"Banach spaces and topology (I) (For Encyclopedia on General Topology -Elsevier)"

June 26, 2001

Banach spaces were defined by S. Banach and others (notably N. Wiener) independently in the 1920's. However it was Banach's 1932 monograph [2] that made the theory of Banach spaces ("espaces du type (B)" in the book) an indispensable tool of modern analysis. The novel idea of Banach is to combine point-set topological ideas with the linear theory in order to obtain such powerful theorems as Banach-Steinhaus theorem, open-mapping theorem and closed graph theorem. Both general topology and theory of Banach spaces continue to benefit from cross-fertilization of analysis and topology. Some of which can be seen in the following pages.

1. Definitions and basic properties. The material of this section is quite standard and details can be found in any textbook on functional analysis. Let X be a linear space (=vector space) over the field $\mathbb{K}$ (the field of **scalars**) where $\mathbb{K}$ is either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. A **norm** $\| \ \|$ on X is a function $x \mapsto \|x\|$ on X into $[0, \infty)$ satisfying the following conditions: (i) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$; (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}, x \in X$; and (iii) $\|x\| = 0$ if and only if $x = 0$. A linear space X equipped with a fixed norm $\| \ \|$ is called a **normed linear space** or **normed space** and is denoted by $(X, \| \ \|)$ or simply by X when no ambiguity is likely. When one wishes to specify $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$), the normed space is referred to as a **real** (or **complex**) **normed (linear) space**. The **unit ball** of the normed space X is the set $B_X \equiv \{x \in X : \|x\| \leq 1\}$ and the **unit sphere** of X is $S_X \equiv \{x \in X : \|x\| = 1\}$. The norm $\| \ \|$ on the normed space X defines the *metric* d (called the **norm metric**) on X by $d(x, y) = \|x - y\|$, $x, y \in X$. The topology of the norm metric is called the **norm topology**, and the normed space is always assumed to have the norm topology unless other topology is specified. Two norms $\| \ \|_1, \| \ \|_2$ on the linear space X are said to be **equivalent** if the norm topologies of the two norms coincide, and this is the case if and only if there are positive numbers a and b such that for each $x \in X$, $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$. A **Banach space** is a *complete* normed space. If Y is a linear subspace of X, where $(X, \| \ \|)$ is a normed space, then we let Y itself be normed with the restriction of $\| \ \|$ to Y. If X is a Banach space and if Y is a closed linear subspace of X, then Y is a Banach space.

A linear map between normed spaces is continuous if it is continuous at a point, and in this case the map is a *Lipschitz map.* A scalar-valued liner map f on a normed space is called a **linear functional** on X. The linear functional f is continuous if and only if the null space $f^{-1}(\{0\})$ is closed, and this is the case if and only if $\|f\| \equiv \sup\{|f(x)| : x \in B_X\} < \infty$. The linear space X^* of all continuous linear functionals on X is a Banach space with the norm $\| \ \|$ given above. The Banach space X^* is called the **dual (space)** of X. One of the fundamental facts on normed spaces is the following **Hahn-Banach extension theorem**. Let X be a normed space and let Y be a linear subspace. If $g \in Y^*$

then there exists an $f \in X$ such that $g = f|_Y$ and $\|f\| = \|g\|$. The theorem in particular implies that, for each $x \in X$, $\|x\| = \sup\{|f(x)| : f \in B_{X^*}\}$. This formula can be interpreted as follows. Let X^{**} be the **bidual** of X, namely $(X^*)^*$. Then there is a canonical linear map $x \mapsto \hat{x}$ from X into X^{**} given by $\hat{x}(f) = f(x)$ for all $x \in X$, $f \in X^*$. The formula above says that the map $x \mapsto \hat{x}$ is an **isometry**, *i.e.* $\|x\| = \|\hat{x}\|$ for each $x \in X$. It is often convenient to identify x and $\hat{x}$ and regard X a subspace of X^{**} (the **canonical embedding** of X into its bidual). If $X = X^{**}$, then X, which is necessarily a Banach space, is said to be **reflexive**.

Two Banach spaces X and Y are said to be **isometric** if there is a one-to-one linear map T of X onto Y such that $\|T(x)\| = \|x\|$ for each $x \in X$. We give some examples of standard Banach spaces.

Example 1. Let T be a *completely regular* topological space and let $C(T)$ (resp. $C_b(T)$) be the space of all scalar-valued continuous (resp. bounded continuous) functions on T. For each $f \in C_b(T)$ the **supremum norm** $\|f\|$ of f is defined by $\|f\| = \sup\{|f(t)| : t \in T\}$. Then $(C_b(T), \| \enspace \|)$ is a Banach space. If βT denote the *Stone-Čech compactification* of T, then the Banach spaces $C_b(T)$ and $C(\beta T)$ are isometric. So, here we restrict our discussion to $C(K)$ where K is a compact Hausdorff space. Then for each $\varphi \in C(K)^*$ there is a unique scalar-valued *Radon measure* μ on K such that $\varphi(f) = \int_K f \, d\mu$ for all $f \in C(K)$ (**Riesz representation theorem**). In this case $\|\varphi\|$ is equal to the total variation of μ on K. Hence $C(K)^*$ is isometric to the space $M(K)$ of all scalar-valued Radon measures on K with the total variation as the norm.

Example 2. (Hilbert spaces) Let H be a linear space over $\mathbb{C}$. An **inner product on H** is a complex-valued function $(x, y) \mapsto \langle x, y \rangle$ on $H \times H$ satisfying (i) $\overline{\langle y, x \rangle} = \langle x, y \rangle$ for each $(x, y) \in H \times H$; (ii) For each $y \in H$, the map $x \mapsto \langle x, y \rangle$ is linear on H; and (iii) $\langle x, x \rangle > 0$ for each $x \in H$ not equal to 0. An **inner product space** or **pre-Hilbert space** is a complex linear space H equipped with a fixed inner product $\langle \, , \, \rangle$. The inner product space H is a normed space with the norm defined by $\|x\| = \langle x, x \rangle^{1/2}$. If H is complete with respect to this norm, H is called a **Hilbert space**. If H is a Hilbert space and if $f \in H^*$, then there is a unique $y \in H$ such that $f(x) = \langle x, y \rangle$ for each $x \in H$. From this it follows that a Hilbert space is automatically a reflexive Banach space.

Example 3. Let Γ be an arbitrary set. For a non-negative function w on Γ, we define

$$\sum_{\gamma \in \Gamma} w(\gamma) = \sup\{\sum_{\gamma \in F} w(\gamma) : F \subset \Gamma, \ F \text{ finite}\}.$$

If w is a scalar-valued function on Γ such that $\sum_{\gamma \in \Gamma} |w(\gamma)| < \infty$, then w is called **summable**. Since in this case the support of w is countable, $\sum_{\gamma \in \Gamma} w(\gamma)$ is well-defined.

For each $p \in [1, \infty)$, let $\ell^p(\Gamma)$ be the space of all scalar-valued functions f such that $\|f\|_p \equiv (\sum_{\gamma \in \Gamma} |f(\gamma)|^p)^{1/p} < \infty$. Then $(\ell^p(\Gamma), \| \enspace \|_p)$ is a Banach space. We also let $\ell^\infty(\Gamma) = C_b(\Gamma)$ with Γ given the discrete topology. Then by Example 1, $\ell^\infty(\Gamma)$ is a Banach space with the supremum norm $\| \enspace \|_\infty$. Now suppose $p \in [1, \infty)$ and $\varphi \in (\ell^p(\Gamma))^*$. Then there is a unique $g \in \ell^q(\Gamma)$, where $1/p + 1/q = 1$ ($q = \infty$ if $p = 1$), such that fg is summable and $\varphi(f) = \sum_{\gamma\in\Gamma} f(\gamma)g(\gamma)$. Furthermore in this case $\|\varphi\| = \|g\|_q$. So $(\ell^p(\Gamma))^*$ and $\ell^q(\Gamma)$ are isometric. It follows that the space $\ell^p(\Gamma)$ is reflexive if $p \in (1, \infty)$. The spaces $\ell^1(\Gamma)$ and $\ell^\infty(\Gamma)$ are not reflexive unless Γ is finite. In case $p = 2$ and $\mathbb{C}$ is the scalar field, $\ell^2(\Gamma)$ is a Hilbert space with the inner product given by $\langle f, g\rangle = \sum_{\gamma\in\Gamma} f(\gamma)\overline{g(\gamma)}$. It can be shown that each Hilbert space H is isometric to $\ell^2(\Gamma)$ for a suitable Γ: if H is infinite dimensional the cardinality of Γ is the *density* of H. Hence the isometric classification of Hilbert spaces is simply a matter of the cardinality of Γ.

Example 4. Let Γ be as in the previous example, and let $c_0(\Gamma)$ be the space of all scalar-valued functions f that **vanish at infinity**, *i.e.* for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma : |f(\gamma)| \geq \varepsilon\}$ is finite. Then $c_0(\Gamma)$ is a closed linear subspace of $\ell^\infty(\Gamma)$, and so it is a Banach space with the supremum norm. If $\varphi \in (c_0(\Gamma))^*$, then there is a unique $g \in \ell^1(\Gamma)$ such that, for each $f \in c_0(\Gamma)$, $\varphi(f) = \sum_{\gamma\in\Gamma} f(\gamma)g(\gamma)$, and in this case $\|\varphi\| = \|f\|_1$. This shows that $(c_0(\Gamma))^*$ is isometric to $\ell^1(\Gamma)$ and that the bidual $(c_0(\Gamma))^{**}$ is isometric to $\ell^\infty(\Gamma)$.

2. The norm topology. In this section, we describe several topics related to the norm topology of the Banach spaces.

(A) Category theorems. Since a Banach space is a complete metric space, it is a Baire space. This fact has the following important and extremely useful consequences.

(a) **Banach-Steinhaus theorem** (or **uniform boundedness principle**). Let $\{T_\alpha : \alpha \in A\}$ be a family of continuous linear maps of a Banach space X into normed spaces. If, for each $x \in X$, $\sup\{\|T_\alpha(x)\| : \alpha \in A\} < \infty$, then $\sup\{\|T_\alpha(x)\| : \alpha \in A, x \in B_X\} < \infty$. In particular, if A is a subset of the dual X^* of a Banach space X such that $\sup\{|f(x)| : f \in A\} < \infty$ for each $x \in X$, then A is **bounded** in the sense that $\sup\{\|f\| : f \in A\} < \infty$. Similarly if A is a subset of a *normed* space X such that $\sup\{|f(x)| : x \in A\} < \infty$ for each $f \in X^*$, then A is bounded.

(b) **Open mapping theorem.** Let T be a continuous linear map of a Banach space X onto a Banach space Y. Then T is an *open* map. Banach spaces X and Y are said to be **isomorphic** if there is a one-to-one continuous linear map T of X onto Y. In this case the inverse map $T^{-1} : Y \to X$ is also continuous by the open mapping theorem, and hence T is a linear homeomorphism.

(c) **Closed-graph theorem.** Let T be a linear map on a Banach space X into a Banach space Y such that the *graph* of T, namely $\{(x, T(x)) : x \in X\}$, is closed in $X \times Y$. Then T is continuous.

(B) Maps into Banach spaces Here we mention two theorems. For the proof and the original sources, we refer the reader to [4] and the bibliography therein.

(a) **Michael's selection theorem**. Let E be a *paracompact* space and let φ be a multivalued map on E taking values among non-empty closed convex subsets of a Banach space X, which is *lower semicontinuous*, *i.e.* for each open subset U of X, the set $\{t \in E : \varphi(t) \cap U \neq \emptyset\}$ is open in E. Then (i) the map φ has a continuous *selector*, *i.e.* a continuous function $f : E \to X$ such that $f(t) \in \varphi(t)$ for each $t \in E$, and (ii) if $A \subset E$ is closed, then each continuous selector for $\varphi|A$ extends to a continuous selector for φ. It follows from this theorem that if X, Y are Banach spaces and if T is a continuous linear map of X onto Y, then there is a continuous map $f : Y \to X$ such that $T(f(y)) = y$ for all $y \in Y$ (**Bartle-Graves's theorem**).

(b) **Borsuk-Dugundji extension theorem**. Let A be a closed subset of a metrizable space M and let X be a normed space. Also let $C(A, X)$ (resp. $C(M, X)$) be the space of all continuous functions on A (resp. M) into X. Then there is a linear map $L : C(A, X) \to C(M, X)$ such that, for each $f \in C(A, X)$, $L(f)$ is an extension of f and the range of $L(f)$ is contained in the convex hull of $f(A)$.

(C) The topological classification. Kadec proved in 1966 that all infinite-dimensional separable Banach spaces are homeomorphic to $\ell^2(\mathbb{N})$, see [4, Corollary 9.1, p. 231]; moreover Anderson, 1966, proved that $\ell^2(\mathbb{N})$ is homeomorphic to the countable infinite product of real lines, i.e. $\mathbb{R}^{\mathbb{N}}$. Finally if X is a separable infinite-dimensional Banach space, then the spaces X, B_X and S_X are all homeomorphic to $\mathbb{R}^{\mathbb{N}}$, see [4, Corollary 5.1, p. 188]. The ultimate answer to the topological classification problem was given by Toruńczyk [12] in 1981: two infinite dimensional Banach spaces are homeomorphic if their *densities* are equal. We also mention that each infinite-dimentional compact convex subset of a Banach space is homeomorphic to the *Hilbert cube* $[0, 1]^{\mathbb{N}}$ (Keller, 1931; see [4, p. 100]).

(D) Spaces of type $C(K)$. According to the **Banach-Stone theorem**, a compact Hausdorff space is characterized by the Banach space of all real-valued continuous functions with the supremum norm: if K, L are compact Hausdorff spaces and if $C(K, \mathbb{R})$ is isometric to $C(L, \mathbb{R})$, then K and L are homeomorphic (see [10, Theorem 7.8.4]). On the other hand, the isomorphic type of $C(K)$ is not sufficient to characterize K, for by **Milutin's theorem**, if K and L are uncountable metrizable compact spaces, then $C(K)$ and $C(L)$ are isomorphic (see [10, 21.5.10]).

(E) Uniform and Lipschitz classification. As said in Section 2(E) the topological structure of a Banach space contains no information on its linear structure. Nonetheless, it was stated by Mazur-Ulam, 1932, that any map preserving the norm metric from a Banach space onto another one and sending 0 to 0 is a linear isometry in the sense of Section 1, [3, Theorem 14.1]. The idea of classifying Banach spaces using Lipschitz or more generally uniformly continuous non-linear maps is a relatively new and very active area of research. The Benyamini-Lindenstrauss monograph [3] is the place to start if one is interested in this subject. We cite only a few results here. Let us say that two metric spaces are **Lipschitz equivalent** (resp. **uniformly equivalent**) if there is a map of one onto the other such that the map and its inverse are both Lipschitz (resp. uniformly continuous). Such a map is called a **Lipschitz equivalence** (resp. **uniform equivalence**).

Let X and Y be Banach spaces and let f be a function defined on an open subset U of X into Y. Then f is said to be **Gâteaux differentiable** at $a \in U$ if there is a continuous linear map $T : X \to Y$ such that for each $h \in X$, $\lim_{t\to 0} t^{-1}(f(a+th) - f(a)) = T(h)$. If this limit converges uniformly for all $h \in S_X$, then f is **Fréchet differentiable** at a. The map T is called the **differential** of f at a and is denoted by $D_a f$.

Now suppose that f is a Lipschitz equivalence of X onto a subset of Y. If f is Gâteaux differentiable at $a \in X$, then $D_a f$ is an isomorphism (*i.e.* linear homeomorphism) of X to a subspace of Y. If X is separable and Y has the *Radon-Nikodým property* (RNP), then any Lipschitz map of X into Y is Gâteaux differentiable at some point (see [3, Theorem 6.42]). Hence in this case X is Lipschitz equivalent to a subset of Y if and only if X is isomorphic to a subspace of Y. Heinrich and Mankiewicz have shown (1980) that the above holds true without the RNP of Y if $Y = Z^*$ for some Banach space Z (see [3, Theorem 7.10]). Thus if a separable Banach space X is Lipschitz equivalent to a subset of a Banach space Y, then X is isomorphic to a subspace of Y^{**}. All reflexive Banach spaces and separable dual Banach spaces have the RNP. Among our examples, $C(K)$ does not have the RNP if K is infinite compact and Hausdorff. For an infinite Γ, $\ell^p(\Gamma)$ has the RNP if $1 \le p < \infty$, and $\ell^\infty(\Gamma)$, $c_0(\Gamma)$ do not have the RNP. Without the RNP and outside dual Banach spaces the situation is quite different: any separable metric space is Lipschitz homeomorphic to a subset of c_0, Aharoni 1974 [3, Theorem 7.11]. The Lipschitz or uniform classification of Banach spaces is known only for a limited classes of Banach spaces. For instance Deville, Godefroy and Zizler have proved (1990) that, for K compact, $C(K)$ is Lipschitz equivalent to some $c_0(\Gamma)$ if and only if $K^{\omega_0} = \emptyset$ (see [3, Theorem 7.13]).

3. Weak and weak* topologies. Let X be a normed space. The **weak topology** for X (resp. **weak***) is the weakest topology for X (resp. X^*) that makes $x \mapsto f(x)$ (resp. $f \mapsto f(x)$) continuous for each $f \in X^*$ (resp. $x \in X$). Note that the dual space X^* also has the weak topology since it is a Banach space. The space X with the weak topology is denoted by (X, w), and similarly (X^*, w^*) is the dual X^* with the weak* topology. If Y is a linear subspace of a normed space X, then, by the Hahn-Banach extension theorem, the weak topology for Y is the restriction of the weak topology for X. Both (X, w) and (X^*, w^*) are Hausdorff locally convex spaces, i.e., 0 has a basis of convex neighborhoods and the vector sum and the multiplication by members of $\mathbb{K}$ are continuous. Clearly on X the weak topology is weaker than the norm topology, and on X^* the weak* topology is weaker than the weak topology. However the Hahn-Banach extension theorem implies that if C is a (norm) closed convex subset of X, then C is weak-closed. A linear functional f on X is weakly continuous if and only if it is norm-continuous, *i.e.* $f \in X^*$. More generally, a linear map from a normed space into another is (norm-norm) continuous if and only if it is weak-weak continuous. A linear functional φ on X^* is weak*-

continuous if and only if there exists an $x \in X$ such that $\varphi(x) = f(x)$ for each $f \in X^*$. When X is embedded in its bidual (*cf.* Section 1), X is weak*-dense in X^{**} and B_X is weak*-dense in $B_{X^{**}}$.

One of the pleasant features of the weak and weak* topologies is that compact sets are relatively easy to come by. Whereas B_X is never norm-compact, unless X is finite dimensional, it is weak-compact if and only if X is reflexive. Thus the unit balls of Hilbert spaces and of $\ell^p(\Gamma)$, with $1 < p < \infty$ and Γ arbitrary, are all weak-compact. Moreover, by *Tychonoff's theorem*, B_{X^*} is always weak*-compact (**Banach-Alaoglu's theorem**). Let X be a normed space and let K be (B_{X^*}, w^*). Then K is a compact Hausdorff space, and there is a natural linear map $\varphi : X \to C(K)$ given by $\varphi(x)(f) = f(x)$ for each $x \in X, f \in K = B_X^*$. Clearly φ is an isometry. If τ_p denotes the *topology of pointwise convergence* for $C(K)$, then φ maps (X, w) homeomorphically onto $(\varphi(X), \tau_p)$ and $\varphi(X)$ *separates* points of K. If X is a Banach space, then $\varphi(X)$ is τ_p-closed in $C(K)$. This shows that the study of the weak topology is very closely related to that of the pointwise topology for the spaces of the type $C(K)$ with K compact. Below we discuss a few topics related to weak and weak* topologies in more detail.

(A) Weak and weak*-compact sets. It had been observed by Šmulian, Eberlein and others that weak-compact subsets of Banach spaces possess properties similar to those of metrizable spaces. This can be summarized by saying that, for each Banach space X, (X, w) is **angelic**, where a regular Hausdorff space is said to be angelic if the closue of each relatively countably compact set A is compact and the closure consists of the limits of sequences in A. This, in turn, is a consequence of the fact that, for each compact Hausdorff space K, $(C(K), \tau_p)$ is angelic, [7]. Corson and Lindenstrauss (1966) conjectured that a weak-compact subset of a Banach space is homeomorphic to a weak-compact subset of $c_0(\Gamma)$ for a suitable set Γ. This conjecture was confirmed in the ground-breaking paper Amir-Lindenstrauss [1], in which the authors have suggested that a space homeomorphic to a weak-compact subset of a Banach space be called an *Eberlein compact* (EC). The confirmation is based on the main theorem of [1]. A Banach space X is said to be **weakly compactly generated** (WCG) if X is generated by a weak-compact subset $K \subset X$, *i.e.* the linear span of K is dense in X. The **Amir- Lindenstrauss theorem** states that if a Banach space X is WCG, then there exist a set Γ and a one-to-one continuous linear map on X into $c_0(\Gamma)$. For properties of Eberlein compacta, see the article on "Eberlein and Corson compacta". Here we mention two properties of weak-compact sets related to the linear structure of Banach spaces. If K is a weak-compact subset of a Banach space, then the closed convex hull of K is weak-compact (Krein-Šmulian, 1940). This can be seen as a consequence of the remarkable, but much harder to prove, **James's theorem** (1972), [8]: if A is a subset of a real-Banach space X such that for each $f \in X^*$ there exists an $a \in A$ satisfying $f(a) = \sup\{f(x) : x \in A\}$, then the weak-closure of A is weak-compact.

There is nothing remarkable about weak*-compact subsets of dual Banach spaces. In fact any compact Hausdorff space is homeomorphic to a weak*-compact subset of a dual Banach space. However, weak*-compact subsets of the dual of particular Banach spaces can be more special. A Banach space X is called an **Asplund space** if each real-valued convex continuous function defined on an open convex subset $U \subset X$ is Fréchet differentiable (*cf.* Section 2(E)) at each point of a dense G_δ subset of U. It is known that X is an Asplund space if and only if X^* has the *Radon-Nikodým property* (RNP). A compact space is said to be *Radon-Nikodým compact* (RN compact) if it is homeomorphic to a weak*-compact subset of the dual of an Asplund space. The properties of RN compact spaces are very similar to those of EC, and the class of EC is properly contained in the class of RN compact spaces. See the article on "Radon-Nikodým" compacta. Section 3(B) below describes another classes of Banach spaces for which their dual spaces have special w^*-compact subsets.

(B) Topological properties of (X, w). The topological study of Banach spaces with the weak topology is a far more subtle matter than that for the norm topology (see Section 2(C)). Here the main problems arise in the non-separable case. In [6] Corson proved that if X is a Banach space, then (X, w) is *paracompact* if and only if (X, w) is *Lindelöf*, and conjectured that (X, w) is Lindelöf if and only if X is WCG. More recently Rezničenko proved that (X, w) is normal if and only if it is Lindelöf. Talagrand proved that if a Banach space X is WCG, then (X, w) is K-*analytic* hence Lindelöf confirming a one-half of the Corson conjecture, see [11] for a detailed account about weakly K-analytic Banach spaces. The converse, however, is false since Rosenthal has constructed a WCG Banach space which has a non-WCG closed linear subspace.

It is proved in [1] that a compact Hausdorff space K is EC if and only if $(C(K), w)$ is WCG. Analogously, it is possible to classify K according to topological properties of $(C(K), w)$. Thus a compact Hausdorff space K is said to be **Talagrand compact** (resp. **Gul'ko compact** if $(C(K), w)$ is K-*analytic* (resp. K-countably determined). Here a *Tychonoff* space T is said to be K**-countably determined** if there are a compactification Z of T and a sequence $\{K_n : n \in \mathbb{N}\}$ of compact sets in Z such that, for each $t \in T$, $t \in \bigcap\{K_n : t \in K_n, n \in \mathbb{N}\} \subset T$. For X Banach space, (X, w) is K-analytic (resp. K-countably determined) if and only if (B_{X^*}, w^*) is Talagrand (resp. Gul'ko) compact space. We have the implications EC $\Rightarrow$ Talagrand compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ *Corson compact*, and none of the arrows can be reversed. (See the article by S. Negrepontis in [KV].)

Finally, we mention a striking result of Rosenthal [9]. By investigating thoroughly what it means for a uniformly bounded sequence of real-valued functions on a set *not* to have a convergent subsequence, he has proved the following: Let X be a separable real Banach space. Then X does not have any isomorphic copy of $\ell^1(\mathbb{N})$ if and only if each member of $B_{X^{**}}$ is the weak*-limit of a sequence in B_X. This means that $B_{X^{**}}$ is a pointwise compact set of *first Baire class* functions on the *Polish space* (B_{X^*}, w^*). A compact Hausdorff space is called **Rosenthal compact** if it is homeomorphic to a pointwise compact subset of the space $B_1(\Omega)$ of first Baire class functions on a Polish space Ω. A real breakthrough in this area was made by Bourgain, Fremlin and Talagrand [5] who proved that, for each Polish space Ω, $B_1(\Omega)$ is angelic with respect to the pointwise topology. Compare the last result with the fact that $(C(K), \tau_p)$ is angelic for each compact Hausdorff space K. See the article on "Rosenthal Compacta".

References

[1] D. Amir and J. Lindenstrauss The structure of weakly compact sets in Banach spaces. *Ann. of Math.*, 88 (1968), 35–46.

[2] S. Banach. *Théorie des óperations linéaires.* Monografje Matematyczne Tom 1, Warszawa, 1932

[3] Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1.* American Mathematical Society, Providence, RI, 2000.

[4] C. Bessaga and A. Pełczyński. *Selected topics in infinite-dimensional topology.* PWN—Polish Scientific Publishers, Warsaw, 1975. Monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58].

[5] J. Bourgain, D. Fremlin, and M. Talagrand. Pointwise compact sets of Baire measurable functions. *Amer J. Math.*, 100:845–886, 1978.

[6] H. H. Corson. The weak topology of a Banach space. *Trans. Amer. Math. Soc.*, 101:1–15, 1961.

[7] A. Grothendieck. Critères de compacité dans les espaces fontionnelles généraux. *Amer. J. Math*, 74:168–186, 1952.

[8] R. C. James. Weakly compact sets. *Trans. Amer. Math. Soc.*, 113:129–140, 1964.

[9] H. P. Rosenthal. Some recent discoveries in the isomorphic theory of Banach spaces. *Bull. Amer. Math. Soc.*, 84 (1978), 803–831.

[10] Z. Semadeni. Banach spaces of continuous functions Vol. 1. Monografie Matematyczne Tom 55, Warszawa, 1971

[11] M. Talagrand. Espaces de Banach faiblement K-analytiques. *Ann. of Math.*, 110:407–438, 1979.

[12] H. Toruńczyk. Characterizing Hilbert space topology. *Fund. Math.*, 111(3):247–262, 1981.

B.Cascales, I. Namioka*, J.Orihuela and M. Raja

Departamento de Mathematicás. Universidad de Murcia. 30100 Espinardo. Murcia. Spain

* University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98185-4350, U.S.A.