |M_n^=(p)[^{-1}_{+1}]| + |M_n^=(p)[^{-1}_{-1}]| + \\ & \quad |M_n^{\neq}(p)[^1_1]| + |M_n^{\neq}(p)[^{+1}_{-1}]| + |M_n^{\neq}(p)[^{-1}_{+1}]| + |M_n^{\neq}(p)[^{-1}_{-1}]| \end{aligned}$$

we conclude that $|M_n^=(p)[^u_v]| \sim \frac{1}{8}|M_n(p)|$ and $|M_n^{\neq}(p)[^u_v]| \sim \frac{1}{8}|M_n(p)|$, and the statement of the lemma will follow from Lemma 6.1. One corollary of Lemma 6.1 that we are going to use in the sequel is that for any fixed q and k, $|M_{n-k}(q)| \sim 9^{-k}|M_n(p)|$. We start by proving that $|M_n^=(p)[^u_v]| \gtrsim \frac{1}{8}M_n(p)$. If we fix $m < n$, we can get a lower bound by counting only those matrices whose first nonzero elements appear in one of the first columns, and we get

$$|M_n^=(p)[^u_v]| \geq \sum_{k=1}^{m} |M_{n-k}(p')|$$

where p' equals either p,$p+1$ or $p-1$, depending on the values of u and v. Asymptotically,

$$|M_n^=(p)[^u_v]| \gtrsim \sum_{k=1}^{m} 9^{-k}|M_n(p)|.$$

Since this happens for every m, and $\sum_{k=1}^{\infty} 9^{-k} = 8^{-1}$, we get that

$$|M_n^=(p)[^u_v]| \gtrsim \frac{1}{8}M_n(p)$$

as desired. We check now that $|M_n^{\neq}(p)[^u_v]| \gtrsim \frac{1}{8}|M_n(p)|$. Given $k, s \geq 1$ with $s + k < n$, the number of matrices in $M_n^{\neq}(p)[^u_v]$ whose first nonzero element of the upper row appears at column s, and the first nonzero element of the lower row appears at column $s + k$ can be computed as

$$\sum_{\xi \in 3^k} |M_{n-s-k}(p_\xi)|$$

where ξ runs over all possible entries in the upper row between $s + 1$ and $s + k$, and p_ξ is a number which depends only on ξ. Similarly, the number of matrices in $M_n^{\neq}(p)[^u_v]$ whose first nonzero element of the lower row appears at column s, and the first nonzero element of the upper row appears at column $s+k$ can be computed as

$$\sum_{\xi \in 3^k} |M_{n-s-k}(p'_\xi)|.$$

In this way, if we fix m, we can estimate for $n > 2m$,

$$\begin{aligned} |M_n^{\neq}(p)[^u_v]| &\geq \sum_{k=1}^{m}\sum_{s=1}^{m}\sum_{\xi \in 3^k} |M_{n-s-k}(p_\xi)| + |M_{n-s-k}(p'_\xi)| \\ &\sim \sum_{k=1}^{m}\sum_{s=1}^{m}\sum_{\xi \in 3^k} \frac{2|M_n(p)|}{9^{s+k}} \sim 2|M_n(p)| \left(\sum_{k=1}^{m}\sum_{s=1}^{m} \frac{3^k}{9^{s+k}}\right) \\ &\sim 2|M_n(p)| \left(\sum_{k=1}^{m} \frac{1}{3^k}\right)\left(\sum_{s=1}^{m} \frac{1}{9^s}\right). \end{aligned}$$

Since this happens for every m, and $\sum_1^\infty 3^{-k} = 2^{-1}$ and $\sum_1^\infty 9^{-s} = 8^{-1}$, we get that $|M_n^{\neq}(p)[^u_v]| \gtrsim \frac{1}{8}|M_n(p)|$ as desired. □

Proposition 6.3.

$$J(n) \sim \frac{3}{8\sqrt{2\pi n}} \cdot 9^n$$

Proof. To each type τ in $n^{<\omega}$ we associate a $2 \times n$ matrix (a^τ_{ij}) with entries in $\{-1, 0, 1\}$. This time it is convenient to enumerate the rows of the matrix by the indices $i = 0, 1$, and the columns by the indices $j = 0, \ldots, n-1$. The matrix is defined as follows:

- $a^\tau_{ij} = 0$ if $j \not\in \tau^i$,
- $a^\tau_{ij} = 1$ if $j \in \tau^i$ and the immediate predecessor of (j, i) in $\lhd$ is of the form (k, i),
- $a^\tau_{ij} = -1$ if $j \in \tau^i$ and the immediate predecessor of (j, i) in $\lhd$ is of the form $(k, 1-i)$, or there is no immediate predecessor.

It is a simple exercise to check that

$$\{(a^\tau_{ij}) : \tau \text{ is a comb type in } n^{<\omega}\} = M^{\neq}_n(0)\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right] \cup M^{\neq}_n(-1)\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right].$$

The cardinality of the set of chain types in $n^{<\omega}$ is $2^n - 1$, so

$$J(n) = |M^{\neq}_n(0)\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]| + |M^{\neq}_n(-1)\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]| + 2^n - 1,$$

and using Lemma 6.2 the proof is over. □

7. Final comments

The results stated in this paper are very far from giving an understanding of the family $\mathfrak{B}(\Gamma)$. One may wonder if it is possible to find an internal characterization of those families of subsets of n that are of the form $\mathfrak{B}(\Gamma)$ for some analytic gap Γ. Even basic questions like the following are unknown to us: Are there always sets in $\mathfrak{B}(\Gamma)$ of any possible cardinality? How many at least? On the other hand, what we do in this paper is to look at several families of subsequences of a given sequence and studying how they can be separated. This seems a basic situation that could arise in different areas of mathematics. The examples provided in ω^* and with ℓ_p-sequences in Banach spaces illustrate this. It is natural to expect that other applications may appear.

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