A biased view of topology as a tool in functional analysis

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Abstract A survey about *"Topology as a tool in functional analysis"* would be such a giant enterprise that we have, naturally, chosen to give here *"Our biased views of topology as a tool in functional analysis"*. The consequence of this is that a big portion of this long paper deals with topics that we have been actively working on during the past years. These topics range from metrizability of compact spaces (and their consequences in functional analysis), networks in topological spaces (and their consequences in renorming theory of Banach spaces), distances to spaces of functions (and their applications to the study of pointwise and weak compactness), James' weak compactness theorem (and their applications to variational problems and risk measures). Some of the results collected here are a few years old while many others are brand new. A few of them are first published here and most of them have been often used in different areas since their publication. The survey is completed with a section devoted to references to some of what we consider the last major achievements in the area in recent years.

1 Introduction

The interaction between functional analysis and topology goes back to their origins and has deepened and widened over the years. Going back to history we have to highlight Banach's 1932 monograph [20] that made the theory of Banach spaces ("espaces du type (B)" in the book) an indispensable tool of modern analysis. The novel idea of Banach is to combine point-set topological ideas with the linear theory in order to obtain such powerful theorems as Banach-Steinhaus theorem, open-mapping theorem and closed graph the-

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orem. For almost a century already general topology and functional analysis continue to benefit from each other.

The aim of this survey is to give *"Our biased views of topology as a tool in functional analysis"*, with particular stress in *"recent"* results. It would be, of course, too pretentious if we even tried to write about the general role of topology as a tool for functional analysis. Without any doubt, there are results much more important than those collected here and, of course, other authors might have different views.

Let us describe very briefly without any further ado the contents of this survey. Here are the different sections of the paper:

1.- Introduction.
2.- Metrizability of compact spaces with applications to functional analysis.
3.- Topological networks meet renorming theory in Banach spaces.
4.- Recent views about pointwise and weak compactness.
5.- Concluding references and remarks.

A subsection devoted to *"Notation and terminology"* follows this short introduction. Then we have 4 more sections. Sections 2, 3 and 4 contain a detailed account about metrizability results and their applications, renorming theory in Banach spaces and pointwise and weak compactness including James' weak compactness and their relatives. In these three sections some of the results are proved while others are only referenced. Now and then we include brand new results or brand new proofs. Section 5 has a different flavor than the previous ones: we, sometimes as mere reporters of words written by others, collect here a good number of comments of what we think as big achievements in this area of topology as a tool for functional analysis; sometimes we ourselves comment on these results but some other times, to do that more properly, we use the wording of other specialists more authoritative than us, and then we literally take some comments from reports, blogs, lectures that we attended and so on. For instance, we take some literal comments from: (1) the final reports, workshop files and videos at http://www.birs.ca/events/2012/5-day-workshops/12w5019 (by R. Anisca, S. Dilworth, E. Odell and B. Sari); (2) Gowers' blog at http://gowers.wordpress.com (by T. Gowers) (3) some survey papers that are conveniently referenced and included in our list of bibliography.

As a general rule each section ends with a subsection devoted to *"Some notes and open problems"* and, since each of them starts with a short introduction of what is presented, there is no need for us to make this general introduction longer and we just start with the content of the paper.

We hope that this survey might serve as a good reference for interested researchers in the area. As authors, this is the first time we attempt to write this sort of survey. As editors we have been always pushing other colleagues to reflect about the importance of the *"interplay between topology and functional analysis"*, see [44] and [45]. We have to apologize to all our colleagues whose results should have been included here but they are not because of the lack of space, time constraints or because of our ignorance.

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1.1 Notation and terminology

Most of our notation and terminology are standard, otherwise it is either explained here or when needed: unexplained concepts and terminology can be found in our standard references for Banach spaces [50, 53, 68, 115], locally convex spaces [127] and topology [65, 119]. We also refer to the very recent book [117] where a variety of topics can be found.

By letters E, K, T, X, etc. we denote sets and sometimes topological spaces. Our topological spaces are assumed to be completely regular. All vector spaces E that we consider in this paper are assumed to be real. Sometimes E is a normed space with the norm $\|\cdot\|$. Given a subset S of a vector space, we write $\operatorname{co}(S)$, $\operatorname{aco}(S)$ and $\operatorname{span}(S)$ to denote, respectively, the convex, absolutely convex and the linear hull of S. If $(E, \|\cdot\|)$ is a normed space then E^* denotes its topological dual. If S is a subset of E^*, then $\sigma(E, S)$ denotes the weakest topology for E that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology for E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (w) and weak* (w^*) topologies respectively. Of course, $\sigma(E, S)$ is always a locally convex topology and it is Hausdorff if, and only if, $E^* = \overline{\operatorname{span} S}^{w^*}$ and similarly for $\sigma(E^*, S)$. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ for the evaluation of x^* at x. If $x \in E$ and $\delta > 0$ we denote by $B(x, \delta)$ (or $B[x, \delta]$) the open (resp. closed) ball centered at x of radius δ: for $x = 0$ and $\delta = 1$ we will simplify our notation and just write $B_E := B[0, 1]$; the unit sphere $\{x \in E : \|x\| = 1\}$ will be denoted by S_E. Recall that a subset B of B_{E^*} is said to be *norming* (resp. 1-*norming*) if

$$\|x\|_F = \sup\{|b^*(x)| : b^* \in B\}$$

is a norm in E equivalent (resp. equal) to the original norm of E. A subspace $F \subset E^*$ is norming (resp. 1-norming) if $F \cap B_{E^*}$ is norming (resp. 1-norming) according with the previous definition.

For a locally convex space E we will use most of the notation explained before for normed spaces, but we will write, according to tradition, E' for its topological dual instead of E^*.

(Z, d) is a metric space (Z if d is tacitly assumed); as usual $\mathbb{R}$ is considered as a metric space endowed with the metric associated to the absolute value $|\cdot|$. The space Z^X is equipped with the product topology τ_p. We let $C(X, Z)$ denote the space of all Z-valued continuous functions on X, and let $B_1(X, Z)$ denote the space of all Z-valued functions of the first Baire class (Baire one functions), *i.e.* pointwise limits of Z-valued continuous functions. When $Z = \mathbb{R}$, we just write $C(X)$ and $B_1(X)$ for $C(X, \mathbb{R})$ and $B_1(X, \mathbb{R})$, respectively.

If $\emptyset \neq A \subset (Z, d)$ we write

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$

For A and B nonempty subsets of (Z, d), we consider the *distance* between A and B given by

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

and the *Hausdorff non-symmetrized distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}. \tag{1}$$

The metric in Z^X is the *standard supremum metric* given for arbitrary functions $f, g \in Z^X$ by

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)) \tag{2}$$

that is allowed to take the value $+\infty$.

In particular, if $f \in \mathbb{R}^X$ we write

$$S_X(f) := \sup_{x \in X} f(x) \in (-\infty, \infty]. \tag{3}$$

$\ell^\infty(X)$ stands for the Banach space of bounded functions in $\mathbb{R}^X$ endowed with the supremum norm, $S_X(|\cdot|)$; whenever $X = K$ is compact, the norm (3) will be denoted as $\|f\|_\infty := S_K(|f|)$.

As usual $\mathbb{N}$ denotes the set of natural numbers. As topological space $\mathbb{N}$ is endowed with the discrete topology and $\mathbb{N}^\mathbb{N}$ has its product topology. Given $\alpha = (a_n)$ and $\beta = (b_n)$ in $\mathbb{N}^\mathbb{N}$ we use the product order

$$\alpha \leq \beta \text{ if } a_n \leq b_n \text{ for every } n \in \mathbb{N}. \tag{4}$$

Given a set X we denote by 2^X the family of all its subsets. If X is a topological space $\mathcal{K}(X)$ stands for the family of all its compact subsets.

2 Metrizability of compact spaces with applications to functional analysis

A topological space (X, τ) is said to be metrizable if there is a metric d on X with τ as associated topology. Metrizatbility results are always notorious results in topology. We mention here two classical ones: Urysohn's metrization theorem and Nagata-Smirnov metrization theorem. The goal of this section is to give a self-contained proof of the metrizability result stated in Theorem 2 below, as well as to offer a retrospective discussion of how often the structures presented there appear in functional analysis and therefore this topological result has had a saying in analysis. As a tool for the proof of Theorem 2 we use upper semicontinuous compact-valued maps, or more precisely a natural way of producing K-analytic structures.

Recall that a topological space X is said to be *Lindelöf Σ-space* (resp. *K-analytic*) if there is a subspace (resp. closed subspace) $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous set-valued map $T : \Sigma \to 2^X$ such that $T(\alpha)$ is a non-empty compact subset of X for each $\alpha \in \Sigma$ and $T(\Sigma) := \bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$. Here the set-valued map T is called *upper semi-continuous* if for each $\alpha \in \Sigma$ and for any open subset U of X such that $T(\alpha) \subset U$ there exists an open neighborhood V of α in Σ with $T(V) \subset U$. A good reference for Lindelöf Σ-spaces is [16]. Oftentimes Lindelöf Σ-spaces are referred to as countably K-determined spaces, as for instance in [183]. Note that Polish spaces (*i.e.* separable and metrizable spaces that are complete for some compatible metric) are continuous images of $\mathbb{N}^{\mathbb{N}}$ and that separable metrizable spaces are continuous images of subspaces Σ of $\mathbb{N}^{\mathbb{N}}$. Therefore in the definition of Lindelöf Σ-space (resp. K-analytic) we can use as domain for T any metric separable space (resp. any Polish space or alternatively just $\mathbb{N}^{\mathbb{N}}$).

The class of K-analytic spaces contains simultaneously the class of Polish spaces and the class of compact spaces. Both classes, K-analytic and Lindelöf Σ-spaces, are stable by closed subspaces, compact-valued upper semicontinuous images, countable products and countable sums. Lindelöf Σ-spaces are Lindelöf (open covers of the space have a countable subcover). On the other hand, if $T : \Sigma \to 2^X$ is upper semicontinuous and compact-valued, for every compact set $K \subset \Sigma$ the image

$$T(K) := \bigcup_{\alpha \in K} T(\alpha)$$

is a compact subset of X. Therefore if $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ is a compact-valued map giving structure of K-analytic space to X and we define

$$A_\alpha := T\big(\{\beta \in \mathbb{N}^{\mathbb{N}} : \beta \leq \alpha\}\big)$$

then we have that:

(A) A_α is compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$;
(B) $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$;
(C) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. (5)

It is natural to ask if, conversely, any topological space X that has a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact subsets satisfying properties (A), (B) and (C) above is K-analytic. The answer in general is no, as examples in [182, 189] and [117, p. 75] show. The first positive answer to this question for the particular case of $X = C_p(K)$ appears in [183, Proposition 6.13], that is, for the space of continuous functions on a compact space K endowed with the pointwise convergence topology. A more general positive result in this direction is presented in Proposition 1 below.

In what follows we use the following notation: given a sequence (x_n) in a topological space X, the set of all cluster points of (x_n) in X is the closed set

$$\mathrm{clust}_X(x_n) := \bigcap_{n\in\mathbb{N}} \overline{\{x_m : m \geq n\}}. \tag{6}$$

Recall that a subset A of X is said to be relatively countably compact X (resp. countably compact) if for every sequence (x_n) in X the set $\mathrm{clust}_X(x_n)$ is not empty (resp. $\mathrm{clust}_X(x_n) \cap A \neq \emptyset$).

If $\alpha = (n_k)$ and $m \in \mathbb{N}$ we write $\alpha|_m := (n_1, n_2, \ldots, n_m)$.

Proposition 1 ([30, 41]). *Let X be a topological space with a family of subsets $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfying conditions (A), (B) and (C) in (5). Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$, define*

$$C_{n_1,n_2,\ldots,n_m} := \bigcup\{A_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|_m = \alpha|_m\}$$

and $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ by the formula

$$T(\alpha) := \bigcap_{k=1}^{\infty} C_{n_1,n_2,\ldots,n_k}.$$

Then:

1. *any sequence in $T(\alpha)$ has cluster points in X all of which remain in $T(\alpha)$; in particular $T(\alpha)$ is countably compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$;*
2. *if we assume that $T(\alpha)$ is compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, then the set-valued map T gives a K-analytic structure to X; this happens in particular when countably compact subsets of X are compact.*

Proof. Both statements in item 1 easily follow from the Claim below.

CLAIM.- Given $\alpha = (n_k)$ *in* $\mathbb{N}^{\mathbb{N}}$ *and* $x_k \in C_{n_1,n_2,\ldots,n_k}$, $k \in \mathbb{N}$, *we have that*

$$\emptyset \neq \text{clust}_X(x_k) \subset \bigcap_{k=1}^{\infty} C_{n_1,n_2,\ldots,n_k}$$

We prove the claim. Let $\beta_k \in \mathbb{N}^{\mathbb{N}}$ be such that

$$x_k \in A_{\beta_k} \text{ and } \beta_k|_k = (n_1, n_2, \ldots, n_k) \text{ for every } k \in \mathbb{N}.$$

Let $\pi_j : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be the j-th projection onto $\mathbb{N}$. For every $j \in \mathbb{N}$ let us define

$$m_j := \max\{\pi_j(\beta_k) : k \in \mathbb{N}\} \text{ and } \beta := (m_j).$$

Note that $\pi_1(\beta) = n_1$ and that condition (B) ensures that $A_{\beta_k} \subset A_\beta$ for every $k \in \mathbb{N}$. Therefore (x_k) is contained in the compact set A_β (condition (A)), and consequently $\emptyset \neq \text{clust}_X(x_k) \subset A_\beta \subset C_{n_1}$. If we repeat the argument above with $(x_k)_{k\geq m}$, for every $m \in \mathbb{N}$, and use that $\text{clust}_X(x_k) = \text{clust}_X(x_k)_{k\geq m}$ we obtain that $\text{clust}_X(x_k) \subset \bigcap_{m=1}^{\infty} C_{n_1,n_2,\ldots,n_m}$ as we want, and the proof of the Claim is over.

To finish we prove the statement in item 2. Assume that each $T(\alpha)$ is compact and let U be an open subset of X such that $T(\alpha) \subset U$. If $\alpha = (n_k)$, then for some $m \in \mathbb{N}$ we have that

$$C_{n_1,n_2,\ldots,n_m} \subset U. \tag{7}$$

If the above were not the case, for every $k \in \mathbb{N}$ there would exist

$$x_k \in C_{n_1,n_2,\ldots,n_k} \setminus U. \tag{8}$$

Note that the Claim says that $\emptyset \neq \text{clust}_X(x_k) \subset T(\alpha)$ but (8) implies that $\text{clust}_X(x_k) \subset X \setminus U \subset X \setminus T(\alpha)$, that is a contradiction that proves the validity of (7). For the open neighbourhood of α defined by

$$V := \{\beta \in \mathbb{N} : \beta|m = (n_1, n_2, \ldots, n_m)\}$$

we clearly have that $T(V) \subset C_{n_1,n_2,\ldots,n_m} \subset U$ and therefore T is upper semicontinous. To finish the proof we observe that $A_\alpha \subset T(\alpha)$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and consequently (C) applies to obtain that $T(\mathbb{N}^{\mathbb{N}}) = X$. With all the above, X is a K-analytic space.

Proposition 1 provides us with a nice tool to produce K-analytic structures as we will see several times in the remaining of this section. Note that in particular Proposition 1, item 2, can be used for the so-called class of *angelic spaces* where all notions of compactness do coincide. Let us recall the definition.

Definition 1 (Fremlin). A regular topological space T is *angelic* if every relatively countably compact subset A of T is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from A.

In angelic spaces the different concepts of compactness and relative compactness coincide: the (relatively) countably compact, (relatively) compact and (relatively) sequentially compact subsets are the same, as seen in [75]. Examples of angelic spaces include spaces $C_p(K)$, when K is a countably compact space, see [92, 120] and all Banach spaces in their weak topologies. More generally if X is Lindelöf Σ-space then $C_p(X)$ is angelic, [152] (see also Corollary 5 in Subsection 4.1): as a consequence of this, w^*-duals of Banach spaces that are weakly Lindelöf Σ-spaces are angelic. Note that in particular, Proposition 1 says that if X is Lindelöf Σ-space then $C_p(X)$ is K-analytic if, and only if, there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of τ_p-compact subsets of $C(X)$ satisfying conditions (A), (B) and (C) in (5). We should mention that this result has been recently proved for any $C_p(X)$ (no restriction on X, just a completely regular space) in [186]. A good reference for many different applications of Proposition 1 is [117].

In some recent literature, see [117], a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ as above is called a *compact resolution* for X. We follow the terminology introduced in [186]:

Definition 2 ([186, 43]). A topological space X with a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfying (A), (B) and (C) in (5) is said to be *dominated by irrationals.*

It is noteworthy to know that a space X is dominated by irrationals if, and only if, there is a Polish space P and a compact cover $\{X_K : K \in \mathcal{K}(P)\}$ of X satisfying $X_K \subset X_L$ whenever $K \subset L$, for $K, L \in \mathcal{K}(P)$, see cite [43, Proposition 2.2].

Using again the terminology of [186, 43] we set the following definitions.

Definition 3 ([186, 43]). Given topological spaces M and Y, an M-ordered compact cover of a space Y is a family $\mathcal{F} = \{F_K : K \in \mathcal{K}(M)\} \subset \mathcal{K}(Y)$ such that

$$\bigcup \mathcal{F} = Y \text{ and } K \subset L \text{ implies } F_K \subset F_L \text{ for any } K, L \in \mathcal{K}(M).$$

Y is said to be *dominated* (resp. *strongly dominated*) by the space M if there exists an M-ordered compact cover $\mathcal{F}$ (resp. that moreover swallows all compact subsets of Y, in the sense that for any compact $C \subset Y$ there is $F \in \mathcal{F}$ such that $C \subset F$) of the space Y.

Results about strongly dominated spaces by Polish spaces are collected in the following two theorems.

Theorem 1 (Christensen, [49], Theorem 3.3). *A second countable topological space is strongly* $\mathbb{N}^{\mathbb{N}}$*-dominated if, and only if, it is completely metrizable.*

Theorem 2. *Let K be a compact space and let Δ be its diagonal. The following statements are equivalent:*

1. *K is metrizable;*
2. *$(C(K), \|\cdot\|_\infty)$ is separable;*
3. *Δ is a G_δ subset of $K \times K$;*
4. *$\Delta = \bigcap_n G_n$ with each G_n open in $K \times K$ and $\{G_n : n \in \mathbb{N}\}$ being a basis of open neighbourhoods of Δ;*
5. *$(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact subsets in $(K \times K) \setminus \Delta$;*
6. *$(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact sets that swallows all the compact subsets in $(K \times K) \setminus \Delta$;*
7. *$(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ a family of compact sets that swallows all the compact subsets in $(K\times K)\setminus\Delta$ such that $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;*
8. *$(K \times K) \setminus \Delta$ is Lindelöf.*

Proof. Although the equivalences between 1, 2, 3, 4, 5, 6 and 8 are classical and well known, and the equivalence of 1 and 7 can be found, with a different formulation in [40, Theorem 1], we give full details for their proofs –the equivalence $1 \Leftrightarrow 3$ is a result due to Šneĭder that can be found in [65, Exercise 4.2.B].

Let us start with $1 \Rightarrow 2$ by assuming that the metric d metrizes the topology of K. Let S be a countable and dense subset of K. For every $s \in S$ let us write $f_s(x) := d(x, s)$, for $x \in K$. Then $\{f_s : s \in S\}$ is a countable subset of $C(K)$ that separates the points of K. Hence, by Stone-Weierstrass' theorem, see [50, V.§8], the algebra $\mathcal{A}$ generated by $\{f_s : s \in S\}$ and the constant functions is dense in $(C(K), \|\cdot\|_\infty)$. Since $\mathcal{A}$ is separable with any vector topology we conclude that $(C(K), \|\cdot\|_\infty)$ is separable too.

Let us prove that $2 \Rightarrow 1$. If $\{f_n : n \in \mathbb{N}\}$ is an enumeration of a countable and dense subset of $(C(K), \|\cdot\|_\infty)$, then the formula

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + f_n(x) - f_n(y)}, \text{ for } x, y \in K,$$

defines a metric on K that metrizes its topology.

The implications $6 \Leftrightarrow 4 \Rightarrow 3 \Leftrightarrow 5$ are clear. On the other hand, $3 \Rightarrow 4$ can be proved with the following arguments. Assume that $\Delta = \bigcap_n G_n$ with each $G_n \subset K \times K$ open. For every $n \in \mathbb{N}$ take an open subset $A_n \subset K \times K$ with $\Delta \subset A_n \subset \overline{A_n} \subset G_n$. Define recursively

$$O_1 := A_1,\, O_2 := O_1 \cap A_2, \ldots, O_n := O_{n-1} \cap A_n, \ldots$$

We have that $\Delta \subset \bigcap_n O_n \subset \bigcap_n \overline{O_n} \subset \bigcap_n G_n = \Delta$ and consequently

$$\Delta = \bigcap_n O_n = \bigcap_n \overline{O_n}.$$

If $G \subset K \times K$ is open with $\Delta \subset G$, then there exists $m \in \mathbb{N}$ with $\overline{O_m} \subset G$. Indeed, if this were not the case $C_n := \overline{O_n} \cap \Big((K \times K) \setminus G\Big) \neq \emptyset$ for every $n \in \mathbb{N}$. Since $\{C_n : n \in \mathbb{N}\}$ is a decreasing sequence of nonempty closed subsets in a compact space we must have $\bigcap_n C_n \neq \emptyset$. This implies that $\bigcap_n \overline{O_n} = \Delta$ is not contained in G, and we reach a contradiction that finishes the proof for the implication $3 \Rightarrow 4$.

The implication $1 \Rightarrow 3$ is rather easy: if d is a metric giving the topology of K then $\Delta = \bigcap_n G_n$ where each G_n is the open set given by

$$G_n := \Big\{(x,y) \in K \times K : d(x,y) < \frac{1}{n}\Big\}.$$

Being $6 \Rightarrow 7$ obvious, we now give a proof for the implication $7 \Rightarrow 2$. Assume that 7 holds and let us define $O_\alpha := (K \times K) \setminus A_\alpha$, $\alpha \in \mathbb{N}$. The family $\mathcal{O} := \{O_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of open neighbourhoods of Δ that satisfies the decreasing condition

$$O_\beta \subset O_\alpha, \text{ if } \alpha \leq \beta \text{ in } \mathbb{N}^{\mathbb{N}}. \tag{9}$$

Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and any $m \in \mathbb{N}$ we write $\alpha|^m := (n_m, n_{m+1}, n_{m+2}, \dots)$ and define

$$B_\alpha := \Big\{f \in n_1 B_{C(K)} : (m \in \mathbb{N}, \text{ and } (x,y) \in O_{\alpha|^m}) \Rightarrow |f(x) - f(y)| \leq \frac{1}{m}\Big\}.$$

Note that each B_α is $\|\cdot\|_\infty$-bounded, closed and equicontinuous as a family of functions defined on K. Therefore, Ascoli's theorem, see [119, p. 234], implies that B_α is compact in $(C(K), \|\cdot\|_\infty)$. The decreasing property (9) implies that $B_\alpha \subset B_\beta$ if $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$. We claim that $C(K) = \bigcup\{B_\alpha : \alpha \in \mathbb{N}\}$. To see this, given $f \in C(K)$ take $M > 0$ such that $\|f\|_\infty \leq M$. On the other hand since $\mathcal{O}$ is a basis of neighborhoods of Δ, there exists a sequence $\big(\alpha_m = (n_k^m)\big)$ in $\mathbb{N}^{\mathbb{N}}$ such that

$$|f(x) - f(y)| \leq \frac{1}{m} \text{ for every } (x,y) \in O_{\alpha_m}.$$

If we define now $n_1 := \max\{n_1^1, M\}$ and $n_k := \max\{n_k^1, n_{k-1}^2, \dots, n_1^k\}$, $k = 2, 3, \dots$, then for the sequence $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we have that $f \in B_\alpha$. The family $\{B_\alpha : \alpha \in \mathbb{N}\}$ of subsets of $(C(K), \|\cdot\|_\infty)$ satisfies the hypothesis of Proposition 1 and since countably compact subsets of $(C(K), \|\cdot\|_\infty)$ are compact, we conclude that $(C(K), \|\cdot\|_\infty)$ is K-analytic, therefore Lindelöf and thus separable. This finishes the proof of $7 \Rightarrow 2$.

With all the above we have proved that statements from 1 to 7 are all equivalent. Note that $5 \Rightarrow 8$ straightforwardly follows from the fact that K_σ-spaces are Lindelöf. To finish the proof of the theorem we prove the implication $8 \Rightarrow 3$. If $x \neq y$, $x, y \in K$, there exist two closed neighbourhoods C_x and C_y of x and y, respectively, such that $C_x \times C_y \subset (K \times K) \setminus \Delta$. Since $(K \times K) \setminus \Delta$ is Lindelöf we can find a countable set D and $x_d \neq y_d$

in K, $d \in D$, such that $(K \times K) \setminus \Delta = \bigcup_{d \in D} C_{x_d} \times C_{y_d}$. Consequently, $\Delta = \bigcap_{d \in D} (K \times K) \setminus (C_{x_d} \times C_{y_d})$ is a G_δ subset, and the proof is over.

More generally one can prove the following result. Recall that the weight $w(X)$ of a topological space X is the smallest cardinal of a basis for its topology.

Proposition 2 ([35]). *Let K be a compact space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

1. $w(K) \leq \mathfrak{m}$*;*
2. *There exists a metric space M with $w(M) \leq \mathfrak{m}$ and a family $\mathcal{O} = \{O_L : L \in \mathcal{K}(M)\}$ of open subsets in $K \times K$ that is basis of the neighborhoods of Δ such that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathcal{K}(M)$;*
3. *$(K \times K) \setminus \Delta$ is strongly dominated by a metric space M with $w(M) \leq \mathfrak{m}$.*

As a consequence of Proposition 2 we can complete Theorem 2 with a couple of new equivalent conditions.

Corollary 1 ([43, 35]). *For a compact space K the following statements are equivalent:*

1. *K is metrizable;*
5. *$(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact subsets in $(K \times K) \setminus \Delta$;*
6. *$(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact sets that swallows all the compact subsets in $(K \times K) \setminus \Delta$;*
7. *$(K \times K) \setminus \Delta = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ a family of compact sets that swallows all the compact subsets in $(K \times K) \setminus \Delta$ such that $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;*
9. *$(K \times K) \setminus \Delta$ is strongly dominated by a Polish space;*
10. *$(K \times K) \setminus \Delta$ is strongly dominated by a separable metric space;*

Proof. Assume that 7 holds. Given a compact set $L \subset \mathbb{N}^\mathbb{N}$ we define $\alpha(L) \in \mathbb{N}^\mathbb{N}$ by the formula

$$\alpha(L) := (\sup \pi_1(L), \sup \pi_2(L), \dots, \sup \pi_n(L), \dots)$$

and $F_L := A_{\alpha(L)}$, where $\pi_n : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is the n^{th}-projection, for every $n \in \mathbb{N}$. The family $\{F_L : L \in \mathcal{K}(\mathbb{N}^\mathbb{N})\}$ strongly dominates $(K \times K) \setminus \Delta$ and therefore 9 holds. The implication $9 \Rightarrow 10$ is obvious and the implication $10 \Rightarrow 1$ follows from Proposition 2. The proof is over.

Let us stress that the power of Theorem 2 and Corollary 1 resides in the fact that the existence of a special uncountable cover as in 7 is the same than the existence of a special countable cover as in 6. In other words, if we want, strongly domination by $\mathbb{N}^\mathbb{N}$ for $(K \times K) \setminus \Delta$ is the same than domination by $\mathbb{N}$ for $(K \times K) \setminus \Delta$. Note that the equivalence $5 \Leftrightarrow 6$ says that for countable covers $\{F_n : n \in \mathbb{N}\}$ of $(K \times K) \setminus \Delta$ it does not matter if the covers swallow all the compact subsets of $(K\times K)\setminus\Delta$ or not. For uncountable covers $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as in 7 it is unknown, to the best of our knowledge, if we can drop the hypothesis that the cover swallows all the compact subsets in $(K \times K) \setminus \Delta$. With regard to this question Theorem 3 collects what can be said assuming MA(ω_1).

Recall that a topological space X is said to have countable tightness if for every subsset A of X and every point $x \in \overline{A}$ there is a countable set $D \subset A$ such that $x \in \overline{D}$.

Theorem 3 ([43]). *Under MA(ω_1), if K is a compact space such that $(K \times K) \setminus \Delta$ is dominated by a Polish space then K is metrizable.*

Proof. The domination by a Polish space of $(K \times K) \setminus \Delta$ is used twice. First, and under MA(ω_1), to ensure that K has a small diagonal and from there deduce that $K \times K$ has countable tightness, [43, Theorem 2.12]. Second, to write $(K\times K)\setminus\Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ also satisfying conditions (A) and (B) in (5).

Now we can make use of Proposition (1) to produce the set-vauled map $T : \mathbb{N}^\mathbb{N} \to 2^{K\times K}$. We claim that for every $\alpha \in \mathbb{N}$ the set $T(\alpha)$ is compact. To see this we simply prove that for the closure $\overline{T(\alpha)}$ in $K \times K$ we have $\overline{T(\alpha)} \subset T(\alpha)$. To this end, take $x \in \overline{T(\alpha)}$, and use the countable tightness of $K \times K$ to produce a countable subset D of $T(\alpha)$ such that $x \in \overline{D}$. Note that for this x there are two possibilities: either $x \in D \subset T(\alpha)$ or $x \in \overline{D} \setminus D$. If the latter is the case, and we enumerate $D := \{y_n : n \in \mathbb{N}\}$, then x is cluster point of the sequence (y_n). We use now item 1 in Proposition (1) to conclude $x \in T(\alpha)$, and from item 2 in the same proposition we obtain that $(K \times K) \setminus \Delta$ is K-analytic. Consequently, $(K \times K) \setminus \Delta$ is Lindelöf and the implication 8 ⇒ 1 in Theorem 2 tells us that K is metrizable.

We go on in this section with a bunch of applications to functional analysis of the results presented until now.

Theorem 4 (Talagrand, [184]). *Every weakly compactly generated Banach space E is weakly Lindelöf.*

Proof. $E = \overline{\text{span} W}$ with W a weakly compact subset of E. By Krein-Šmulian theorem, [50, Theorem 3.14], we can assume that W is moreover absolutely convex and therefore $E = \overline{\bigcup_n nW}$. Given $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, let us define

$$A_\alpha := \big(n_1 W + B_{E^{**}}\big) \cap \big(n_2 W + \frac{1}{2} B_{E^{**}}\big) \cap \cdots \cap \big(n_k W + \frac{1}{k} B_{E^{**}}\big) \cap \ldots,$$

It is easy to prove that every $A_\alpha \subset E$ is weakly compact, $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$ and $E = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Since (E, w) is angelic, see for instance Theorem 28, weakly countably compact subsets of E are weakly compact. Thus (E, w) is K-analytic after Proposition 1, hence (E, w) is Lindelöf, and the proof is over.

Theorem 5 (Dieudonné, Theorem §.2.(5) [127]). *Every Fréchet-Montel space E is separable (in particular, for any open set $\Omega \subset \mathbb{C}$ the space of holomorphic functions $(\mathcal{H}(\Omega), \tau_k)$ with its compact-open topology is separable).*

Proof. Fix $U_1 \supset U_2 \supset \cdots \supset U_n \ldots$ a basis of absolutely convex closed neighborhoods of 0. Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, let us define

$$A_\alpha := \bigcap_{k=1}^{\infty} n_k U_k.$$

The family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is made up of closed bounded sets, covers E and satisfies $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$. Since E is Montel, each A_α is compact and since E is Fréchet it is metrizable and therefore Proposition 1 enters our game again to say that E is K-analytic. Note that therefore E is metrizable and Lindelöf, hence separable.

Recall that a topological space is said to be *analytic* (or Souslin) if it is the continuous image of a Polish space. Note that analytic spaces are always continuous images of $\mathbb{N}^{\mathbb{N}}$.

Theorem 6. *The dual E' of an inductive limit $E = \varinjlim E_m$ of a sequence (E_m) of separable Fréchet (in particular Fréchet-Montel) spaces is analytic when endowed with the topology τ_c of uniform convergence on compact sets of E.*

Proof. Fix $U_1^m \supset U_2^m \supset \cdots \supset U_n^m \ldots$ a basis of absolutely convex closed neighborhoods of 0 in E_m. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ let us define

$$U_\alpha := \overline{\operatorname{aco} \bigcup_{k=1}^{\infty} U_{n_k}^k} \tag{10}$$

Note that $U_\beta \subset U_\alpha$ if $\alpha \leq \beta$ and that $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhood of 0 in E, see [127, p. 215]. Therefore if we take polars in $\langle E, E' \rangle$ and define $A_\alpha := U_\alpha^o$, then each A_α is compact in (E', τ_c) see [127, §21.6.(3)], $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$. On the other hand, since each E_m is separable, E is separable, and therefore (E', τ_c) has a coarser Hausdorff topology that is metrizable. This implies that (E', τ_c) is angelic, see [75, Lemma 3.1]. Consequently, (E', τ_c) is K-analytic after Proposition 1 and since it has a coarser Hausdorff metrizable topology it is analytic according to a result by Choquet, see [48].

If $\Omega \subset \mathbb{R}^n$ is an open set we denote by $\mathcal{D}'(\Omega)$ the space of distributions endowed with its strong topology of uniform convergence on bounded (compact) sets of the test-functions space $\mathcal{D}(\Omega)$, see [50, Chapter IV. Section 5] and [175].

Corollary 2 (Schwartz, [174]). *Any linear map $T : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ whose graph is a Borel subset of $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ is continuous.*

Proof. On the one hand $\mathcal{D}'(\Omega)$ ultrabornological and on the other hand $\mathcal{D}'(\Omega)$ is analytic according to Theorem 6. The Corollary follows from a famous result by Schwartz, see [174], that establishes the validity of a Borel graph theorem for linear maps between ultrabornological locally convex spaces and analytic locally convex spaces.

It is worth mentioning that the results in [174] offered the first important (partial) positive answer to a question raised by Grothendieck, see [93], whose final aim was to obtain the validity of the Closed Graph theorem for linear maps $T : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

To finish the section we introduce the class $\mathfrak{G}$ of locally convex spaces, whose properties and the consequences that can be derived from them were the true motivation to establish the implication $7 \Rightarrow 1$ collected in Theorem 2 as originally proved in [40, Theorem 1].

Definition 4 ([40]). A locally convex space E belongs to the class $\mathfrak{G}$ if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of E' satisfying the properties:

(a) for any $\alpha \in \mathbb{N}^\mathbb{N}$ the countable subsets of A_α are equicontinuous;
(b) $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$;
(c) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

The good news for class $\mathfrak{G}$ is that it is a very wide stable class for the usual operations in functional analysis of countable type (completions, closed subspaces, quotients, direct sums, products, etc.) that contains metrizable locally convex spaces and their duals and for which $7 \Rightarrow 1$ collected in Theorem 2 implies:

Theorem 7 ([40]). *If E is a locally convex space in class $\mathfrak{G}$, then its compact (even its precompact) subsets are metrizable.*

Proof. See [40, Theorem 2].

Let us mention that Theorem 7 implies, in particular, that precompact subsets of countable inductive limits of Fréchet spaces are metrizable, that was first proved in [39], answering a question posed by Floret in [74]. The question raised by Floret was motivated by the fact that the good behaviour of compact subsets of inductive limits should imply reasonable consequences about localization and retractivity properties in inductive limits that were important while studying certain abstract settings for partial differential equations. We refer to the recent book [117] for more applications and consequences of the ideas presented here in both topology and functional analysis.

2.1 Some notes and open problems

The paper [43] contains a long list of questions related to this section. We isolated the following one that has been presented above.

Question 1. if K is a compact space such that $(K \times K) \setminus \Delta$ is dominated by a Polish space, is K metrizable?

We note that the answer given in Theorem 3 under $\mathrm{MA}(\omega_1)$ to this question is based in [43, Theorem 2.12] that makes use of the fact that in this situation the compact space K must have a small diagonal: other arguments from here allow then to conclude that K is metrizable. More in general we have the following question:

Question 2 (Husek, 1977). Is every compact space with small diagonal metrizable?

This problem has been throughly explained by Gruenhage in [99]. Here is another problem that to the best of our knowledge is open

Question 3 ([176]). Let E be a locally convex space and $K \subset E$ a convex compact set that is perfectly normal. Is K metrizable?

Let us stress that Helly compact (increasing functions from $[0, 1]$ into $[0, 1]$ with the topology of pointwise convergence) is convex, compact, separable and not metrizable. A natural question connected with the above problems is the following one.

Question 4. Are there natural (and useful) conditions that we can impose to a convex compact space K in order it to be homeomorphic to a compact set of a space of the class $\mathfrak{G}$?

Good references for metrizability of compact convex sets are [51, 135, 143, 170].

3 Topological networks meet renorming theory in Banach spaces

Banach spaces have offered historically one of the most fruitful frameworks in mathematical analysis. Renorming theory tries to find isomorphisms for Banach spaces that improve their norms. That means to make the geometrical and topological properties of the unit ball of a given Banach space as close as possible to those of the unit ball of a

Hilbert space. The existence of equivalent *good* norms on a particular Banach space depends on its structure and has in turn a deep impact on its geometrical properties. Questions concerning renormings in Banach spaces have been of particular importance to provide smooth functions and tools for optimization theory. An excellent monograph of renorming theory up to 1993 is [53]. In order to have an up-to-date account of the theory we should add [104, 81, 190, 139, 178].

Surprisingly enough tools coming from pure set-theoretical topology, like the concept of *network*, are of great importance to study successfully renorming theory in Banach spaces. In this section we report on results obtained in the last years as a complement to those that can be found in the given references. We shall do it with new proofs of the main theorems based on new analysis and new results

3.1 Locally uniformly rotund renormings

If $(E, \|\cdot\|)$ is a normed space, the norm $\|\cdot\|$ is said to be locally uniformly rotund (**LUR** for short) if

$$\left[\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0\right] \Rightarrow \lim_n \|x - x_n\| = 0$$

for any sequence (x_n) and any x in E. The construction of these kind of norms in separable Banach spaces led Kadec to the proof of the existence of homeomorphisms between any two separable Banach spaces, [23, Chapter VI-9]. For a non separable Banach space is not always possible to have such an equivalent norm: the space ℓ^∞ does not have it, see for instance [53, p. 74]. If such a norm exists, its construction is usually based on a good system of coordinates that is known on the normed space E from the very beginning, for instance a biorthogonal system

$$\{(x_i, f_i) \in E \times E^* : i \in I\},$$

with some additional properties such as being a strong Markushevich basis, [101, Chapter III]. When such a system does not exist the norm can be constructed providing enough convex functions on the given space E and adding all of them up with the powerful lemma of Deville, Godefroy and Zizler, see [53, Lemma VII.1.1],[190]. It reads as follows:

Lemma 1 (Deville, Godefroy and Zizler decomposition method). *Let $(E, \|\cdot\|)$ be a normed space, let I be a set and let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be families of non-negative convex functions on E which are uniformly bounded on bounded subsets of E. For every $x \in E$, $m \in \mathbb{N}$ and $i \in I$ define*

$$\varphi(x) = \sup\{\varphi_i(x) : i \in I\}, \quad (11)$$
$$\theta_{i,m}(x) = \varphi_i(x)^2 + 2^{-m}\psi_i(x)^2, \quad (12)$$
$$\theta_m(x) = \sup\{\theta_{i,m}(x) : i \in I\}, \quad (13)$$
$$\theta(x) = \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}(\theta_m(x) + \theta_m(-x)). \quad (14)$$

Then the Minkowski functional of $B = \{x \in E : \theta(x) \leq 1\}$ *is an equivalent norm* $\|\cdot\|_B$ *on* E *such that if* $x_n, x \in E$ *satisfy the* LUR *condition:*

$$\lim_n [2\|x_n\|_B^2 + 2\|x\|_B^2 - \|x_n + x\|_B^2] = 0,$$

then there is a sequence (i_n) *in* I *with the properties:*

1. $\lim_n \varphi_{i_n}(x) = \lim_n \varphi_{i_n}(x_n) = \lim_n \varphi_{i_n}((x + x_n)/2) = \sup\{\varphi_i(x) : i \in I\}$
2. $\lim_n \left[\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(x) - \psi_{i_n}^2\left(\frac{1}{2}(x_n + x)\right)\right] = 0.$

The previous Lemma is the core of the decomposition method for renormings of nonseparable Banach spaces as described in [53, Chapter VII]. It has been extensively used by R. Haydon in his seminal papers [104, 106] as well as in [105]. Lemma 1 was first introduced by R. Deville and it is based on the construction of an equivalent **LUR** norm on Banach spaces with strong M-basis, [101, Theorem 3.48], where the convex functions are given by distances to suitable finite dimensional subspaces as well as evaluations on coordinate functionals in the dual space E^*. Let us note that if we add lower semicontinuity properties on the involved functions $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ we obtain lower semicontinuity for the new norm $\|\cdot\|_B$.

In order to deal with renorming results valid for the very different weak topologies that appear in Banach spaces we fix, from now on, $(E, \|\cdot\|)$ a Banach space together with a norming subspace $F \subset E^*$ in the dual space. Dealing with norming subspaces we will unify many results about dual norms or pointwise lower semicontinuos norms for $C(K)$ spaces. A topologist reader might be comfortable looking at our Banach space E always contained in some space $\ell^\infty(\Gamma)$ and then thinking that we want to find, if possible, pointwise lower semicontinuous equivalent norms only.

A first topological connection between the existence of biorthogonal systems of functions and LUR renormings follows, see [157]:

Theorem 8 (Topological Coordinates System). *Let* E *be a Banach space and let* F *be a a norming subspace of* E^**.* E *has an equivalent* $\sigma(E,F)$*-lower semicontinuous and* ***LUR*** *norm if, and only if, there are countably many families of convex and* $\sigma(E,F)$*-lower semicontinuous functions*

$$\{\varphi_i^n : E \to \mathbb{R}^+; i \in I_n\}_{n=1}^{\infty}$$

together with norm open sets

$$G_i^n \subset \{\varphi_i^n > 0\} \cap \{\varphi_j^n = 0 : j \neq i, j \in I_n\}$$

such that $\{G_i^n : i \in I_n, n \in \mathbb{N}\}$ *a basis for the norm topology of* E

The above topological approach to **LUR** renormings is mainly based on Stone's theorem about paracompactness of metric spaces and follows the pioneer work by R. Hansell, see [103, 141]: the σ-discrete basis for the norm topology of a normed space E can be refined and then modified to obtain the basis described in Theorem 8. More recent contributions that show the interplay between this topological method and the one based on Deville's lemma are [106, 136]. A straightforward proof of the main renorming constructions in [139, 165] can be found in [158].

These approaches for **LUR** renormings are strongly based on the topological concept of *network*. A family of subsets $\mathcal{N}$ in a topological space (T, τ) is a network for the topology τ if for every $W \in \tau$ and every $x \in W$, there is some $N \in \mathcal{N}$ such that $x \in N \subset W$. A central result for the theory is the following one, [165, 139]:

Theorem 9 (Slicely Network). *Let* E *be a normed space,* F *a norming subspace of* E^* *and* $\mathcal{H}$ *the family of all* $\sigma(E,F)$*-open half spaces in* E*. Then* E *admits a* $\sigma(E,F)$*-lower semicontinuous equivalent* ***LUR*** *norm if, and only if, there is a sequence* (A_n) *of subsets of* E *such that the family of sets*

$$\{A_n \cap H : H \in \mathcal{H}, n \in \mathbb{N}\}$$

is a network for the norm topology in E

The first proof of this result used martingale constructions, [139]. Without martingales a delicate process of convexification of the sets A_n is needed to construct a countable family of seminorms which controls the claim, [165]. Stone's theorem is required if additional information on the structure of the sets A_n is needed, see [139, Chapter 3]. After the Connection Lemma in [157] or the Slice Localization Theorem of [158] the convexification process is not necessary any more. The main construction is now done with the use of Deville's Master Lemma together with Corollary 3, which seems to be a main tool for the matter. It says that given any family of slices of a bounded set A of a normed space E, it is always possible to construct an equivalent norm such that the **LUR** condition for a sequence (x_n), and a fixed point x in A, implies that the sequence eventually belongs to slices containing the point x. When the slices involved have small diameter, then the sequence is eventually close to x. If the diameter can be made small enough, then the sequence (x_n) converges to x and the new norm is going to be locally uniformly rotund at the point x.

Our presentation here is done through Theorem 10, for which we give a formulation that includes cases where Deville's Lemma must be recursively used and the Slice Localization theorem appears as Corollary 3. With this approach, complete proofs for the main theorems in [106] and [105] follow from Theorem 11, see [136]. We shall present and easy proof for Theorem 11 that follows from Theorem 10: the original proof, formulated in [78], was much more involved with arguments intended to use Theorem 9. More measures of non-compactness has been successfully studied by M. Raja, [168]. Some open problems around these questions can be found in [139, Section 6.3].

Theorem 10 (Slice Convex Localization). *Let E be a normed space with a norming subspace F in E^*. Let A be a bounded subset of E and $\mathcal{H}$ a family of $\sigma(E,F)$-open half spaces with nonempty slices $A \cap H \neq \emptyset$ for every $H \in \mathcal{H}$. Given a uniformly bounded on bounded sets family*

$$\{\psi_H^A : E \to [0,+\infty) : H \in \mathcal{H}\}$$

of convex and $\sigma(E,F)$-lower semicontinuous functions there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ with the property:

If $x \in A \cap H$ for some $H \in \mathcal{H}$ and (x_n) is a sequence in E such that

$$\lim_n \left(2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x + x_n\|^2_{\mathcal{H},A}\right) = 0,$$

then there is a sequence of open half spaces $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ satisfying:

1. There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.
2. $\lim_{n\to\infty} \left[\frac{1}{2}\psi^A_{H_n}(x_n)^2 + \frac{1}{2}\psi^A_{H_n}(x)^2 - \psi^A_{H_n}(\frac{x+x_n}{2})^2\right] = 0.$

Proof. Let us consider the family of $\sigma(E,F)$-lower semicontinuous and convex functions $\{\varphi_H : H \in \mathcal{H}\}$ defined by

$$\varphi_H(x) = \inf\left\{\|x - d\|_F : d \in \overline{(E \setminus H) \cap co(A)}^{\sigma(E^{**},E^*)}\right\}, \ x \in E,$$

see [157, Definition 2.2]. We can apply now Devilles's Lemma 1 to obtain an equivalent norm $\|\cdot\|_{\mathcal{H},A}$ on E such that if a sequence (x_n) and x in E satisfy

$$\lim_n \left(2\,\|x_n\|^2_{\mathcal{H},A} + 2\,\|x\|^2_{\mathcal{H},A} - \|x_n + x\|^2_{\mathcal{H},A}\right) = 0,$$

then there exists a sequence (H_n) in $\mathcal{H}$ such that

(a) $\lim_n \varphi_{H_n}(x) = \lim_n \varphi_{H_n}(x_n) = \lim_n \varphi_{H_n}((x+x_n)/2) = \sup\{\varphi_H(x) : H \in \mathcal{H}\}.$

and

2. $\lim_n \left[(1/2)\psi^A_{H_n}(x_n)^2 + (1/2)\psi^A_{H_n}(x)^2 - \psi^A_{H_n}((x_n + x)/2)^2\right] = 0.$

Since the given point x belongs to one of the open half spaces $H_0 \in \mathcal{H}$, we have that $\varphi_{H_0}(x) > 0$ and we conclude that

$$\sup\{\varphi_H(x) : H \in \mathcal{H}_\varepsilon\} \geq \varphi_{H_0}(x) > 0.$$

Now condition (a) provides us with an integer n_0 such that

$$\varphi_{H_n}(x) > 0, \varphi_{H_n}(x_n) > 0, \varphi_{H_n}((x+x_n)/2) > 0$$

whenever $n \geq n_0$, from where we conclude that 1 in the statement of the theorem holds and the proof is over.

As a corollary we have the main result in [158]:

Corollary 3 (Slice Localization Theorem, [158]). *Let E be a normed space with a norming subspace F in E^*. Let A be a bounded subset in E and $\mathcal{H}$ a family of $\sigma(E,F)$-open half spaces such that for every $H \in \mathcal{H}$ the set $A \cap H$ is non-empty. Then there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ such that for every sequence (x_n) in E and $x \in A \cap H$ for some $H \in \mathcal{H}$, if*

$$\lim_n \left(2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x+x_n\|^2_{\mathcal{H},A}\right) = 0,$$

then there is a sequence of open half spaces $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ such that

1. *There is $n_0 \in \mathbb{N}$ such that $x \in H_n$ and $x_n \in H_n$ if $x_n \in A$ for $n \geq n_0$.*
2. *For every $\delta > 0$ there is some n_δ such that*

$$x, x_n \in \overline{\left(co(A \cap H_n) + B(0,\delta)\right)}^{\sigma(E,F)}$$

for all $n \geq n_\delta$.

Proof. Let us choose a point $a_H \in H \cap A$ and set $D_H = co(H \cap A)$ for every $H \in \mathcal{H}$, and $D_H^\delta := D_H + \delta B_E$, where $H \in \mathcal{H}$. We denote by p_H^δ the Minkowski functional of the convex body $\overline{D_H^\delta}^{\sigma(E,F)} - a_H$ and we define the $\sigma(E,F)$-lower semicontinuous norm p_H by the formula

$$p_H(x)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \left(p_H^{1/n}(x)\right)^2$$

for every $x \in E$. Finally we define the non-negative, convex, and $\sigma(E,F)$-lower semicontinuous function ψ_H^A as $\psi_H^A(x) := p_H(x - a_H)$ for every $x \in E$. Since A is bounded the family of functions $\{\psi_H^A : H \in \mathcal{H}\}$ is uniformly bounded on bounded sets and Theorem 10 gives us an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ such that if $x \in A \cap H$ for some $H \in \mathcal{H}$ and (x_n) is a sequence in E with

$$\lim_n \left(2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x+x_n\|^2_{\mathcal{H},A}\right) = 0,$$

then there is a sequence of open half spaces $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ satisfying:

(a) There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.
(b) $\lim_{n\to\infty} \left[\frac{1}{2}\psi_{H_n}^A(x_n)^2 + \frac{1}{2}\psi_{H_n}^A(x)^2 - \psi_{H_n}^A(\frac{x+x_n}{2})^2\right] = 0$

Convexity arguments and the above definitions allow us to finish the proof, see [158, Theorem 3].

Remark 1. Corollary 3 provide a straightforward proof of Theorem 9. Indeed, for every $m, p \in \mathbb{N}$ we fix the family $\mathcal{H}_{m,p}$ of $\sigma(E,F)$-open half spaces H such that diameter of $A_p \cap H$ is less than $1/m$. If we apply now Corollary 3 we get an equivalent norm $\|\cdot\|_{m,p}$ that verifies its conclusions. Thus for any sequence (x_n) and x such that

$$\lim_n \left(2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x + x_n\|_{m,p}^2\right) = 0,$$

we have

$$\|x - x_n\| \leq 1/m + \delta \text{ for } n \geq n_\delta, \tag{15}$$

whenever $x \in A_p \cap H$ for some $H \in \mathcal{H}_{m,p}$. Let us take $c_{m,p}$ such that $\|\cdot\|_{m,p} \leq c_{m,p}\|\cdot\|$. If we set

$$\|x\|_0^2 := \sum_{m,p=1}^{\infty} \frac{1}{c_{m,p}2^{m+p}} \|x\|_n^2,\ x \in E,$$

we obtain the renorming that we are looking for. Indeed, given any sequence (x_n) and x in E such that

$$\lim_n \left(2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2\right) = 0$$

by convexity arguments we have that

$$\lim_n \left(2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x + x_n\|_{m,p}^2\right) = 0,$$

for every $m, p \in \mathbb{N}$. If we fix $\varepsilon > 0$ and we take $m \in \mathbb{N}$ with $1/m < \varepsilon/2$, it follows from our hypothesis that there exists $p \in \mathbb{N}$ such that $x \in H \cap A_p$ and $H \cap A_p$ has diameter less than $1/m$, thus $H \in \mathcal{H}_{m,p}$. Inequality (15) says that $\|x - x_n\| \leq 1/m + \varepsilon/2 < \varepsilon$ for $n \geq n_{\varepsilon/2}$, so the proof is over.

Let us present now our proof for the following theorem. It is a fundamental result within the theory obtained in [78].

Theorem 11. *Let E be a normed space with a norming subspace F in E^*. If there is a sequence of sets (A_n) such that for every $x \in E$ and every $\varepsilon > 0$ there is $p \in N$ and a $\sigma(E,F)$-open half space H such that $x \in H \cap A_p$ is not empty and can be covered by finitely many sets of diameter less than ε, then E admits an equivalent $\sigma(E,F)$-lower semicontinuous and **LUR** norm.*

Our proof is based on the following Lemma together with Theorem 10.

Lemma 2. *Let E be a normed space with a norming subspace F in E^*. Let A be a subset of E, $\varepsilon > 0$ and $\mathcal{H}$ a family of $\sigma(E,F)$-open half spaces such that for every $H \in \mathcal{H}$ the slice $H \cap A$ is not empty and covered by finitely many sets of diameter less than ε. Then there is a family $\{\psi_H : H \in \mathcal{H}\}$ of non-negative, convex and $\sigma(E,F)$-lower semicontinuous functions such that,* *given sequences* $(x_n) \subset E$ *and* $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ *with* $x \in A \cap H_n$, *for every* $n \in \mathbb{N}$, *it follows that*

$$\|x_n - x\| < 3\varepsilon$$

for n *big enough, whenever we have*

$$\lim_n \left[(\frac{1}{2}\psi_{H_n}(x_n)^2 + \frac{1}{2}\psi_{H_n}(x)^2 - \psi_{H_n}(\frac{x_n + x}{2})^2 \right] = 0.$$

Proof. Let us fix a basis $\mathcal{B}$ for the norm topology on E and choose a point $a_B \in B$ for every $B \in \mathcal{B}$. Since there is a finite set $S \subset E$ such that

$$A \cap H \subset S + B(0, \varepsilon),$$

for fixed $0 < \delta < \varepsilon$, we find finite subfamilies $\mathcal{F}^\delta \subset \mathcal{B}$ of sets with diameter less than δ so that

$$A \cap H \subset S + B(0, \varepsilon) \subset \bigcup \mathcal{F}^\delta + B(0, \varepsilon).$$

For every $F \in \mathcal{F}^\delta$ we set $D_{F,\varepsilon} = \mathrm{co}(F) + B(0, 2\varepsilon)$ and denote by p_{F^δ} the Minkowski functional of the convex body

$$\overline{D_{F,\varepsilon}}^{\sigma(E,F)} - a_F.$$

Then, we define the non-negative, convex and $\sigma(E, F)$-lower semicontinuous function ψ_H by the formula

$$\psi_H(x)^2 = \sum_{F \in \mathcal{F}^\delta} p_{F^\delta}(x - a_F)^2, x \in E.$$

Let us observe that ψ_H is well defined since the sum has finite support. Let us prove that $\{\psi_H : H \in \mathcal{H}\}$ is the family that we are looking for. Fix sequences $(x_n) \subset E$ and $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ such that $x \in A \cap H_n$ for every $n \in \mathbb{N}$. If we have that

$$\lim_n \left[(\frac{1}{2}\psi_{H_n}(x_n)^2 + \frac{1}{2}\psi_{H_n}(x)^2 - \psi_{H_n}(\frac{x_n + x}{2})^2 \right] = 0,$$

we obtain that

$$\lim_n \left[(2p_{F_n^\delta}(x_n - a_{F_n^\delta})^2 + 2p_{F_n^\delta}(x - a_{F_n^\delta})^2 - p_{F_n^\delta}((x_n + x) - 2a_{F_n^\delta})^2 \right] = 0,$$

for every $F_n^\delta \in \mathcal{F}_n^\delta$ that we might choose for $n = 1, 2, \ldots$; here $\mathcal{F}_n^\delta$ denotes the finite family fixed above and made up with sets of the basis $\mathcal{B}$ to ensure that

$$A \cap H_n \subset \bigcup \mathcal{F}_n^\delta + B(0, \varepsilon).$$

In particular, since $x \in A \cap H_n$ for every $n \in \mathbb{N}$, we can take a set $F_n^\delta \in \mathcal{F}_n^\delta$ so that $x \in F_n^\delta + B(0,\varepsilon)$. Then we have that

$$\lim_n [p_{F_n^\delta}(x_n - a_{F_n^\delta}) - p_{F_n^\delta}(x - a_{F_n^\delta})] = 0.$$

Since $x \in F_n^\delta + B(0,\varepsilon)$ we conclude that

$$P_{F_n^\delta}(x - a_{F_n^\delta}) < 1 - \varepsilon$$

and consequently there is n_0 such that

$$P_{F_n^\delta}(x_n - a_{F_n^\delta}) < 1 - \varepsilon,$$

for $n \geq n_0$. We have proved then that

$$x_n, x \in D_{F_n^\delta}$$

for $n \geq n_0$, and since $\operatorname{diam}(D_{F_n^\delta}) \leq 2\varepsilon + \delta$ we conclude that

$$\|x_n - x\| < 3\varepsilon$$

whenever $n \geq n_0$ and the proof is over.

We arrive now to our proof of Theorem 11:

Proof (Theorem 11). Let us consider the family $\mathcal{H}_{m,p}$ of all $\sigma(E,F)$-open half spaces such that $A_p \cap H$ is not empty and can be covered with finitely many sets of diameter less than $1/m$. If we apply the previous Lemma for the family $\mathcal{H}_{m,p}$ and the set A_p, we obtain a family of non-negative, convex and $\sigma(E,F)$-lower semicontinuous functions $\{\psi_H^{m,p} : H \in \mathcal{H}_{m,p}\}$ such that if

$$\lim_n \left[\frac{1}{2}\psi_{H_n}^{m,p}(x_n)^2 + \frac{1}{2}\psi_{H_n}^{m,p}(x)^2 - \psi_{H_n}^{m,p}\left(\frac{x_n + x}{2}\right)^2\right] = 0,$$

for sequences $(x_n) \subset E$, $\{H_n \in \mathcal{H}_{m,p} : n = 1, 2, ...\}$ and $x \in A \cap H_n$ for every $n \in \mathbb{N}$, then $\|x_n - x\| \leq 3/m$ for n big enough. Without loss of generality we can and do assume that the sets A_p are bounded and therefore the families $\{\psi_H^{m,p} : H \in \mathcal{H}_{m,p}\}$ as defined in Lemma 2 are uniformly bounded on bounded sets. Thus we can apply Theorem 10 to obtain an equivalent norm $\|\cdot\|_{m,p}$ that verifies its claims 1 and 2 for any sequence (x_n) and x such that

$$\lim_n \left(2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x + x_n\|_{m,p}^2\right) = 0.$$

Let us take $c_{m,p}$ with $\|\cdot\|_{m,p} \leq c_{m,p}\|\cdot\|$. If we set

$$\|x\|_0^2 := \sum_{m,p=1}^{\infty} \frac{1}{c_{m,p}2^{m+p}} \|x\|_n^2,\ x \in E,$$

we obtain the renorming that we are looking for. Indeed, given any sequence (x_n) and x in E such that

$$\lim_n \left(2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2\right) = 0,$$

by convexity arguments we have that

$$\lim_n \left(2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x + x_n\|_{m,p}^2\right) = 0, \tag{16}$$

for every $m, p \in \mathbb{N}$. If we fix $\varepsilon > 0$ and we take $m \in \mathbb{N}$ with $3/m < \varepsilon$, and it follows from our hypothesis that there exists $p \in \mathbb{N}$ such that $x \in H \cap A_p$ and $H \cap A_p$ can be covered by finitely many sets of diameter less than $1/m$, thus $H \in \mathcal{H}_{m,p}$. Identity (16) and Theorem 10 imply that there is a sequence $\{H_n \in \mathcal{H}_{m,p} : n = 1, 2, ...\}$ such that

1. There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.
2. $\lim_{n\to\infty} \left[\frac{1}{2}\psi_{H_n}^{m,p}(x_n)^2 + \frac{1}{2}\psi_{H_n}^{m,p}(x)^2 - \psi_{H_n}^{m,p}(\frac{x+x_n}{2})^2\right] = 0$

Now we apply Lemma 2, used to obtain the functions $\psi_H^{m,p}$, to deduce from assertion 2 above that $\|x - x_n\| \leq 3/m < \varepsilon$ for n big enough, so the proof is over.

Theorem 11 has been used in [136] to prove renormings in spaces $C(K)$ based on the uniform structure of the compact space itself. If $K \subset [0,1]^\Gamma$, the uniform continuity of a given $x \in C(K)$ can be described in terms of the set Γ of coordinates functionals. When we have a descriptive process to do so we will be able to produce a **LUR** renorming of $C(K)$, [136]:

Theorem 12. *Let $K \subset [0,1]^\Gamma$ be a compact space such that there is a sequence of sets (A_n) in $C(K)$ with the property, that for every $x \in C(K)$ and every $\varepsilon > 0$ there exist $p \in N$ and a pointwise open half space H together with a finite subset $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ of coordinates in Γ such that $x \in H \cap A_p$, and for every $y \in A \cap H_p$ there exists $\delta_y > 0$ so that*

$$|y(s) - y(t)| < \varepsilon,$$

whenever $|s(\gamma_i) - t(\gamma_i)| < \delta_y$ for $i = 1, 2, \ldots, N$.

Then $C(K)$ admits a pointwise lower semicontinuous equivalent **LUR** *norm.*

The previous theorem provides a tool to prove the following result, see [136].

Corollary 4. *$C(K)$ admits a pointwise lower semicontinuos* **LUR** *norm in the following cases:*

1. K is σ-discrete.
2. K is the w^* dual unit ball of a dual Banach space with a dual **LUR** *norm.*

3. *$K \subset [0,1]^P$ is separable, where P is a Polish space and every $s \in K$ has at most countably many discontinuities.*

To establish the link between the theory of generalized metric spaces and **LUR** renormings we recall the following definitions.

Definition 5. Let E be a normed space and F a norming subspace in the dual E^*. A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets on E is called $\sigma(E,F)$-slicely isolated (or $\sigma(E,F)$-slicely relatively discrete) if it is a disjoint family of sets such that for every

$$x \in \bigcup \{B_i : i \in I\}$$

there exist a $\sigma(E,F)$-open half space H and $i_0 \in I$ such that

$$H \cap \bigcup \{B_i : i \in I, i \neq i_0\} = \emptyset \text{ and } x \in B_{i_0} \cap H.$$

If H is such that it meets a finite number of elements in $\mathcal{B}$ we say that $\mathcal{B}$ is $\sigma(E,F)$-slicely relatively locally finite.

The connection between generalized metric spaces, see [95, 97, 98], and **LUR** renormings is described in the work of A. Moltó, S. Troyanski, M. Valdivia and the second named author here, see [139, Chapter 3] and the references therein; in this monograph the network point of view for **LUR** renormings is the central one with an extensive use of Stone's theorem. The seminal papers by R. W. Hansell [102, 103] together with those by J. E. Jayne, I. Namioka and C. A. Rogers [111, 110, 112] are an essential part of this development. The connection between both theories was well established in [140]. A main result in this area is the following one, see [139, Chapter III, Theorem 3.1], which is equivalent to Theorem 9 when we have in mind Stone's theorem on the paracompactness of a metric space. Thanks to Theorem 11 we also have this locally finite version obtained in [78].

Theorem 13. *Let E be a normed space and F a norming subspace in the dual E^*. The following statements are equivalent:*

1. *The space E admits an equivalent $\sigma(E,F)$-lower semicontinuous and **LUR** norm*
2. *The norm topology has a network $\mathcal{N}$ that can be written as $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ where each of the families $\mathcal{N}_n$ is $\sigma(E,F)$- slicely relatively locally finite.*
3. *The norm topology has a network $\mathcal{N}$ that can be written as $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ where each of the families $\mathcal{N}_n$ is $\sigma(E,F)$- slicely isolated of sets which are difference of convex and $\sigma(E,F)$-closed sets.*

Proof. Let us show first that 2 implies 1. If we have a network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ for the norm topology as described in 2, we set

$$A_{p,q} := \bigcup \{N : N \in \mathcal{N}_q \| \cdot \| - \text{diam}(N) < 1/p\},$$

for $p, q \in \mathbb{N}$. Given $x \in E$ and $\varepsilon \geq 1/p > 0$, if we take q and $M \in \mathcal{N}_q$ with $x \in M \subset B(x, 1/(2p))$, then by the slicely locally finite property of the family $\mathcal{N}_q$ there is a $\sigma(E,F)$-open half space H with $x \in H \cap A_{p,q}$ and $H \cap A_{p,q}$ is covered by finitely many members of $\mathcal{N}_q$, all of them with diameter less than or equal to $1/p \leq \varepsilon$. Theorem 11 gives us the equivalent **LUR** norm. The construction to prove 1 implies 3 follows [158]: all points in the unit sphere of a **LUR** norm are denting points, then for $\varepsilon > 0$ fixed we will have a family of $\sigma(E,F)$-open half spaces $\mathcal{H}_\varepsilon$, covering the unit sphere S_E of our $\sigma(E,F)$-lower semicontinuous and LUR norm, and such that $\|\cdot\| - \operatorname{diam}(H \cap B_E) < \varepsilon$ for every $H \in \mathcal{H}_\varepsilon$. Let us choose a well order relation for the elements in $\mathcal{H}_\varepsilon$ and let us write

$$\mathcal{H}_\varepsilon = \{H_\gamma : \ \gamma < \Gamma\}$$

where we denote $H_\gamma = \{x \in E : f_\gamma(x) > \lambda_\gamma\}$, $f_\gamma \in B_{E^*} \cap F$.

We set

$$M_\gamma := H_\gamma \cap B_E \setminus \left(\bigcup \{H_\beta \cap B_E : \ \beta < \gamma\} \right),$$

for every $\gamma < \Gamma$. Let us define the sets $M_\gamma^n := \{x \in M_\gamma : f_\gamma(x) \geq \lambda_\gamma + 1/n\}$. It follows that, when $x \in M_\gamma^n$ and $y \in M_\beta^n$ for $\gamma \neq \beta$ then we have either

$$f_\gamma(x) - f_\gamma(y) \geq \ 1/n \quad (\text{ when } \gamma < \beta), \tag{17}$$

or

$$f_\beta(y) - f_\beta(x) \geq \ 1/n \quad (\text{ when } \beta < \gamma). \tag{18}$$

In in any case

$$\|x - y\| \geq 1/n \tag{19}$$

because the linear functionals f_γ and f_β are assumed to be in $B_{E^*} \cap F$. If we fix $x \in S_E$ the **LUR** condition of the norm gives a slice

$$G = \{y \in B_E : g(y) > \mu\},$$

with $g(x) > \mu$, $g \in B_{E^*} \cap F$ and $\|\cdot\| - \operatorname{diam}(G) < 1/n$, thus G meets at most one member of the family of sets $\{M_\gamma^n : \gamma < \Gamma\}$ by(19).

These families of closed and convex subsets of E cover the unit sphere S_E and they suffice to describe the network there. To describe the network for the whole space E we need to make differences of closed convex sets. To this end, take $x \in E \setminus \{0\}$, and $y := x/\|x\|$. If we take $\gamma_0 < \Gamma$ so that $y \in M_{\gamma_0}$ and n big enough to have $f_{\gamma_0}(y) > \lambda_{\gamma_0} + 1/n$, we will have a rational number $0 < \mu_x < 1$, close enough to one, such that $f_{\gamma_0}(\mu_x y) > \lambda_{\gamma_0} + 1/n$. The **LUR** condition of the norm tells us that there is $\delta_x > 0$ such that $\|(y+z)/2\| > 1 - \delta_x$ implies that $\|y - z\| < 1/n$ whenever $\|z\| \leq 1$.

Let us take a rational number ρ such that

$$\rho > \|x\| > \rho(1-\delta_x) \text{ and } \rho\mu_x < \|x\|.$$

Then $x \in \rho M^n_{\gamma_0}$ and $\|\cdot\| - \text{diam}(\rho M^n_{\gamma_0}) < \rho\varepsilon$. Moreover, if we choose $g_x \in B_{E^*} \cap F$ such that $g_x(x) > \rho(1-\delta_x)$ then, for any $z \in \bigcup\{\rho M^n_\gamma : \gamma < \Gamma\}$ with $g_x(z) > \rho(1-\delta_x)$, we will have

$$g_x(z/\rho) > 1-\delta_x \text{ and } g_x(y) > \rho(1-\delta_x)/\|x\| > 1-\delta_x.$$

Thus $\left\|\frac{y+z/\rho}{2}\right\| > 1-\delta_x$, and we have that $\|y - z/\rho\| < 1/n$, and therefore $\gamma = \gamma_0$. Consequently, if we consider sets $M^{n,p}_\gamma := \{x \in M^n_\gamma \cap S_E : \delta_x > 1/p\}$ and we take the family

$$\{\rho M^{n,p}_\gamma \setminus \rho(1-1/p)B_E : \gamma < \Gamma\},$$

for rational numbers ρ and integers p, n fixed, we form an slicely isolated family of sets. All together, with the same construction done for every $\varepsilon > 0$ we obtain a family

$$\bigcup\Big\{\{\rho M^{n,p}_\gamma(\varepsilon) \setminus \rho(1-1/p)B_E : \gamma < \Gamma\} : \rho \in \mathbb{Q}, n, p \in \mathbb{N}, \varepsilon > 0\Big\}$$

which is a network for the norm topology. Taking $\varepsilon = 1/r, r = 1, 2, ...$ we obtain the network for the norm that we are looking for. The fact that 3 implies 2 is just by definition and the proof is over.

Remark 2. The same construction can be adapted to provide a $\sigma(E,F)$-slicely isolated network for the $\sigma(E,F)$-topology when the given norm $\|\cdot\|$ on E only verifies that $\sigma(E,F) - \lim_n x_n = x$ whenever we have

$$\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0,$$

i.e. when $\|\cdot\|$ is a $\sigma(E,F)$-**LUR** norm, see the Main Lemma 3.19 and Theorem 3.21 in Section 3.3 of [139]. The existence of such a network was first described in [140]. Stone's theorem plays again an important role for a deep knowledge of the connection with **LUR**-renormings on Banach spaces with a w-**LUR** norm, as well as, dual **LUR**-renorming on the dual of an Asplund space with a w^*-**LUR** norm; note that the unit sphere in this class of spaces belongs to the generalized metric space class of Moore spaces, see [95], for the w or the w^* topology, respectively, see [140, 139].

The following result that answered a long standing open problem shows the way to construct a slicely isolated network for the norm topology from a slicely isolated network in the weak topology.

Theorem 14 ([140]). *A normed space E has a $\sigma(E,E^*)$-**LUR** norm if, and only if, it has an equivalent **LUR** norm*

The proof of this result has a key point in the following:

Proposition 3. *Let E be a normed space and d any metric on it generating a topology finer than the weak topology. For every norm discrete family of sets*

$$\{D_\gamma : \gamma \in \Gamma\},$$

each D_γ can be decomposed as $D_\gamma = \bigcup_{n=1}^\infty D_\gamma^n$ in such a way that each family $\{D_\gamma^n : \gamma \in \Gamma\}$, $n \in \mathbb{N}$, is discrete for the d-topology.

Proof. By Corollary 2.36 and Theorem 2.28 in [139] the family $\{D_\gamma : \gamma \in \Gamma\}$ can be decomposed as $D_\gamma = \bigcup_n^\infty D_\gamma^n$ where families $\{D_\gamma^n : \gamma \in \Gamma\}$ are relatively discrete in their union for the metric d and fixed $n \in \mathbb{N}$. The families

$$\left\{D_\gamma^{n,m} := \{x \in D_\gamma^n : d(x,y) \geq 1/m \text{ for every } y \notin D_\gamma^n\} : \gamma \in \Gamma\right\}$$

are d-discrete for fixed $n, m \in \mathbb{N}$. Since for every $\gamma \in \Gamma$ we have $D_\gamma = \bigcup_{n,m=1}^\infty D_\gamma^{n,m}$ the proof is over.

A sketch for the proof of Theorem 14 now follows. The network constructed for the weak topology of E based on Remark 2 also provides us with a metric d on E, generating a topology finer than the weak topology, and such that there is a sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of E satisfying that the family of sets

$$\{A_n \cap H : H \text{ a weak open half space }, n \in \mathbb{N}\}$$

is a network for the topology generated by d, see [141, Theorem 3.21]. Let us take a σ-discrete basis $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ for the norm topology of E. Every discrete family $\mathcal{B}_n$ can be decomposed by Proposition 3 in countably many families $\mathcal{B}_{n,m}$ where each of them must be d-discrete. Thus we see that

$$C_{n,m.p} = \bigcup \mathcal{B}_{n,m} \cap A_p$$

is a sequence of subsets of E such that

$$\{C_{n,m,p} \cap H : H \text{ a weak open half space }, n, m, p \in \mathbb{N}\}$$

is a network for the norm topology of E and Theorem 9 finishes the proof.

In dual Asplund spaces we have the following result of M. Raja, [166]:

Theorem 15. *The dual Banach space E^* of an Asplund space E has a* $\sigma(E, E^*)$-**LUR** *norm if, and only if, it has an equivalent dual* ***LUR*** *norm*

A proof of this result is given in [139, Corollary 3.24].

The previous constructions finally lead to characterizations through the basis of the norm topology, [157].

Theorem 16 (LUR-Basis). *Let E be a normed space with a norming subspace $F \subset E^*$. E admits an equivalent $\sigma(E, F)$-lower semicontinuous and*

***LUR** norm if, and only if, the norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is $\sigma(E,F)$-slicely isolated and norm discrete.*

The above Theorem is a great culmination after years of research on the interplay between Topology and Renorming Theory. Indeed, relative discretness with slices is the necessary and sufficient condition to be added to norm discretness in the Bing metrization Theorem, [65, Theorem 4.4.8], in order to have an equivalent **LUR**-norm. For a dual space E^*, the w^*-compactness of the unit ball plays its role and the result is valid with w^*-relatively discreteness instead of slicely isolatedness.

Theorem 17 (LUR*-Basis). *Let E^* be dual Banach space. E^* admits an equivalent dual and **LUR** norm if, and only if, the norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that each family $\mathcal{B}_n$ is relatively weak*-discrete in its union and norm discrete on the whole space E^*.*

Proof. Dual **LUR**-renorming on E^* is equivalent to have a sequence (A_n) of subsets of E^* such that $\{A_n \cap W : W \text{ is } w^* - \text{open}\}$ is a network of the norm topology by Raja's Theorem, [163]. If we have a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that each family $\mathcal{B}_n$ is relatively *weak**-discrete in its union we set $A_n := \bigcup\{B : B \in \mathcal{B}_n\}$ and we have the network condition satisfied. The converse implication follows from Theorem 16.

3.2 Strictly convex renorming

A norm $\|\cdot\|$ in a vector space E is said to be strictly convex if the unit sphere doest not contain non trivial segments, *i.e.*

$$\left\|\frac{x+y}{2}\right\| < 1 \text{ whenever } \|x\| = \|y\| = 1 \text{ and } x \neq y.$$

A norm $\|\cdot\|$ is strictly convex if, and only if

$$\left[2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0\right] \text{ implies } x = y.$$

The topological property strongly connected with strictly convex norms is the following one, introduced in [159].

Definition 6. We say that a topological space (X,τ) is a $T_0(*)$-space, (resp. is an $T_1(*)$-space) if there are families of open sets $\mathcal{W}_n$, $n = 1, 2, \ldots,$ such that for $x \neq y$ there are some $p \in \mathbb{N}$ and either we have $y \notin \text{Star}(x, \mathcal{W}_p) \neq \emptyset$ or $x \notin \text{Star}(y, \mathcal{W}_p) \neq \emptyset$ (resp. $x \notin \text{Star}(y, \mathcal{W}_p \neq \emptyset)$).

For a family $\mathcal{F}$ of subsets of X our definition of Star is as follows:

$$\mathrm{Star}(x,\mathcal{F}) := \bigcup\{F : x \in F \in \mathcal{F}\}.$$

When a norm in $(E,\|\cdot\|)$ is strictly convex the weak topology is $T_0(*)$. Indeed, $\{E \setminus \rho B_E : \rho \in \mathbb{Q}\}$ is a countable family of weak open sets that $T_0(*)$-separates any pair of points x, y with $\|x\| \neq \|y\|$. Moreover, if two different points $x, y \in E$ are such that $\|x\| = \|y\| = r$, we can choose $n \in \mathbb{N}$ such that for any $g \in B_{E^*}$ we have either $g(x) \leq r - 1/n$ or $g(y) \leq r - 1/n$. Indeed, if not there is a sequence $g_n \in B_{E^*}$ such that

$$g_n(x) > r - 1/n, g_n(y) > r - 1/n, n = 1, 2, \ldots$$

If g is a w^* cluster point of the sequence (g_n), then we have $g(x) = g(y) = r$ and therefore

$$g\left(\frac{x+y}{2}\right) = r \text{ and } \left\|\frac{x+y}{2}\right\| = r,$$

a contradiction with the strict convexity of the given norm. Thus the families $\mathcal{H}_n^r$ of all open half spaces of the form $H = \{x \in E : g(x) > r - 1/n\}$ for $g \in B_{E^*}$ verifies the separation property $T_0(*)$ for the weak topology on E. More important is the fact that the converse is true and it provides an answer to an old question of Lindenstrauss, see [133, Question 18] and [159, Theorem 2.7]:

Theorem 18 (Strictly Convex Renorming). *Let E be a normed space with a norming subspace $F \subset E^*$. Then E admits an equivalent $\sigma(E,F)$-lower semicontinuous and strictly convex norm if, and only if, there are families $\mathcal{H}_n$, $n = 1, 2, \ldots,$ of $\sigma(E,F)$-open half spaces that $T_0(*)$ separates points of E.*

Proof. Let us fix $n, m \in \mathbb{N}$ and apply the Slice Localization Corollary 10 to the family of slices given by $\{mB_E \cap H : H \in \mathcal{H}_n\}$ to get the equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{n,m} \leq c_{n,m}\|\cdot\|$. If we set

$$|||\cdot|||^2 := \sum_{n,m=1}^{\infty} \frac{1}{c_{n,m}2^{n+m}}\|\cdot\|_{n,m}^2,$$

we obtain the equivalent strictly convex norm we are looking for. Indeed, the condition

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right| = |||x||| = |||y|||,$$

implies that

$$2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2 = 0,$$

and thus

$$2\|x\|_{m,n}^2 + 2\|y\|_{m,n}^2 - \|x+y\|_{m,n} = 0,$$

for every $m, n \in \mathbb{N}$. The Slice Localization Theorem allows us to find some $H_n \in \mathcal{H}_n$ such that $x, y \in H_n$ for every $n \in \mathbb{N}$ if both x and y belong to mB_E; that is a contradiction with the $T_0(*)$ separation assumed by hypothesis.

For a scattered compact space K the previous theorem provides a dual strictly convex renorming characterization for $C(K)^*$ in terms of the topology of the compact space itself only, leading then to a result in the vein of Lindenstrauss question cited above.

Theorem 19 (Scattered case). *Let K be a scattered compact space. Then the dual space $C(K)^*$ admits a strictly convex dual (resp.* **LUR***) norm if, and only if, K is a $T_0(*)$-space, (resp. σ-discrete).*

The result above corresponds with Theorem 3.1 in [159]. Raja's Corollary 4.4 in [166], is the **LUR** case: in this case K must be σ-discrete and compact. Another result in the same line says that a compact space K has a σ-isolated network, *i.e.* is descriptive, if, and only if, the dual space $C(K)^*$ admits a w^*-**LUR** dual equivalent norm, [167, Theorem 1.3].

When the $T_1(*)$ separation property is stressed asking for a network condition we arrive to the following result, see [100]:

Theorem 20. *Let E be a normed space with a norming subspace $F \subset E^*$. The following statements are equivalent:*

1. *E admits an equivalent $\sigma(E,F)$-lower semicontinuous and $\sigma(E,F)$-***LUR** *norm.*
2. *$\sigma(E,F)$ admits a network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ where $\mathcal{N}_n$ is $\sigma(E,F)$-slicely isolated for every $n \in \mathbb{N}$.*
3. *There are families $\mathcal{H}_n$ of $\sigma(E,F)$-open half spaces and non void sets $A_p \subset E$ such that*
$$\{\mathrm{Star}(x, \mathcal{H}_n) \cap A_p : n, p \in \mathbb{N}\}$$
is a network of the $\sigma(E,F)$-topology on $E \setminus \{0\}$.

Proof. $1 \Rightarrow 2$ A $\sigma(E,F)$-lower semicontinuous and $\sigma(E,F)$-**LUR** norm gives us a $\sigma(E,F)$-slicely isolated network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ as observed in Remark 2, [140, 139, 167].

$2 \Rightarrow 3$ If we take $A_n = \bigcup \mathcal{N}_n$ and $\mathcal{H}_n$ is the family of $\sigma(E,F)$-open half spaces meeting at most one element of $\mathcal{N}_n$, we have that

$$\{\mathrm{Star}(x, \mathcal{H}_n) \cap A_n : x \in A_n\}$$

is a refinement of $\mathcal{N}_n$, from where the network condition follows.

$3 \Rightarrow 1$ Let us fix $n, p \in \mathbb{N}$ and apply the Slice Localization Theorem 3 to the family of slices given by $\{A_n \cap H : H \in \mathcal{H}_p\}$ to get an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{n,p} \leq c_{n,p}\|\cdot\|$. If we set

$$|||\cdot|||^2 := \sum_{n,p=1}^{\infty} \frac{1}{c_{n,p}2^{n+p}}\|\cdot\|_{n,p}^2$$

we obtain the equivalent $\sigma(E,F)$-**LUR** norm we are looking for. Indeed, in case we have

$$\lim_n (2|||x_n|||^2 + 2|||x|||^2 - |||x+x_n|||^2) = 0$$

standard convex arguments imply that

$$\lim_n (2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x+x_n\|_{m,p}^2) = 0,$$

for every $m,p \in \mathbb{N}$. Let us fix $f_i \in F \cap B_{E^*}$, $\varepsilon/2 > \delta > 0$ and

$$W = \{y \in E : |f_i(y) - f_i(x)| \leq \varepsilon + \delta, i = 1,2,\dots,m\}.$$

By definition of network there are $m,p \in \mathbb{N}$ and such that

$$x \in \mathrm{Star}(x,\mathcal{H}_m) \cap A_p \subset \{y \in E : |f_i(y) - f_i(x)| < \varepsilon/2, i = 1,2,\dots,m\}.$$

Since

$$\lim_n (2\|x_n\|_{m,p}^2 + 2\|x\|_{m,p}^2 - \|x+x_n\|_{m,p}^2) = 0,$$

the Slice Localization Corollary 10 tells us that there is a sequence of open half spaces

$$\{H_n \in \mathcal{H} : n = 1,2,...\} \subset \mathcal{H}_m$$

such that:

1. There is $n_0 \in \mathbb{N}$ such that $x \in H_n$ for $n \geq n_0$.
2. There is some n_δ such that

$$x, x_n \in \overline{(\mathrm{co}(A_p \cap H_n) + B(0,\delta))}^{\sigma(E,F)},$$

   for every $n \geq n_\delta$.

Since

$$A_p \cap H_n \subset \mathrm{Star}(x,\mathcal{H}_m) \cap A_p \subset \{y \in E : |f_i(y) - f_i(x)| < \varepsilon/2, i = 1,2,\dots,m\},$$

we have that $\big(co(A_p \cap H_n) + B(0,\delta)\big) \subset W$ for n big enough. Finally

$$x_n \in \overline{(\mathrm{co}(A_p \cap H_n) + B(0,\delta))}^{\sigma(E,F)} \subset W$$

for n big enough and the proof is over.

3.3 Some notes and open problems

Networks have become an essential tool in renorming theory of Banach spaces. As a set theoretical tool networks have allowed to go farther than traditional decomposition methods based on system of coordinates on the space. Networks have been a scalpel where previous machinery did not work at all: a good example of this is the way in which R. Haydon proved that a Banach space E admits an equivalent **LUR** norm if its dual E^* has a dual **LUR** norm, [106]. Haydon approach goes over the structure of compact spaces living in the dual space (E^*, w^*) and makes use of networks together with the fact that E is an Asplund space. From a topological point of view an Asplund space E is nothing else than a Banach space such that its dual space E^* is a Lindelöf space for the topology $\gamma(E^*, E)$ of uniform convergence on separable bounded subsets of E, [153]: recall that w^*-compact spaces in such dual spaces are called Radon-Nikodým compacta, [145, 146]. More references for the topology γ can be found in Section 5.1.

A compact topological space K is called descriptive if it has a σ-isolated (*i.e.* relatively discrete) network, Namioka-Phelps if it is homeomorphic to a w^*-compact subset of a dual Banach space with a dual **LUR**-norm. M. Raja showed in [166] that a compact space K is descriptive if, and only if, $C(K)^*$ admits an equivalent dual w^*-**LUR** norm. Moreover, K is descriptive and scattered if, and only if, $C(K)^*$ admits an equivalent dual **LUR** norm, and if, and only if, K is σ-discrete, see [166]. Even more, a compact space K is Namioka-Phelps if, and only if, it is descriptive and Radon-Nykodym, see [166].

A compact space K is called Gruenhage compact, [96], if there is a sequence $(\mathcal{U}_n)_{n=1}^\infty$ of open sets such that for two different elements $x, y \in K$ there is $p \in \mathbb{N}$ and $U \in \mathcal{U}_p$ such that

1. $U \cap \{x, y\}$ is a singleton;
2. either x lies in only finitely many $V \in \mathcal{U}_p$, or y lies in only finitely many $V \in \mathcal{U}_p$.

Descriptive compact spaces are Gruenhage compacta. When K is descriptive (resp. Gruenhage) the w^* dual unit ball $B_{C(K)^*}$ is descriptive too (resp. Gruenhage). Gruenhage compact spaces are $T_0(*)$ but they are different classes of compact spaces, [159, 177]. If a Banach space E has a $T_0(*)$ dual unit ball with the w^*-topology, we conjecture that E^* admits an equivalent dual strictly convex norm, see[73]. A main question that remains open is the following:

Question 5. If K is a compact space with $T_0(*)$ separation property, is the same true for $B_{C(K)^*}$ with its w^*-topology?

A recent survey for $C(K)$-renormings can be found in the paper [178].

Network characterizations of classical classes of compact spaces coming form Functional Analysis, such as Eberlein, Talagrand or Gul'ko compacta, have been obtained, see [94, 77, 67]. Some of these characterizations are given in terms of covering properties of $K^2 \setminus \Delta$ related with σ-metacompactness. The following question seems to be open:

Question 6. Is there a network characterization of Corson compact spaces?

Concerning renorming properties remaining open questions where new topological methods could be needed are, for instance:

Question 7. If K is a descriptive compact space, is there an equivalent strictly convex norm on $C(K)$?

Question 8. If K is a compact space such that $C(K)$ admits an equivalent Frechet differentiable norm, is there an equivalent **LUR** norm on it?

Chapter 6 in [141] contains around 30 open problems related with the material collected in this paper for any interested reader.

4 Recent views about pointwise and weak compactness

It is commonly accepted that the study of compactness in Banach spaces, or more generally in functional analysis, is of great importance because of its applicability. Here are two well known examples that any reader might come up with. First, when dealing with normal operators on Hilbert spaces, the compactness of the operator ensures its diagonalization than can be used to solve some concrete differential equations. Second, for spaces $C(K)$, their pointwise compact subsets are sequentially compact, what with the help of Lebesgue Convergence and Riesz theorems is used to prove that weakly compact subsets of $C(K)$ are precisely those that are uniformly bounded and pointwise compact: note that sequential behaviour is needed here to be able to use Lebesgue Convergence theorem.

Our aim in this Section is double. On the one hand, Subsection 4.1 is devoted to present a recent *quantitative* approach to classical compactness results. This approach gives an extra insight to the classical results as well as triggers a number of open questions in different exciting research branches. We will give, for instance, *quantitative* versions of the angelicity of spaces $C_p(X)$ for X Lindelöf Σ-space, Grothendieck's characterization of weak compactness in spaces $C(K)$ and also of the Eberlein-Šmulian and Krein-Šmulian theorems. The above results specialized in Banach spaces lead to several equivalent measures of *non-weak compactness.* We also propose a method to measure the distance from a function $f \in \mathbb{R}^X$ to $B_1(X)$, that allows us to obtain, when X is Polish, a quantitative version of the well known Rosenthal's result stating that in $B_1(X)$ the pointwise relatively countably compact sets are pointwise compact. On the other hand, Subsection 4.2 contains quite recent results about the celebrated James' compactness theorem motivated by their applications to financial mathematics.

The state of the art of the topics presented in this section can be found, amongst others, in the references [2, 3, 4, 5, 32, 33, 42, 69, 90, 88, 91, 89]

4.1 A quantitative approach to compactness

In order to fix ideas, let us give an example of what we understand as a *quantitative* approach to the study of compactness. For a compact Hausdorff space K let us consider $C(K)$ embedded in $\mathbb{R}^K$, and let d be the metric of uniform convergence in $\mathbb{R}^K$. Let H be a uniformly bounded subset of $\mathbb{R}^K$. Observe that by Tychonoff's theorem $\overline{H}^{\mathbb{R}^K}$ is τ_p-compact and therefore, for H to being τ_p-relatively compact in $C(K)$ we just need that $\overline{H}^{\mathbb{R}^K}$ remains inside $C(K)$. If we write $\hat{d}$ to denote the *worst* distance of $\overline{H}^{\mathbb{R}^K}$ to $C(K)$ as defined by (1), *i.e.*

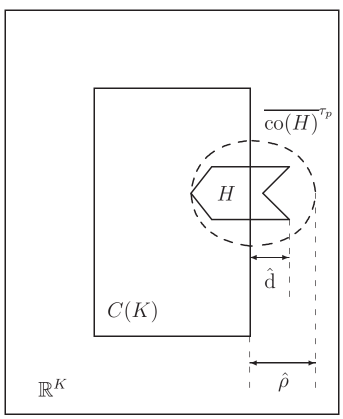

Fig. 1 Quantities and compactness

$$\hat{d} := \sup\left\{d(f, C(K)) : f \in \overline{H}^{\mathbb{R}^K}\right\},$$

then $\hat{d} = 0$ if, and only, if $\overline{H}^{\mathbb{R}^K}$ is contained in $C(K)$. It is natural to ask about formulas to compute d, $\hat{d}$ and useful estimates involving $\hat{d}$ that are equivalent to qualitative properties of the sets H's. As we will see these formulas and estimates can be given and, for instance, one can prove inequalities of the kind

$$\hat{d} \leq \hat{\rho} \leq 5\hat{d}, \qquad (20)$$

where $\hat{\rho}$ is the *worst* distance from the closed convex hull $\overline{\mathrm{co}(H)}^{\mathbb{R}^K}$ to $C(K)$ – see Figure 1. A moment of thought will suffice the reader to understand that inequalities (20) imply Krein-Šmulian theorem as we will explain later.

The *quantitative* approach to the study of compactness that we propose can be done indeed in more general situations than those of spases $C(K)$

as explained in the pages that follow. In general the distance to spaces of continuous functions is given by the formula (21) below and illustrated by Figure 2.

Theorem 21. *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2}\operatorname{osc}(f) \tag{21}$$

where

$$\operatorname{osc}(f) = \sup_{x\in X}\operatorname{osc}(f,x) = \sup_{x\in X}\inf\{\operatorname{diam} f(U) : U \subset X \text{ open}, x \in U\}.$$

A proof for the above result, when X is a paracompact space and the functions involved are assumed to be bounded, can be found in [22, Proposition 1.18]. In this proof, paracompactness of X and boundedness of the functions are used because Michael's selection theorem, see [138], is applied to prove, as an intermediate step, a particular case of the following result:

Theorem 22 ([109, Theorem 12.16]). *Let X be a normal space and let $f_1 \le f_2$ be two real functions on X such that f_1 is upper semicontinuous and f_2 is lower semicontinuous. Then, there exists a continuous function $f \in C(X)$ such that $f_1(x) \le f(x) \le f_2(x)$ for every $x \in X$.*

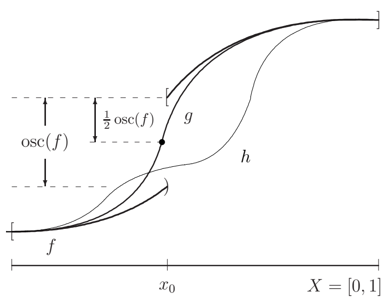

Fig. 2 Distance to C(X)

It was indeed pointed out in the remarks about Proposition 1.18 in [22] that Theorem 21 holds true for normal spaces and the fact that the boundedness of the functions involved does not make a real difference can be read in [7]. It should be noted also that the validity of Formula (21) characterizes normality of the space X, as the reader can easily check, see [7].

In the following definition we introduce a quantity that measures how far from $C(X)$ can go the cluster points of sequences in sets $H \subset \mathbb{R}^X$.

Definition 7. Let X be a topological space and (Z, d) a metric space. If H is a subset Z^X we define

$$\operatorname{ck}(H) := \sup_{(f_n)\subset H} d(\operatorname{clust}_{\mathbb{R}^X}(f_n), C(X,Z)).$$

Our convention is that $\inf \emptyset := +\infty$.

The following result links the quantity $\operatorname{ck}(H)$ with the worst distance $\hat{d}$ defined according to formula (1) by

$$\hat{d}(\overline{H}^{Z^X}, C(X,Z)) := \sup\{d(f, C(X,Z)) : f \in \overline{H}^{Z^X}\}.$$

Theorem 23 ([3]). *Let X be a Lindelöf Σ-space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{Z^X}, C(X,Z)) \leq 3\operatorname{ck}(H) + 2\hat{d}(H, C(X,Z)) \leq 5\operatorname{ck}(H).$$

Note that this result says that the worst distance to $C(X,Z)$ of cluster points of sequences in H controls the worst distance to $C(X,Z)$ of limits of converging nets in H. Observe that if $H \subset C(X,Z)$ is a τ_p-relatively countably compact subset of $C(X,Z)$ then $\operatorname{ck}(H) = 0$ and therefore one has $\hat{d}(\overline{H}^{Z^X}, C(X,Z)) = 0$. Consequently, the inequalities in Theorem 23 are a *quantitative* counterpart of the result saying that in the above conditions any subset H of $C(X,Z)$ that is τ_p-relatively countably compact in $C(X,Z)$ is τ_p-relatively compact.

Theorem 23 is based in the following one that is, in spirit, the *quantitative* counterpart of Theorem 1 in [152].

Theorem 24 (Quantitative angelicity [3]). *Let X be a Lindelöf Σ-space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then, for any $f \in \overline{H}^{Z^X}$ there exists a sequence (f_n) in H such that*

$$\sup_{x \in X} d(g(x), f(x)) \overset{(a)}{\leq} 2\operatorname{ck}(H) + 2\hat{d}(H, C(X,Z)) \overset{(b)}{\leq} 4\operatorname{ck}(H) \tag{22}$$

for any cluster point g of (f_n) in Z^X.

The proof of Theorem 24 uses as an important tool $\gamma_K(H)$ as defined below. Whereas (b) in Equation 22 is obvious, inequality (a) is proved by establishing a rather involved lemma stating that with the notation above

$$\sup_{x \in X} d(g(x), f(x)) \leq \sup_{K \subset X,\ \text{compact}} \gamma|_K(H),$$

and then proving that

$$\sup_{K \subset X,\ \text{compact}} \gamma|_K(H) \leq 2\operatorname{ck}(H) + 2\hat{d}(H, C(X,Z)).$$

Definition 8. Let X be a topological space and (Z,d) a metric space. If H is a subset Z^X and K a subset of X we write

$$\gamma_K(H) := \sup\left\{d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) : (f_m) \subset H, (x_n) \subset K\right\},$$

assuming the involved limits exist.

Note that $\gamma_K(H) = 0$ means in the language of [92] that H interchanges limits with K.

Corollary 5 ([152]). *Let X be a Lindelöf Σ-space. Then $C_p(X)$ is an angelic space.*

As of now the readers should be able to deduce by themselves how Corollary 5 follows from Theorem 23 and Theorem 24, whose combination is, therefore, a quantitative version of the angelicity of spaces $C_p(X)$.

We should stress that the previous results can be proved in the more general setting of spaces X being *web-compact, quasi-Souslin, etc.*, see [3, 6, 152]. From the angelicity of these $C_p(X)$ spaces non-trivial applications can be obtained, as for instance, regarding the study of compactness for the weak topology in locally convex spaces. Indeed, being aware of the fact that if $(E, \mathfrak{T})$ is a locally convex space then (E, w) embeds as a subspace of $C_p(E', \sigma(E', E))$, if $C_p(E', \sigma(E', E))$ is angelic, then its subspace (E, w) is also angelic. This is what happens when $(E, \mathfrak{T})$ is a locally convex space in class $\mathfrak{G}$, see Definition 4 and [40, 152]. Consequently, from the above and Theorem 2 it follows, as explained in [40], that *"dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even have a good behaviour for the weak topology: they are weakly angelic and their weakly compact subsets are metrizable if, and only if, they are separable".* A good idea of the impact that these techniques have had and still have in the theory of locally convex spaces can be guessed from their many applications over the years: see the the recent book [117] for a comprehensive collection of these applications.

If we deal with a compact space $X = K$ instead of a Lindelöf Σ-space in the previous results, some of the constants involved in the inequalities can be sharpened and other applications obtained.

Theorem 25 ([3, 33]). *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. We have*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \gamma_K(H) \leq 2\operatorname{ck}(H).$$

and for any $f \in \overline{H}^{\mathbb{R}^K}$, there is a sequence (f_n) in H such that

$$\sup_{x \in K} |g(x) - f(x)| \leq 2\operatorname{ck}(H)$$

for any cluster point g of (f_n) in $\mathbb{R}^K$.

The following theorem is a quantitative version of the Krein-Šmulian theorem: see next section for its consequences in Banach spaces.

Theorem 26 ([33]). *Let K be a compact topological space and let H be a uniformly bounded subset of $\mathbb{R}^K$. Then*

$$\gamma_K(H) = \gamma_K(co(H)) \tag{23}$$

and as a consequence we obtain that if $H \subset C(K)$ we have that

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{24}$$

In general, for $H \subset \mathbb{R}^K$ we have that

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 5\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{25}$$

The equality (23) is rather involved: the proof offered in [33] uses some ideas from the proof of the Krein-Šmulian theorem in Kelley-Namioka's book [120, Ch 5. Sec. 17]. We note that a version for Banach spaces of the above result, less general than the one here, was proved first in[69] using Ptak's combinatorial lemma, see [127, §24.4.6]. Inequality (24) easily follows from (23) and Theorem 25:

$$\hat{d}(\overline{\mathrm{co}(H)}^{\mathbb{R}^K}, C(K)) \leq \gamma_K(\mathrm{co}(H)) = \gamma_K(H) \leq 2\,\mathrm{ck}(H) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)).$$

When $H \subset \mathbb{R}^K$, we naturally approximate H by a set in $C(K)$, then use inequality (24) and, after playing some games with the sets, and **5** in (25) appears as $5 = 2 \times 2 + 1$: see [33] for details.

We refer to the literature referenced in this section for the details about sharpeness of the constant involved in the presented inequalities.

4.1.1 Distance to Banach spaces

We can export the results obtained when using distances to spaces of continuous functions to the context of Banach spaces. The tool to do so is the following result that has been established in [33].

Theorem 27 (Quantitative Grothendieck's completeness theorem). *Let E be a Banach space and let B_{E^*} be the closed unit ball in the dual E^* endowed with the w^*-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embeddings. Then, for every $x^{**} \in E^{**}$ we have*

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*}))\,.$$

Observe that Grothendieck's completeness theorem, [127, §21.9.4], when specialized in Banach spaces, says that $j(x^{**})$ is w^*-continuous when restricted to B_{E^*} (*i.e.* $d(j(x^{**}), C(B_{E^*})) = 0$) implies $x^{**} \in i(E)$ (*i.e.*

$d(x^{**}, i(E)) = 0$). Therefore Theorem 27 can certainly be looked at as a quantitative version of Grothendieck's completeness theorem.

If E is Banach space and H is a bounded subset of E and we write $\overline{H}^{w^*}$ for the w^*-closure of H in E^{**}, we can measure how far H is from being w-relatively compact in E using

$$k(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{y \in \overline{H}^{w^*}} \inf_{x \in E} \|y - x\|. \tag{26}$$

If we consider $\ell_\infty(B_{E^*})$ as a subspace of $(\mathbb{R}^{B_{E^*}}, \tau_p)$, then the natural embedding $j : (E^{**}, w^*) \to (\ell_\infty(B_{E^*}), \tau_p)$ is continuous. For a bounded set $H \subset E^{**}$, the closure $\overline{H}^{w^*}$ is w^*-compact and therefore the continuity of j gives us that $\overline{j(H)}^{\tau_p} = j(\overline{H}^{w^*})$. So according with Theorem 27 we have that

$$\begin{aligned}\hat{d}(\overline{j(H)}^{\tau_p}, C(B_{E^*}, w^*)) &= \hat{d}(j(\overline{H}^{w^*}), C(B_{E^*}, w^*)) \\ &= \sup_{z \in \overline{H}^{w^*}} d(j(z), C(B_{E^*}, w^*)) \\ &= \sup_{z \in \overline{H}^{w^*}} d(z, i(E)) = \hat{d}(\overline{H}^{w^*}, i(E)). \end{aligned} \tag{27}$$

Similarly we have

$$d(\overline{j(H)}^{\tau_p}, C(B_{E^*}, w^*)) = d(\overline{H}^{w^*}, i(E)). \tag{28}$$

In what follows we will no make distinction between E and $i(E)$. Please note that according to the previous definitions a bounded subset H of E is weakly relatively compact in E if, and only if, $k(H) = 0$. Therefore it is natural to refer to this $k(H)$ as a *measure of weak noncompactness* of H, meaning by that, that the farther $k(H)$ is from zero the farther is H from being weakly relatively compact in E. We refer the interested reader to [21, 128], where measures of weak noncompactness are axiomatically defined. A measure of weak noncompactness is a non-negative function μ defined on the family $\mathcal{M}_E$ of bounded subsets of a Banach space E, with the following properties:

(i) $\mu(A) = 0$ if, and only if, A is weakly relatively compact in E,
(ii) if $A \subset B$ then $\mu(A) \leq \mu(B)$,
(iii) $\mu(\mathrm{co}(A)) = \mu(A)$,
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(vi) $\mu(\lambda A) = |\lambda| \mu(A)$, $\lambda \in \mathbb{R}$.

Beyond the formalities we will refer in general to measures of weak noncompactness to quantities as above fulfilling property (i), and sometimes a few of the others. These measures of noncompactness or weak noncompactness have been successfully applied to the study of compactness, operator theory, differential equations and integral equations, see for instance [2, 4, 18, 24, 32, 33, 69, 88, 90, 91, 129, 128, 130].

The following definition collects several measures of weak noncompactness that appeared in the aforementioned literature.

Definition 9. Given a bounded subset H of a Banach space E we define:

$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset E \text{ is } w\text{-compact}\},$$

$$\gamma(H) := \sup\left\{|\lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n)| : (x_m^*) \subset B_{E^*}, (x_n) \subset H\right\},$$

assuming the involved limits exist,

$$\mathrm{ck_E}(H) := \sup_{(x_n)\subset H} d\left(\mathrm{clust}_{(E^{**},w^*)}(x_n), E\right),$$

$$\mathrm{k}(H) := \widehat{\mathrm{d}}(\overline{H}^{w^*}, E) = \sup_{x^{**}\in \overline{H}^{w^*}} d(x^{**}, E),$$

$$\mathrm{Ja_E}(H) = \inf\{\varepsilon > 0 : \text{ for every } x^* \in E^*, \text{ there exists } x^{**} \in \overline{H}^{w^*} \text{ such that } x^{**}(x^*) = S_H(x^*) \text{ and } d(x^{**}, E) \le \varepsilon\}.$$

and

$$\sigma(H) := \sup_{(x_n^*)\subset B_{E^*}} d_{\|\cdot\|_H}\left(\mathrm{clust}_{(E^*,w^*)}(x_n^*), \mathrm{co}\{x_n^* : n \in \mathbb{N}\}\right).$$

where $d_{\|\cdot\|_H}(\cdot,\cdot)$ stands for the distance between two sets associated to the seminorm $\|x^*\|_H := \sup_{x\in H}|x^*(x)|$, $x^* \in E^*$.

Note that with the proper identifications $\gamma(H) := \gamma_{B_{E^*}}(H)$ where the latter has the meaning of Definition 8. The function ω was introduced by de Blasi [24] as a measure of weak noncompactness that is somehow the counterpart for the weak topology of the classical Kuratowski measure of norm-noncompactness. Properties for γ can be found in [4, 18, 33, 69, 128] and for $\mathrm{ck_E}$ in [4] –note that $\mathrm{ck_E}$ is denoted as ck in that paper. The quantity k has been used in [4, 33, 69, 90]. A thorough study for $\mathrm{Ja_E}$ has been done in [32] to prove, amongst other things, a quantitative version of James' weak compactness theorem, whose statement is presented as part of Theorem 28 bellow. Theorem 28 tells us that *all classical approaches used so far to study weak compactness in Banach spaces (Tychonoff's theorem, Eberlein-Šmulian's theorem, Eberlein-Grothendieck double-limit criterion and James' theorem) are qualitatively and quantitatively equivalent.* The quantity σ, inspired by Simons' inequality, has been very recently introduced in [42, Section 3].

Theorem 28 (Quantitative characterizations of weak compactness). *For any bounded subset H of a Banach space E the following inequalities hold* *true:*

$$\begin{array}{ccccccccc} & & & & & & \sigma(H) & \leq & 2\omega(H) \\ & & & & & \rotatebox{45}{$\leq$} & & & \rotatebox{90}{$\geq$} \\ \frac{1}{2}\gamma(H) & \leq & \mathrm{Ja_E}(H) & \leq & \mathrm{ck}_E(H) & \leq & k(H) & \leq & \gamma(H). \end{array} \tag{29}$$

Moreover for any $x^{**} \in \overline{H}^{w^*}$, *there exists a sequence* (x_n) *in* H *such that*

$$\|x^{**} - y^{**}\| \leq \gamma(H) \tag{30}$$

for any w^**-cluster point* y^{**} *of* (x_n) *in* E^{**}.

Furthermore, H *is weakly relatively compact in* E *if, and only if, one (equivalently, all) of the numbers* $\gamma(H), \mathrm{Ja_E}(H), \mathrm{ck}_E(H), \mathrm{k}(H), \sigma(H)$ *and* $\omega(H)$ *is zero.*

A full proof with references to prior work for the inequalities

$$\frac{1}{2}\gamma(H) \leq \mathrm{ck}_E(H) \leq k(H) \leq \gamma(H) \leq 2\omega(H)$$

and (30) can be found in [4, Theorem 2.3], see also [69, 90]. The inequalities

$$\frac{1}{2}\gamma(H) \leq \mathrm{Ja_E}(H) \leq \mathrm{ck}_E(H)$$

are established in Theorem 3.1 and Proposition 2.2 of [32] -this is a quantitative version of James' compactness theorem. For a proof of $\mathrm{ck}_E(H) \leq \sigma(H)$ and $\sigma(H) \leq 2\omega(H)$ we refer to [42, Theorem 3.7]. The fact that $\omega(H) = 0$ if, and only if, H is weakly relatively compact in E follows from a well-known result of Grothendieck, see [54, Lemma 2, p. 227]. Clearly, $\mathrm{k}(H) = 0$ if, and only if, $\overline{H}^{w^*} \subset E$, that is equivalent to the fact that H is weakly relatively compact by Tychonoff's theorem. Keeping in mind the last considerations and the chain of inequalities (29), one (equivalently, all) of the numbers $\gamma(H), \mathrm{Ja_E}(H), \mathrm{ck}_E(H), \mathrm{k}(H), \sigma(H)$ and $\omega(H)$ is zero if, and only if, H is weakly relatively compact.

Let us note that the inequalities

$$\mathrm{ck}_E(H) \leq k(H) \leq 2\,\mathrm{ck}_E(H),$$

that follow from (29), offer a quantitative version of the Eberlein's theorem saying that weakly relatively countably compact sets in Banach spaces are weakly relatively compact, see [62]. Note also that (30) implies that points in the weak closure of a weakly relatively compact set of a Banach space are reachable by weakly convergent sequences from within the set. Summing up, these inequalities are a *quantitative* version of the angelicity of Banach spaces with their weak topologies, see Definition 1), and hence they imply a quantitative version of Šmulian's theorem, see [179], that says that weakly relatively compact subsets of a Banach space are weakly relatively sequentially compact

The aforementioned references contain examples showing when the inequalities in (29) are sharp, as well as sufficient conditions of when the inequalities become equalities.

With regard to convex hulls, the quantities in Theorem 28 behave quite differently. For an arbitrary bounded set H of a Banach space E, the following statements hold:

$$\begin{aligned} \gamma(\mathrm{co}(H)) = \gamma(H), &\quad \mathrm{Ja_E}(\mathrm{co}(H)) \leq \mathrm{Ja_E}(H); \\ \mathrm{ck}_E(\mathrm{co}(H)) \leq 2\,\mathrm{ck}_E(H), &\quad \mathrm{k}(\mathrm{co}(H)) \leq 2\,\mathrm{k}(H); \\ \sigma(\mathrm{co}(H)) = \sigma(H), &\quad \omega(\mathrm{co}(H)) = \omega(H). \end{aligned} \tag{31}$$

Constant 2 for ck_E and k is sharp, [32, 90, 88], and it is unknown if $\mathrm{Ja_E}$ might really decrease when passing to convex hulls. The equality $\gamma(A) = \gamma(\mathrm{co}(A))$ is a bit delicate and has been established in [33, 69, 90]. Note that inequalities (31) immediately imply Krein-Šmulian theorem for Banach spaces that states that the closed convex hull of a weakly compact set is again weakly compact.

As the reader should have observed the inequalities (29) say that the measures of weak noncompactness γ, $\mathrm{Ja_E}$, ck, k and γ are equivalent; on the other hand no information about the equivalence with them of σ and ω has been provided. As of now, we do not now if σ is equivalent to the other ones but we do know that ω is not.

Corollary 6 ([4, 18]). *The measures of weak noncompactness γ and ω are not equivalent, meaning, there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have $\omega(H) \leq N\gamma(H)$.*

Corollary 6 can be obtained combining an example of a separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \leq w(T_n(B_E)) \leq \frac{1}{n},$$

see [18, Theorem 4] and the following *quantitative* version of Schauder's theorem:

Theorem 29 (Quantitative Schauder's theorem, [4]). *Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then*

$$\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E)).$$

The following result is a quantitative strengthening of the classical Grothendieck's characterization of weakly compact sets in spaces $C(K)$.

Theorem 30 (Quantitative Grothendieck's theorem, [4]). *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Then we* *have*

$$\gamma_K(H) \le \gamma(H) \le 2\gamma_K(H).$$

Note that this result implies that if H is a uniformly bounded subset of $C(K)$, then H is relatively weakly compact (*i.e.* $\gamma(H) = 0$) if, and only if, H is relatively τ_p-compact (*i.e.* $\gamma(H) = 0$). It is worth mentioning that the proof we provided in [4] does not use the Lebesgue Convergence theorem as the classical proof of Grothendieck's theorem does: our proof relies on purely topological arguments.

4.1.2 Distances to spaces of Baire one functions

The game that we just played when using distances to spaces of continuous functions to study compactness in $C_p(X)$ and in Banach spaces can be played with other spaces of functions as well. It is known that if X is a topological space and E is a Banach space, then uniform limits of sequences of Baire one functions from X with values in E are Baire one functions again. Hence, for a function $f \in E^X$ we have that $f \in B_1(X, E)$ if, and only if, $d(f, B_1(X, E)) = 0$. For any subset $A \subset E^X$ we have $\hat{d}(A, B_1(X, E)) = 0$ if, and only if, $A \subset B_1(X, E)$. In this way, and similarly to the case of continuous functions, when $E = \mathbb{R}$ and $H \subset \mathbb{R}^X$ is pointwise bounded, the number $\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X))$ gives us a measure of non τ_p-compactness of H relative to $B_1(X)$.

In order to succeed with the plan of *quantitatively* study pointwise compactness relative to $B_1(X, E)$ we need a formula to compute distances to spaces of Baire one functions. A formula of this sort is given using the concept of fragmented and σ-fragmented maps as introduced in [113]. Recall that for a given $\varepsilon > 0$, a metric space-valued function $f : X \to (Z, d)$ is said to be ε-fragmented if for each non-empty subset $F \subset X$ there exists an open subset $U \subset X$ such that $U \cap F \neq \emptyset$ and $\operatorname{diam}(f(U \cap F)) \le \varepsilon$. Given $\varepsilon > 0$, we say that f is ε-σ-fragmented by *closed sets* if there is a countable closed covering $(X_n)_n$ of X such that $f|_{X_n}$ is ε-fragmented for each $n \in \mathbb{N}$.

Definition 10 ([5, 91]). Let X be a topological space, (Z, d) a metric space and $f \in Z^X$ a function. We define:

$$\operatorname{frag}(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-fragmented}\},$$
$$\sigma\text{-frag}_c(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-}\sigma\text{-fragmented by closed sets}\},$$

where by definition, $\inf \emptyset = +\infty$.

The indexes frag and σ-frag_c are related to each other as follows:

Theorem 31 ([5]). *Let X be a topological space and (Z, d) a metric space. If $f \in Z^X$ then the following inequality holds*

$$\sigma\text{-frag}_{\mathrm{c}}(f) \le \operatorname{frag}(f).$$

If moreover X is complete, then

$$\sigma\text{-frag}_{\mathrm{c}}(f) = \operatorname{frag}(f).$$

With frag and σ-frag$_{\mathrm{c}}$ one can estimate distances to $B_1(X,E)$.

Theorem 32 ([5]). *Let X be a metric space and E a Banach space. If $f \in E^X$ then*

$$\frac{1}{2}\,\sigma\text{-frag}_{\mathrm{c}}(f) \le d(f, B_1(X,E)) \le \sigma\text{-frag}_{\mathrm{c}}(f).$$

In the case $E = \mathbb{R}$ we have the equality

$$d(f, B_1(X)) = \frac{1}{2}\,\sigma\text{-frag}_{\mathrm{c}}(f).$$

Corollary 7 ([5]). *If X is a complete metric space and $f \in \mathbb{R}^X$, then*

$$d(f, B_1(X)) = \frac{1}{2}\operatorname{frag}(f).$$

Note that the corollary above extends [91, Proposition 6.4.], where this result is only proved when X is Polish.

Bearing in mind the definitions involved one can prove:

Lemma 3 ([5]). *Let X be a separable metric space, (Z,d) a metric space and H a pointwise relatively compact subset of (Z^X, τ_p). Then,*

$$\sup_{f \in \overline{H}^{Z^X}} \operatorname{frag}(f) = \sup_{(f_n) \subset H} \inf \{\operatorname{frag}(f) : f \in \operatorname{clust}(f_n)\}. \tag{32}$$

Proof. Let α be the right hand side of (32). Clearly

$$\beta := \sup_{f \in \overline{H}} \operatorname{frag}(f) \ge \alpha.$$

If $\beta = 0$ we are done. Otherwise, the equality (32) will be established if we prove that each time $\beta > \varepsilon > 0$ we also have $\alpha \ge \varepsilon$. Assume $\beta > \varepsilon > 0$ and pick $f \in \overline{H}^{Z^X}$ such that $\operatorname{frag}(f) > \varepsilon$. Then there exists a non-empty subset $F \subset X$ such that $\operatorname{diam} f(F \cap U) > \varepsilon$ for each open set $U \subset X$ with $U \cap F \ne \emptyset$. Let us fix $\{U_n : n \in \mathbb{N}\}$ a basis for the topology in X and write $B := \{n \in \mathbb{N} : U_n \cap F \ne \emptyset\}$. For every $n \in B$ we can choose $x_n, y_n \in U_n \cap F$ such that $d(f(x_n), f(y_n)) > \varepsilon$. Let us write $C := \{x_n : n \in B\} \cup \{y_n : n \in B\}$. Since $C \subset X$ is countable and $f \in \overline{H}^{Z^X}$ there exists a sequence (f_n) in H such that $\lim_n f_n(x) = f(x)$ for every $x \in C$. Therefore, if g is an arbitrary τ_p-cluster point of (f_n) then $g|_C = f|_C$ and in particular we have that

$$d(g(x_n), g(y_n)) > \varepsilon, \text{ for every } n \in B. \tag{33}$$

If U is an open set such that $U \cap C \neq \emptyset$ then there exists $n \in \mathbb{N}$ such that $\emptyset \neq U_n \cap C \subset U \cap F$. Hence, $n \in B$ and since $x_n, y_n \in U \cap C$ we conclude

$$\operatorname{diam} g(U \cap C) \geq d(g(x_n), g(y_n)) \overset{(33)}{>} \varepsilon.$$

We have proved that

$$\inf\{\operatorname{frag}(f) : f \in \operatorname{clust}(f_n)\} \geq \varepsilon$$

and therefore $\alpha \geq \varepsilon$ so the proof is complete.

Observe that the quantity

$$\operatorname{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\operatorname{clust}_{Z^X}(\varphi), B_1(X, E)).$$

gives an estimate of how far a set $H \subset E^X$ from being τ_p-relatively countably compact with respect to $B_1(X, E)$.

The following result is the quantitative version of a well known result due to Rosenthal [171]: note that when $H \subset B_1(X)$ is τ_p-relatively compact in $B_1(X)$, the inequalities below and Tychonoff's theorem imply that H is τ_p-relatively compact in $B_1(X)$.

Theorem 33 (Quantitative Rosenthal's theorem, [5]). *Let X be a Polish space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{E^X}, B_1(X, E)) \leq 2 \operatorname{ck}(H).$$

In the particular case when $E = \mathbb{R}$ we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X)) = \operatorname{ck}(H).$$

Let us finish this section by saying that the above results can be used to give a quantitative version of a Srivatsa's result, [180], that states that whenever X is metric any weakly continuous function $f \in E^X$ belongs to $B_1(X, E)$. The quantitative counterpart to Srivatsa's result says that for an arbitrary $f \in E^X$ we have

$$d(f, B_1(X, E)) \leq 2 \sup_{x^* \in B_{E^*}} \operatorname{osc}(x^* \circ f).$$

As a consequence, it can be proved that for functions in two variables $f : X \times K \to \mathbb{R}$, X complete metric and K compact, there exists a G_δ-dense set $D \subset X$ such that the oscillation of f at each $(x, k) \in D \times K$ is bounded by the oscillations of the *partial* functions f_x and f^k. Using games, it is established indeed that if X is a σ-β-unfavorable space and K is a compact space, then there exists a dense G_δ-subset D of X such that, for each $(y,k) \in D \times K$,

$$\operatorname{osc}(f,(y,k)) \leq 6 \sup_{x\in X} \operatorname{osc}(f_x) + 8 \sup_{k\in K} \operatorname{osc}(f^k),$$

that is a quantitative Namioka's type theorem. Indeed, when the right hand side of the above inequality is zero we are dealing with separately continuous functions $f : X \times K \to \mathbb{R}$ and we obtain as particular cases some well-known results obtained by I. Namioka in the mid of the 1970's, see [144].

4.2 Last news on James' compactness Theorem

From now on X will denote a non-empty set. Given a pointwise bounded sequence (f_n) in $\mathbb{R}^X$, we define

$$\operatorname{co}_{\sigma_p}\{f_n\colon\ n \in \mathbb{N}\} := \left\{\sum_{n=1}^{\infty} \lambda_n f_n : \lambda_n \geq 0,\ \text{for } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1\right\},$$

where the functions $\sum_{n=1}^{\infty} \lambda_n f_n \in \mathbb{R}^X$ above are pointwise defined on X, *i.e.* for every $x \in X$ the absolutely convergent series

$$\sum_{n=1}^{\infty} \lambda_n f_n(x)$$

defines the function $\sum_{n=1}^{\infty} \lambda_n f_n : X \to \mathbb{R}$.

The following result contains an extended version of Simons' inequality together with the Inf-liminf statement proved in [155]. The corresponding result for supremum instead of infimum has been recently proved in [42] where we refer the interested reader. Our proof of Theorem 34 is based on E. Oja's approach for the classical inequality, see [68, Lemma 3.123]).

Theorem 34 (Simons' Theorem in $\mathbb{R}^{\mathbf{X}}$). *Let X be a nonempty set, let (f_n) be a pointwise bounded sequence in $\mathbb{R}^X$ and let Y be a subset of X such that for every $g \in \operatorname{co}_{\sigma_p}\{f_n\colon n \in \mathbb{N}\}$ there exists $y \in Y$ with*

$$g(y) = \inf\{g(x) : x \in X\}.$$

Then the following statements hold true:

$$\sup\{\inf_{x\in X} g(x) : g \in \operatorname{co}_p\{f_n\colon n \in \mathbb{N}\}\} \geq \inf_{y\in Y}\{\liminf_n f_n(y)\} \tag{34}$$

and

$$\inf\{\liminf_n f_n(x) : x \in X\} = \inf\{\liminf_n f_n(y) : y \in Y\}. \tag{35}$$

Proof. We set

$$C_k := \Big\{ \sum_{n=k}^{\infty} \lambda_n f_n : \lambda_n \geq 0, \sum_{n=k}^{\infty} \lambda_n = 1 \Big\},$$

for $k = 1, 2, \ldots,$ and let us fix $\varepsilon > 0$. By induction it is possible to choose $g_k \in C_k, k = 1, 2, \ldots$ such that

$$\inf_X (2^k v_k + g_{k+1}) \geq \sup_{g \in C_{k+1}} \inf_X (2^k v_k + g) - \frac{\varepsilon}{2^{k+1}},$$

where $v_0 = 0$ and $v_k = \sum_{n=1}^{k} \frac{1}{2^n} g_n$. Indeed, for every $k \in \mathbb{N}, \mu \geq 1$ and

$$v \in \frac{1}{2} C_1 + \frac{1}{2^2} C_1 + \cdots + \frac{1}{2^k} C_1 \text{ or } v = 0,$$

we have that

$$\sup\{\inf\{(\mu v + g)(X) : g \in C_1\}\} < +\infty$$

because once $x_0 \in X$ is fixed we have that

$$\begin{aligned} \sup\{\inf\{(\mu v + g)(X) : g \in C_1\}\} &< \sup\{(\mu v + g)(x_0) : g \in C_1\}\} \\ &\leq \left(\mu \frac{2^n - 1}{2^n} + 1\right) \sup\{f_n(x_0) : n = 1, 2, \ldots\}. \end{aligned}$$

Let us write now $v = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n$, and let us observe that $v \in C_1$. Since

$$g_{k+1} = 2^{k+1} v_{k+1} - 2^{k+1} v_k$$

it follows that

$$2^{k+1} v_{k+1} - 2^k v_k = 2^k v_k + g_{k+1}.$$

Then

$$\inf_X (2^{k+1} v_{k+1} - 2^k v_k) \geq \inf_X (2^k v_k + (2^k v - 2^k v_k)) - \frac{\varepsilon}{2^{k+1}}$$

by the choice that we have done for g_{k+1}. So

$$\inf_X (2^{k+1} v_{k+1} - 2^k v_k) \geq \inf_X (2^k v_k + (2^k v - 2^k v_k)) - \frac{\varepsilon}{2^{k+1}} = \inf_X (2^k v) - \frac{\varepsilon}{2^{k+1}}.$$

Since $v \in C_1$, our hypothesis says that there is $t \in Y$ with $v(t) = \inf v(X)$. Thus,

$$2^m v_m(t) = \sum_{k=0}^{m-1} (2^{k+1} v_{k+1} - 2^k v_k)(t) \geq \sum_{k=0}^{m-1} \inf_X (2^{k+1} v_{k+1} - 2^k v_k) \geq$$

$$\geq \sum_{k=0}^{m-1} 2^k \inf_X v - \frac{\varepsilon}{2^{k+1}} \geq (2^m - 1) \inf_X v - \varepsilon = 2^m v(t) - \inf_X v - \varepsilon.$$

So we have

$$\inf_X v \geq 2^m v(t) - 2^m v_m(t) - \varepsilon = 2^m (v - v_m)(t) - \varepsilon,$$

for every $m \in \mathbb{N}$. Then we arrive to

$$\sup_{g \in C_1} \inf_{x \in X} g(x) \geq \inf_X v \geq \liminf_{m \to \infty} 2^m (v - v_m)(t) - \varepsilon \geq \liminf_{m \to \infty} f_m(t) - \varepsilon$$

where the last inequality follows from the fact that $2^m(v - v_m) \in C_{m+1}$. Since our argument is valid for every $\varepsilon > 0$ the proof of (34) is over. For the equality (35), we observe that (34) says that

$$\inf_{y \in Y} \liminf_{n \to \infty} f_n(y) \leq \sup_{g \in C_1} \inf_{x \in X} g(x).$$

If we fix $x \in X$ and we assume that

$$\inf_{y \in Y} \liminf_{n \to \infty} f_n(y) > \liminf_{n \to \infty} f_n(x),$$

then we can take a subsequence to have

$$\inf_{y \in Y} \liminf_{n \to \infty} f_n(y) > \sup_{n \in \mathbb{N}} f_n(x).$$

But then

$$\inf_{y \in Y} \liminf_{n \to \infty} f_n(y) > \sup_{g \in C_1} g(x) \geq \sup_{g \in C_1} \inf_{x \in X} g(x),$$

which is a contradiction with the above inequality that finishes the proof.

Corollary 8. *Let X be a nonempty set and (f_n) a pointwise bounded sequence in $\mathbb{R}^X$. If Y is a subset of X such that for every function g in $\mathrm{co}_{\sigma_p}\{f_n : n \in \mathbb{N}\}$ there exists $y \in Y$ with*

$$g(y) = \sup_{x \in X}(g(x)).$$

Then we have that

$$\inf \big\{ \sup\{g(x) : x \in X\} : g \in \mathrm{co}_p\{f_n : n \in \mathbb{N}\} \big\} \leq \sup_{y \in Y}(\limsup_n f_n(y)),$$

and

$$\sup\{\limsup_n f_n(x) : x \in X\} = \sup\{\limsup_n f_n(y) : y \in Y\}.$$

The following corollary generalizes Rainwater's Theorem [68, Theorem 3.134], which asserts that a sequence (x_n) in a Banach space E is weakly null if it is bounded and for each extreme point e^* of B_{E^*},

$$\lim_n e^*(x_n) = 0.$$

Given a bounded sequence (x_n) in a Banach space E, we define

$$\operatorname{co}_\sigma\{x_n : n \in \mathbb{N}\} := \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \text{ for all } n \geq 1,\ \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\}$$

Note that series are clearly norm-convergent and that

$$\operatorname{co}_\sigma\{x_n : n \in \mathbb{N}\} = \operatorname{co}_{\sigma_p}\{x_n : n \in \mathbb{N}\}$$

when for the second set we look at the x_n's as functions defined on B_{E^*}.

Corollary 9 (Unbounded Rainwater-Simons' theorem). *If E is a Banach space, $B \subset C$ are nonempty subsets of E^* and (x_n) is a bounded sequence in E such that for every $x \in \operatorname{co}_\sigma\{x_n : n \in \mathbb{N}\}$ there exists $b^* \in B$ with*

$$\langle x, b^* \rangle = \sup\{\langle x, c^* \rangle : c^* \in C\},$$

then

$$\sup_{b^* \in B} \left(\limsup_n \langle x_n, b^* \rangle \right) = \sup_{c^* \in C} \left(\limsup_n \langle x_n, c^* \rangle \right).$$

As a consequence

$$\sigma(E, B) - \lim_n x_n = 0 \ \Rightarrow\ \sigma(E, C) - \lim_n x_n = 0.$$

The unbounded Rainwater-Simons theorem (or the Simons' inequality in $\mathbb{R}^X$) not only gives as special cases those classical results that follow from the Simons' inequality (some of them are discussed here, besides the already mentioned ones that can be found in [53, 79]), but it also provides new applications, see [155, 31]. Let us remark that W. Moors has recently obtained, see [142, Corollary 1], a particular case of the unbounded Rainwater–Simons' theorem that allowed him to give a proof of James' theorem for Banach spaces whose dual unit balls are w^*-sequentially compact. A more general class of spaces is considered in the following definition.

Definition 11. Given a sequence (v_n) in the vector space E, we say that another sequence (u_n) is a *convex block sequence of* (v_n) if there is a sequence of finite subsets of integers (F_n) such that

$$\max F_1 < \min F_2 \leq \max F_2 < \min F_3 \cdots < \max F_n < \min F_{n+1} < \cdots$$

together with sets of positive numbers $\{\lambda_i^n : i \in F_n\} \subset (0, 1]$ satisfying

$$\sum_{i \in F_n} \lambda_i^n = 1 \text{ and } u_n = \sum_{i \in F_n} \lambda_i^n v_i.$$

When E is a Banach space and each sequence (x_n^*) in B_{E^*} has a convex block w^*–convergent sequence we say that B_{E^*} is w^*-convex block compact.

Let us observe that every subsequence of a given sequence (v_n) is a convex block sequence too, thus w^*-sequentially compact sets are w^*-convex block compact. J. Bourgain proved in [29] that if the Banach space E does not contain a copy of $\ell^1(\mathbb{N})$, then its dual unit ball is w^*–convex block compact. This result was extended for spaces not containing a copy of $\ell^1(\mathbb{R})$ under Martin's axiom and the negation of the Continuum Hypothesis in [107].

The sequential lemma below is taken from [155], see also [42].

Lemma 4. *Suppose that the dual unit ball of E is w^*-convex block compact and that A is a nonempty, bounded subset of E. Then A is weakly relatively compact if, and only if, each w^*-null sequence in E^* is also $\sigma(E^*, \overline{A}^{w^*})$-null.*

Proof. If A is weakly relatively compact, then we have $A = \overline{A}^{w^*}$ and the conclusion follows. According to Theorem 28 above, to see the converse implication we have to check the validity of the identity that

$$\operatorname{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \operatorname{co}\{x_n^* : n \in \mathbb{N}\}) = 0, \tag{36}$$

for every bounded sequence (x_n^*) in E^*. Thus, let us fix (x_n^*) a bounded sequence in B_{E^*}. Since B_{E^*} is w^*-convex block compact, there exists a block sequence (y_n^*) of (x_n^*) and an $x_0^* \in B_{E^*}$ such that

$$w^*\text{-}\lim_n y_n^* = x_0^*.$$

Then, by assumption, (y_n^*) also converges to x_0^* pointwise on $\overline{A}^{w^*} \subset E^{**}$. Mazur's theorem applied to the sequence of continuous functions (y_n^*) restricted to the w^*-compact space $\overline{A}^{w^*}$ tell us that

$$0 = \operatorname{dist}_{\|\cdot\|_{\overline{A}^{w^*}}}(x_0^*, \operatorname{co}\{y_n^* : n \in \mathbb{N}\}) = \operatorname{dist}_{\|\cdot\|_A}(x_0^*, \operatorname{co}\{x_n^* : n \in \mathbb{N}\}) \geq 0,$$

It is not difficult to check that $x_0^* \in L\{x_n^*\}$ and (36) is proved. The proof is over.

A nonlinear version of classical James' compactness theorem is the following one. It has been recently obtained and applied in different contexts by [154, 155] and in full generality by Saint Raymond in [172].

Theorem 35. *Let E be a Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper map such that*

$$\text{for every } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Then

for every $c \in \mathbb{R}$, *the sublevel set* $f^{-1}((-\infty, c])$ *is weakly relatively compact.*

Following [155, 42], we present a proof for the wide class of Banach spaces with w^*-convex block compact dual unit balls:

Proof. Let us consider the epigraph of f, *i.e.*

$$\operatorname{epi}(f) = \{(x,t) \in E \times \mathbb{R} \colon f(x) \leq t\}$$

We first claim that for every $(x^*, \lambda) \in E^* \times \mathbb{R}$ with $\lambda < 0$, there exists $x_0 \in E, f(x_0) < +\infty$ such that

$$\sup\{(x^*, \lambda)(x,t) : (x,t) \in \operatorname{epi}(f)\} = x^*(x_0) + \lambda f(x_0). \tag{37}$$

In fact, the optimization problem

$$\sup_{x \in E}\{\langle x, x^* \rangle - f(x)\} \tag{38}$$

may be rewritten as

$$\sup_{(x,t) \in \operatorname{epi}(f)} \{(x^*, -1), (x,t)\} \tag{39}$$

and the sup in (38) is attained if and only if the sup in (39) is attained.

Let us fix $c \in \mathbb{R}$ and assume that $A := f^{-1}((-\infty, c])$ is nonempty. The uniform boundedness principle and the optimization assumption on f imply that A is bounded. In order to obtain the relative weak compactness of A we apply Lemma 4. Thus, let us consider a w^*-null sequence (x_n^*) in E^* and let us prove that it also is $\sigma(E^*, \overline{A}^{w^*})$-null.

It follows from the unbounded Rainwater-Simons' theorem (Corollary 9), taking the Banach space $E^* \times \mathbb{R}$,

$$B := \operatorname{epi}(f) \subset C := \overline{\operatorname{epi}(f)}^{\sigma(E^{**} \times \mathbb{R}, E^* \times \mathbb{R})}$$

and the bounded sequence

$$\left(x_n^*, -\frac{1}{n}\right),$$

that

$$\sigma(E^* \times \mathbb{R}, B) - \lim_n \left(x_n^*, -\frac{1}{n}\right) = \sigma(E^* \times \mathbb{R}, C) - \lim_n \left(x_n^*, -\frac{1}{n}\right),$$

But $w^* - \lim_{n \geq 1} x_n^* = 0$, so we have that

$$\sigma(E^* \times \mathbb{R}, C) - \lim_n \left(x_n^*, -\frac{1}{n}\right) = 0.$$

As a consequence, since $A \times \{c\} \subset B$, then $\overline{A}^{w^*} \times \{c\} \subset C$, and so

$$\sigma(E^*, \overline{A}^{w^*}) - \lim_n x_n^* = 0,$$

as announced.

4.3 Some notes and open problems

To handle compactness in the setting of infinite dimensional function can be difficult. As commented before, sometimes applications require of characterizations via sequences instead of nets (think about using Lebesgue dominated convergence theorem). Research about some kind of sequential behaviour of compact sets has always attracted analysts. Two examples follow. First, D. Fremlin's dichotomy theorem saying that in a perfect probability space a sequence of measurable functions either has some subsequence with no measurable cluster point, or has a subsequence almost everywhere pointwise convergent. Second, Komlós' theorem saying that every L^1-bounded sequence of real functions contains a subsequence such that the arithmetic means of all its subsequences converge pointwise almost everywhere. The angelic character of different function spaces has been always a very exciting topic of research. As said previously, the recent book [117] is a good reference for the topic. James' compactness theorem is an optimization result with plenty of different applications; the papers [154, 155, 42] contain applications to variational problems and the so called Lebesgue measures of risk in mathematical finance are treated.

Let us finish the section with a couple of open questions:

Question 9. Is there any quantitative approach to describe convergent sequences to closure points of relatively compact subsets of Baire one functions on a Polish space?

Question 10. Is there any characterization of Banach spaces with w^*-angelic dual unit ball?

5 Concluding references and remarks

We have been dealing in the previous sections with different questions where we, the authors, have done some work, always using topology as a tool for functional analysis. Since the outstanding chapter by S. Negrepontis in the first Handbook of Set-theoretic Topology [148], an unbelievable amount of research has been done in this vast area. We, sometimes as mere reporters of words written by others, collect in this final section notes, comments, references and open problems around topics that might be of interest for the reader. We remark that a special issue of RACSAM entitled “Open problems in infinite dimensional Geometry and Topology” and edited by the authors of this survey paper, [45], offers a wide selection of problems fitting very well within the contents of this section.

5.1 Compactness, Lindelöfness and other covering properties in Banach spaces

Fragmentability is a very useful topological concept in Banach space theory. It was introduced by J. E. Jayne and C. A. Rogers to deal with Borel selectors of certain set valued maps, see [114]. It is also the right concept to understand Asplund spaces and differentiability properties of convex functions defined on Banach spaces. The survey paper by I. Namioka, see [147], is a recommendable place to read about fragmentability. If (X, τ) is a topological space and ρ is a pseudo-metric on it, we say that (X, τ) is *fragmented by* ρ *(resp.*σ*-fragmented)* if the identity map on X is ε fragmented (respectively $\varepsilon - \sigma$-fragmented) for every $\varepsilon > 0$, see Subsection 4.1.2.

Here are some results related to fragmentability. If a Banach space E has a Gateaux differentiable norm then the w^*-dual unit ball B_{E^*} is fragmented by some metric, and every proper continuous convex function on E is Gateaux differentiable in a G_δ subset of the interior of its domain. The monograph [66], by M. Fabian, explores in great detail all connections between topology and analysis in this topic. Asplund spaces are characterized as those Banach spaces for which their w^* dual unit balls are fragmented by the dual norm. A compact space K is called a Radon-Nikodým compact (shortly, RN compact) if it is homeomorphic to a norm-fragmented w^* compact subset of a dual Banach space. Eberlein compact spaces (shortly, EC compact), *i.e.* compacta homeomorphic to weakly compact sets of a Banach space, and scattered compact spaces are RN-compact. The class of RN-compact spaces has properties very similar to the class of EC compact spaces, however it is not stable by continuos images. This fact has been recently been established by Avilés and Koszmider, see [19], solving a long standing open problem asked by I. Namioka. If K is continuous image of a RN-compact and a Corson compact, *i.e.* it embeds in a Σ product, then K must be an EC, see [17, 156]. On compact spaces fragmentability by a lower semicontinuous metric can be characterized by the Lindelöf property. Indeed if K is a compact subset of the cube $[-1, 1]^D$, then K is fragmented by the norm of $\ell^\infty(D)$ if, and only if, $(K, \gamma(D))$ is Lindelöf, where $\gamma(D)$ is the topology of uniform convergence on countable subsets of D, see [37]. If we set a K-analytic subset $X \subset [-1, 1]^D$ the previous result extends to say that $(X, \gamma(D))$ is Lindelöf if, and only if, X

is σ-fragmented by the norm of $\ell^\infty(D)$, [36]. The notion of σ-fragmentability was introduced and strongly developed by J.E. Jayne, I. Namioka and C.A. Rogers, see [110, 111, 112]. The paper [36] analyzes the relationship between the Lindelöf property and σ-fragmentability. A very fruitful approach based on games is due to Kenderov and Moors, [121], and yet another approach based on the concept of network is due to Hansell, [102, 103]. When $C_p(K)$ is σ-fragmented by the supremum norm the compact K has the so called Namioka's property: every separately continuous function $f : B \times K \longrightarrow R$, where B is a Baire space, has a dense G_δ subset $T \subset B$ such that f is jointly continuous at $T \times K$. If $C(K)$ has an equivalent pointwise lower semicontinuous and **LUR** norm, then $C_p(K)$ is σ-fragmented by the supremum norm and the compact K verifies Namioka's property, see section 7, Chapter VII in [53]. For instance, every pointwise compact subset $K \subset \mathbb{R}^X$ made of Baire one functions defined on a Polish space X and such that every $f \in K$ has at most countably many discontinuities verifies that $C_p(K)$ is σ-fragmented by the norm. Nevertheless it is an open problem to know if the σ-algebras of Borel sets for the pointwise and norm topologies coincide on $C(K)$. Moreover $C(K)$ has an equivalent pointwise **LUR** norm if K is separable too, [105]. If there is a sequence of subsets (A_n) in $C(K)$ such that the family $\{A_n \cap W : W \text{ pointwise open}, n \in \mathbb{N}\}$ is a network for the norm topology on $C(K)$ the we have:

1. Borel sets for the pointwise and norm topologies coincide on $C(K)$.
2. There is an equivalent F-norm such that pointwise and norm topologies coincide on the unit sphere.
3. The compact K has the Namioka's property.

The above results obtained via networks were established in [164]. They have been recently improved in [72]. It is a tantalizing conjecture that the above network property could be characterized with item 2 but using a norm, instead of only an F-norm. A positive result in this direction was obtained by Raja in [164] when the sets (A_n) are convex. If we have two metrics ρ and d defined on a set X, the fact that we have a sequence of subsets (A_n) such that the family $\{A_n \cap W : W \text{ d-open}, n \in \mathbb{N}\}$ is a network for the ρ-topology is equivalent to have countable sets S_x for every $x \in X$ such that

$$x \in \overline{\cup\{S_{x_n} : n = 1, 2, \dots\}}^{\rho} \text{ whenever } d - \lim_n x_n = x, \tag{40}$$

see [141, Theorem 2.32]. This is an essential property to understand how maps from a normed space E to a metric space X provide an equivalent **LUR** norm on E: this is the basis for the non-linear transfer studied in [141]. If the d-limit of the sequence (x_n) in (40) is taken in a non-necessarily metrizable topology τ, then it is said that we have the linking separability property (LSP) between the topology τ and the metric ρ. This property has been deeply studied by L. Oncina, who showed that a compact space K is EC if, and only if, it has the LSP with respect to a lower semicontinuous metric, and, also, that a RN-compact space is EC-compact if, and only, it has the LSP, [151]. These ideas were subsequently applied by Dow, Junilla and Pelant to clarify the relationship between Gul'ko and Corson compacta, [61]. In a different setting we can describe a similar property to (40) that is fulfilled by any Borel selector $f : E \longrightarrow E^*$ for the attaining set-valued map

$$F(x) = \{x^* \in E^* : x^*(x) = \sup\{k^*(x) : k^* \in K\}\},$$

where K is a convex and w^*-compact subset of E^* fragmented by the norm. In this case we have the identity $\overline{\mathrm{co}(f(E))}^{\|\cdot\|} = K$, that leads to a characterization of strong boundaries of Asplund spaces, [34, 31]. Similar topological conditions lead to non-linear transfer properties either for strictly convex norms or pointwise-**LUR** norms, [73, 100].

Let us finish this subsection recalling a well known open problem:

Question 11. Let K and L be compact spaces. If K is a Corson and $C(K)$ is isomorphic to $C(L)$, must L be Corson compact too?

If the dual ball $B_{C(K)^*}$ is Corson compact the answer is yes after a result by Kalenda, who characterizes when this happens through the existence of Projectional Resolutions of the Identity for every equivalent norm on $C(K)$, see [118].

5.2 Hereditarily indecomposable Banach spaces

Our first paragraph here is taken from our article for the Encyclopedia of General Topology with I. Namioka and M. Raja [38]: *"A sequence of vectors (x_n) is called a basis of a Banach space E if every $x \in E$ has a unique representation as $x = \sum a_i x_i$ with scalars a_i. If the convergence of the series is unconditional the basis is called an unconditional basis. In that case every infinite subset M of integers gives a continuous linear projection $P_M(\sum a_i x_i) = \sum_{i \in M} a_i x_i$. Each infinite dimensional Banach space contains an infinite dimensional subspace with a basis and Banach asked if each separable Banach space has a basis. A famous counterexample of Enflo [63] solved even a stronger version of the problem dealing with the approximation property of Grothendieck. After Enflo's counterexample, and for a long time, it was conjectured that each infinite dimensional Banach space contains copies of c_0 or ℓ^p or, at least, an infinite dimensional subspace with an unconditional basis. This is the case for Banach spaces with a C^∞-smooth bump function, [52] and for the class of Orlicz spaces (Lindenstrauss and Tzafriri [132]). Nevertheless Tsirelson [188] constructed a reflexive Banach space T not containing ℓ^p for $1 < p < +\infty$".*

Tsirelson's construction has been modified by Schlumprecht [173] opening the door for the construction by Gowers and Maurey [87] of a separable reflexive Banach space E that does not have any infinite-dimensional subspace with an unconditional basis. Gowers-Maurey's example GM has the property that, for each infinite dimensional closed subspace Z admits only trivial projections, *i.e. any continuous linear projection* $P : Z \longrightarrow Z$ *is trivial: either* $\dim \text{Im } P < \infty$ *or* $\dim \text{Ker} P < \infty$. A Banach space with this property is said to be *hereditarily indecomposable, H.I. for short.* This is equivalent to the following remarkable geometric property: for any two infinite dimensional subspaces the distance between their unit spheres is zero (the angle zero property). GM space has the property that every operator in GM is of the form $\lambda I + S$, where λ is a scalar, I is the identity, and S is a strictly singular operator. The formal definition of a strictly singular operator is that you cannot restrict to an infinite-dimensional subspace on which it is an isomorphic embedding. Every compact operator is strictly singular. An obvious question that this raised was whether GM admitted an operator that was strictly singular but not compact. Androulakis and Schlumprecht showed in [1], that GM space does have non-compact strictly singular operators. This led to new concepts defining a class of Banach spaces with very remarkable properties, for instance every H.I. Banach space is arbitrarily distortable, [187], and it is not isomorphic to any proper subspace, answering in the negative the long standing hyperplane problem. Fortunately for the mathematical community Spiros Argyros has made this area his own, and with various collaborators proved a variety of remarkable results, both positive and negative, about spaces of this kind. In particular they showed that the class of H.I. spaces is extensive, [10, 8], and they have been able to develop a method to construct non separable H.I. Banach spaces, see [15]. Argyros and Tolias constructed a nonseparable H.I. Banach space which is the dual, as well as the second dual, of a separable H.I. Banach space, with space of bounded operators of the form $\lambda I + W$ where W is weakly compact and hence with separable range. Then they obtain the complete dichotomy for quotients of H.I. spaces. Namely, they prove that every separable Banach space E, not containing isomorphically ℓ^1, is a quotient of a H.I. Banach space X with E^* isometric to a complemented subspace of X^*. Argyros and Raikoftsalis have shown that every separable reflexive Banach space is a quotient of a reflexive H.I. space, which yields that every separable reflexive Banach is isomorphic to a subspace of a reflexive indecomposable space, [14]. Finally, according to [162] *"Argyros and Motakis have just given another remarkable example of an H.I. reflexive space E so that every* $T \in \mathcal{L}(E)$ *admits a nontrivial invariant subspace. Moreover this holds for all* $T \in \mathcal{L}(X)$*, for any closed subspace* $X \subset E$*. The strictly singular operators on every subspace of E form a nonseparable ideal but every* $T \in \mathcal{L}(E)$ *either commutes with a non-zero compact operator or else* $T^3 = 0$*, [13]. This example solves another open problem on spreading models. The construction uses Tsirelson ideas under constraints, motivated by earlier constructions in [149, 150], see next subsection for more complete details on the subject".*

5.3 Bourgain-Delbaen constructions of Bananch spaces with very few operators

The key ideas for the new examples we comment on here go back to the remarkable construction in 1980 of J. Bourgain and F. Delbaen, [28]. As reported in [162] *"they constructed a Banach space E with E^* isomorphic to ℓ^1, yet c_0 doest not embed into E. It seems that this example struck researchers as quite special and too limited to be useful in solving other open problems".* We ourselves witnessed that almost 25 years later R. Haydon suggested, at the end of S. Argyros's talk at the V Conference of Banach Spaces, Cáceres, Spain, 2006, the idea of using Bourgain-Delbaen construction to help in some way Argyros's school of methods for the theory of Hereditarely Indecomposable Banach spaces, [15]. Despite the answer was that this might not help, after Cáceres meeting they began to think otherwise. Two years later, S. Argyros and R. Haydon used the BD-construction to solve a famous problem in Banach spaces. They presented their construction for the first time in the Spring School at Paseky, 2008. Given a specific classical example of a Banach space E, it is usually quite easy to construct many nontrivial bounded linear operators $T \in \mathcal{L}(E)$. But just given that E is separable and infinite dimensional, this is not at all clear. It can be read in [162] that *"over 35 years ago Lindenstrauss [133] asked if such an E existed so that*

$$\mathcal{L}(E) = \{\lambda I + K : \lambda \in \mathbb{C}, K \text{ compact operator } \}.$$

In their remarkable example of a space E not containing an unconditional basic sequence W.T. Gowers and B. Maurey [87] proved that for their space all operators had the form $\lambda I + S$ where S is strictly singular, as previously said. But the scalar plus compact problem remained open. Then Argyros and Haydon [12] constructed a space AH with the scalar plus compact property. AH space is formed using the Bourgain Delbaen technique and thus AH^* is isomorphic to ℓ^1. AH space can be constructed to be H.I. too. Shortly after that D. Freeman, E. Odell and Th. Schlumprecht [76] proved that if E is a separable Banach space then E embeds into an isomorphic predual of ℓ^1. The proof, again, adopted the Bourgain Delbaen construction".

Let us remark, following Gowers's webblog, *"that one of the biggest problems in functional analysis is the invariant subspace problem, which asks whether for every operator T on a Hilbert space H there is a proper closed subspace $Z \subset H$ such that $T(Z) \subset Z$. Even the corresponding question for Banach spaces is very hard, but operators without invariant subspaces have been constructed for various Banach spaces in amazing work of P. Enflo, and subsequently C. Read, [64, 169]. Nonetheless, the problem for Hilbert spaces remains stubbornly open. Now one might speculate that the result is hard to prove because it is in fact false. And one might even speculate that it is false for every Banach space. However, the example of Argyros and Haydon shows that the situation is more complicated. A famous result of Lomonosov shows* *that every operator that commutes with a non-zero compact operator must have an invariant subspace, [134]. And obviously K commutes with $\lambda I + K$ if K is non-zero, or trivially has an invariant subspace if $K = 0$. From this we conclude that every continuous linear operator on the space of AH has a non-trivial invariant subspace. AH has been the first space for which such a result is known. What this shows, as T. Gowers says, is that you cannot hope to find a counterexample for a general Banach space, because in a sense a general Banach space does not have to have enough operators for there to be any chance at all of a counterexample. Argyros-Haydon space has very definitely taken over as the new nastiest known Banach space".*

Thus, according to [162] *"Banach spaces E satisfying the "scalar plus compact" property are of interest to operator theorists since every operator $T \in \mathcal{L}(E)$ must admit a nontrivial invariant subspace. Furthermore $\mathcal{L}(E)$ is separable, and from the construction, is amenable as a Banach algebra. Argyros, Freeman, Haydon, Odell, Raikoftsalis, Schlumprecht and Zisimopoulou finally joined efforts to show how to construct extensions $\mathcal{L}^{\infty}_{E,hi}$ of a Bourgain-Delbaen space $\mathcal{E}$ that contains E in such a way that, for instance we have $\mathcal{L}^{\infty}_{E,hi}/E$ is H.I. and has the "scalar plus compact" property whenever E has a separable dual. Thus, any separable superreflexive space can be embedded into an isomorphic predual E of ℓ^1 with the "scalar plus compact" property. E shares the properties of [12]. Namely all $T \in \mathcal{L}(E)$ admits nontrivial invariant subspaces, $\mathcal{L}(E)$ is separable and amenable. Furthermore E is somewhat reflexive (every infinite dimensional subspace of E contains an infinite dimensional reflexive subspace), [9]. Matthew Tarbard, motivated by the question as to whether any H.I. isomorphic predual of ℓ^1, with the "scalar plus strictly singular" property must have the "scalar plus compact" property, has recently showed this to be false. Indeed one can obtain such spaces with the Calkin algebra, $\mathcal{L}(E)/\mathcal{K}(E)$ having any finite dimension. Here $\mathcal{K}(E)$ denotes the ideal of compact operators on E, [185]. Therefore the lattice of closed ideals in the algebra of bounded linear operators on these spaces can have any given finite cardinality. An important open question is the following:*

Question 12. Is it possible to construct a reflexive "scalar-plus-compact" space?"

5.4 Ramsey methods in Banach spaces

Ramsey theory is a branch of combinatorics that has been successfully applied to Banach space theory in the last decades. Indeed, H.P. Rosenthal's characterization of Banach spaces containing ℓ^1: (every bounded sequence in a Banach space E has a subsequence which is either weakly Cauchy or equivalent to the unit vector basis of ℓ^1, [171]) can be seen as the first result of this kind, [86]. Results in this direction are saying that if a Banach space fails to contain a subspace with some good symmetry property, then it must have a subspace which lacks symmetry in a very extreme way. The prominent example in this line is Gowers' dichotomy theorem saying that every infinite dimensional Banach space E has an infinite dimensional subspace X which either has an unconditional basis or is hereditarely indecomposable, [85]. Chapter 24 in the Handbook of the geometry of Banach spaces, [86], is an excellent point to read about the matter. As reported in [162], *"based on these results Gowers began a program of isomorphic classification of Banach spaces. The aim of this program is to find a classification of Banach spaces up to subspaces, by producing a list of classes of Banach spaces such that:*

1. *if a space belongs to a class, then every subspace belongs to the same class, or maybe, in the case when the properties defining the class depend on a basis of the space, every block subspace belongs to the same class,*
2. *the classes are inevitable,* i.e. *every Banach space contains a subspace in one of the classes,*
3. *any two classes in the list are disjoint,*
4. *belonging to one class gives a lot of information about operators that may be defined on the space or on its subspaces.*

Such a list is referred as a list of inevitable classes of Gowers. One of the motivations of of Gowers' program is the classification of those spaces (such as Tsirelson's space T) which do not contain a copy of c_0 or ℓ^p, [71]. First two examples of inevitable classes are H.I. spaces and spaces with unconditional basis. The second dichotomy result by Gowers says that any Banach space contains a subspace with a basis such that either no two disjointly supported block subspaces are isomorphic, or such that any two subspaces have further subspaces which are isomorphic. He called the second property quasi minimality, and H. Rosenthal had defined a space to be minimal if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace is called strictly quasi minimal, so Gowers again divided the class of quasi minimal spaces into the class of strictly quasi minimal spaces and the class of minimal spaces. Gowers therefore produced a list of four classes of Banach spaces, corresponding to classical examples, or more recent couterexamples to classical questions: HI spaces, such as GM; spaces with bases such that no disjointly supported subspaces are isomorphic, such as the counterexample of Gowers to the hyperplane's problem of Banach; strictly quasi minimal spaces with an unconditional basis, such as T; and, minimal spaces, such as c_0 or ℓ^p. By further dichotomies results 19 inevitable classes have been described by now, [71].

A lot of interesting open problems fitting in this area can be found in [56].

5.5 Banach spaces $C(K)$ with few operators

If we want to talk about spaces $C(K)$ with few operators the right author to be referenced is be P. Koszmider. Following his very recent a interesting survey paper [123] we say that Banach space $C(K)$ has few operators if for every linear bounded operator T on $C(K)$ we have that $T = gI + S$ or $T^* = g^*I + S$ where g is continuous on K, g^* is Borel on K and S are weakly compact on $C(K)$ or $C(K)^*$ respectively. Let us remark that weakly compact operators coincide with the strictly singular ones for $C(K)$ spaces, [160]. $C(K)$ spaces with few operators share some common properties with the spaces of Gowers and Maurey, but their norm is simpler. For example, some of them are indecomposable Banach spaces and are not isomorphic to their hyperplanes, [122]. It follows that there are examples of $C(K)$ spaces which are not isomorphic to any $C(L)$ for L totally disconnected, [122, 161]. Banach spaces with few operators have been used as ingredients of other interesting constructions during the last years:

- A Banach space E is called extremely non complex if, and only if every linear bounded operator $T \in \mathcal{L}(E)$ satisfies the norm equality $\|T^2 + I\|^2 = 1 + \|T^2\|$. A real Banach space has complex structure if, and only if, there is on it an operator T satisfying $T^2 = -I$. $C(K)$ spaces with few operators are extremely non complex, and infinite sums of incomparable $C(K)$ spaces with few operators provide examples of extremely non-complex Banach spaces with many operators, [126]. Examples of Banach spaces with a trivial group of onto isometries now follow, [125].
- Even Banach spaces are those real Banach spaces which admits complex structure but their hyperplanes do not. Several examples of even Banach spaces of the form $C(K)$ are constructed with spaces with few operators, [70].
- Another remarkable example, constructed with a compactification of infinite unions of countably many copies of spaces K such that $C(K)$ has few operators, is a totally disconnected compact space K_1 which has compact subsets $K_2 \subset L_1 \subset K_1$ such that $C(K_1)$ is isomorphic as Banach space to $C(K_2)$, but not to $C(L_1)$. Thus we have two non-isomorphic Banach spaces of the form $C(K)$ which are isomorphic to complemented subspaces of each other, providing a solution to Shroeder-Bernstein problem of the form $C(K)$, [124].

We isolate below the following problem that is due to S. Argyros.

Question 13. Is there any bound of the densities of indecomposable Banach spaces

5.6 Descriptive set theory in Banach spaces

We start by crediting the recent monograph by P. Dodos [58], together with the paper of G. Godefroy [83] as the right places to understand the nature of problems, last results and open questions on the subject on descriptive theory in Banach spaces. These two references have been our reading to organize, select and comment on the results about this topic that now follows.

One of the central sources connecting descriptive set-theoretic topology and the geometry of Banach spaces are universality problems. The natural question is: Let $\mathcal{C}$ be a class of separable Banach spaces such that every space E in the class $\mathcal{C}$ has a certain property, say property (P). When can we find a separable Banach space X which has property (P) and contains an isomorphic copy of every member of $\mathcal{C}$? Classical properties of Banach spaces, such as "being reflexive", "having separable dual", "not containing an isomorphic copy of c_0" have a positive answer if, and essentially only if, the class $\mathcal{C}$ is analytic in a natural "coding" of separable Banach spaces. It was B. Bossard who made clear how to deal with Borel and analytic classes of Banach spaces in [26], where he proved that the relation of linear isomorphism between Banach spaces is analytic non-Borel without analytic selection. For every separable Banach space E the set of all closed linear subspaces of E endowed with the relative Effros-Borel σ-algebra is standard. Since $C(2^{\mathbb{N}})$ is isometrically universal for all separable Banach spaces we can consider the set $\mathcal{SB}$ of all closed linear subspaces of $C(2^{\mathbb{N}})$ as the standard Borel space of all separable Banach spaces, [25]. With this identification, properties of separable Banach spaces become sets in SB where the complexity can be measured. For instance, given an infinite dimensional separable Banach space the class $\mathcal{C}_E$ of separable Banach spaces which contains an isomorphic copy of E is analytic non-Borel. This work opened the way to several applications of descriptive set theory to Banach spaces, for instance G. Godefroy showed that there is not a separable Banach space E so that every separable and strictly convex space X embeds isometrically into E, [82], solving a long standing open problem of Lindenstrauss. The use of (transfinite) uniform boundedness principles has been used in the last few years by S. Argyros, P. Dodos and their coauthors who deepened the theory with the discovery of "amalgamation" methods which tighten the links between set-theoretical and linear operations. These new techniques provide the right approach to universality problems. For instance, the class $\mathcal{SD}$ of all $E \in \mathcal{SB}$ with E^* separable is coanalytic and the Szlenk index is a Π_1^1 rank on it, [25]. It is also possible to code basic sequences, [26], proving that shrinking basic sequences $\mathcal{S}$ is coanalytic and the Szlenk index is again a Π_1^1 rank on $\mathcal{S}$. Szlenk index were introduced in [181] to prove that there is not universal space for the class $\mathcal{SD}$. J. Bourgain considerably strengthened Szlenk's result by showing that if a separable Banach space X is universal for all separable reflexive space, then X must contain $C(2^{\mathbb{N}})$, and so, it is universal for all separable

Banach spaces, [27] Bossard refined Bourgain's result showing that if $\mathcal{A}$ is an analytic set in $\mathcal{SB}$ such that for every reflexive Banach space there is $Z \in \mathcal{A}$ with Z isomorphic to X, then there exists $X \in \mathcal{A}$ which is universal, [26]. Argyros and Dodos say that a class $\mathcal{C} \subset \mathcal{SB}$ is strongly bounded if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ that contains an isomorphic copy of every $X \in \mathcal{A}$. Examples of strongly bounded classes of separable Banach spaces are reflexive spaces, spaces with separable dual as well as the one of non universal spaces, [58]. The last one is an old problem of S.A. Kechris. These results follow from the constructions done in [11] for the same classes adding to have shrinking basis. The general case is covered in [59, 57]. To finish let us point out that Dodos and Lopez Abad have shown how the class of Banach spaces not containing H.I. subspaces is strongly bounded, [60]. Hence if we decide once and for all to live in a universe from where H.I. spaces are banned, then strong boundedness holds for ever, [83]. Let us finish selecting the following open problem asked in [83]:

Question 14. Let E be an infinite dimensional separable Banach space which is not isomorphic to ℓ^2. Does E contains infinite dimensional subspaces $\{\mathbb{E}_m : m \in \mathbb{N}\}$ such that E_n is not isomorphic to E_k if $n \neq k$?

The recent paper [84] contains more interesting open problems on the matter.

5.7 Nonlinear geometry of Banach spaces

It seems very natural to finish this section presenting a short report on achievements on nonlinear geometry of Banach spaces. Besides our own knowledge and in order to properly present the most important results on this topic we have used the survey paper by N. Kalton [118] from where some literal comments are mixed below with our own words.

A Banach space is, by its nature, also a metric space. When we identify a Banach space with its underlying metric space, we choose to forget its linear structure. A fundamental question of nonlinear geometry of Banach spaces is to determine to what extent the metric structure of a Banach space already determines its linear structure. Another one is concerning nonlinear embeddings of one Banach space into another, and more generally of metric spaces into Banach spaces. The book of Benyamini and Lindenstrauss, [22], gave a definitive form to the subject driving a lot of research in the area during the last years. Other areas of mathematics such as theoretical computer or C^*-algebras have a strong interplay with the matter looking at the problem to determine how well a metric space can be embedded in a particular Banach space. Since any two separable infinite-dimensional Fréchet spaces are homeomorphic by the beautiful theorem of Anderson-Kadets, see Chapter VI, Theorem 5.2 in [23], it seems that just topology say nothing on the linear structure of the spaces involved. But there is still something else to mention, as Kalton said, leaving the realm of locally convex spaces. Indeed, there are two remarkable results of Cauty: there is a separable F-space (complete metric linear) which is not homeomorphic to a separable Banach space, [46], and every compact convex subset of an F-space has the Schauder fixed point property, [47] (it seems that some controversy remains on that result). This problem had been open since 1930, when Schauder proved the original fixed point theorem. It doest not seem to be known if an infinite-dimensional compact convex set is necesarily homeomorphic to the Hilbert cube (for subsets of Banach spaces this corresponds with Keller's theorem, see Chapter III, Theorem 3.1 in [23]. The homeomorphic theory of non-locally convex F-spaces seems to be a very rich and interesting area for further research.

Parallel to the linear theory, the main focus is the nonlinear classification of Banach spaces. The linear operators are replaced by Lipschitz or uniformly continuous maps. The problems of interest are Lipschitz, uniform or coarser embedding of metric spaces in normed spaces, or such an embedding of a Banach space into another. As an example, let us consider the following question: If E and F are separable Banach spaces which are Lipschitz isomorphic, are E and F linearly isomorphic? Since every space $C(K)$ is Lipschitz homeomorphic to $c_0(\Gamma)$ whenever the compact space K has ω_0-derived set $K^{(\omega_0)} = \emptyset$, [53, Chapter VI, Theorem 8.9], the previous question reduces to the separable case. Indeed, a Ciesielski-Pol compact space K, [53, Chapter VI,Theorem 8.8.3], gives an example with $K^3 = \emptyset$ and no linear continuous injection of $C(K)$ into any $c_0(\Gamma)$. When $E = L^p$ or $E = l^p$, $1 < p < \infty$ the answer is yes by a result of Heinrich and Mankiewicz, [108]. If $E = c_0$ the answer is yes and it is due to Godefroy, Kalton and Lancien, [80]. Johnson, Lindenstrauss and Scheteman achieved a major breakthrough in 1996 showing that If $1 < p < \infty$ and E is uniformly homeomorphic to l^p, then E is linearly isomorphic to l^p, [116], notice that for $p = 1$ there is no answer even for Lipschitz isomorphism. By using a metric notion of cotype, Mendel and Naor showed that L^q uniformly embeds into L^p if, and only if, either we have $p \leq q \leq 2$ or $q \leq p$, [137] Recently, Lima and Randrianarivony,[131] proved that the uniform quotients of $l^p, 1 < p < 2$ are the same up to isomorphism as the linear quotients of l^p answering a problem that had been open for over a decade. Their proof made essential use of Property (β) of Rolewicz, which is an asymptotic property of Banach spaces whose definition involves the metric but not the linear structure of the space, and which therefore lends itself nicely to the nonlinear theory. S. J. Dilworth, D. Kutzarova, G. Lancien, and N. L. Randrianarivony have shown that if $T : E \longrightarrow F$ is a uniform quotient then the modulus of asymptotic smoothness of F essentially dominates the (β)-modulus of E. It follows that the separable spaces that are isomorphic to spaces with Property (β) are precisely the reflexive spaces E such that both E and E^* have Szlenk index equal to ω, [55]. A main open problem here is:

Question 15. If E is a Banach space uniformly homeomorphic to c_0, doest it follow that E is linearly isomorphic to c_0?

References

1. G. Androulakis and T. Schlumprecht. Strictly singular, non-compact operators exist on the space of Gowers and Maurey. *J. London Math. Soc. (2)*, 64(3):655–674, 2001.
2. C. Angosto. *Distance to function spaces.* PhD thesis, Universidad de Murcia, 2007.
3. C. Angosto and B. Cascales. The quantitative difference between countable compactness and compactness. *J. Math. Anal. Appl.*, 343(1):479–491, 2008.
4. C. Angosto and B. Cascales. Measures of weak noncompactness in Banach spaces. *Topology Appl.*, 156(7):1412–1421, 2009.
5. C. Angosto, B. Cascales, and I. Namioka. Distances to spaces of Baire one functions. *Math. Z.*, 263(1):103–124, 2009.
6. C. Angosto, J Kąkol, and M. López Pellicer. A quantitative approach to weak compactness in Fréchet spaces and spaces C(X). Preprint, 2012.
7. Carlos Angosto and Bernardo Cascales. A new look at compactness via distances to function spaces. In *Advanced courses of mathematical analysis III*, pages 49–66. World Sci. Publ., Hackensack, NJ, 2008.
8. S. A. Argyros and V. Felouzis. Interpolating hereditarily indecomposable Banach spaces. *J. Amer. Math. Soc.*, 13(2):243–294 (electronic), 2000.
9. S. A. Argyros, D. Freeman, R. G. Haydon, E. Odell, Tf. Raikoftalis, Th. Schlumprecht, and D.Z. Zisimopoulou. Hi extensions of $\mathcal{L}_\infty$-spaces with very few operators. Preprint, 2012.
10. Spiros A. Argyros. A universal property of reflexive hereditarily indecomposable Banach spaces. *Proc. Amer. Math. Soc.*, 129(11):3231–3239 (electronic), 2001.
11. Spiros A. Argyros and Pandelis Dodos. Genericity and amalgamation of classes of Banach spaces. *Adv. Math.*, 209(2):666–748, 2007.
12. Spiros A. Argyros and Richard G. Haydon. A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem. *Acta Math.*, 206(1):1–54, 2011.
13. Spiros A. Argyros and P. Motakis. The invariant subspace property of the strictly singular operators in a reflexive spaces. Preprint, 2012.
14. Spiros A. Argyros and Theocharis Raikoftsalis. The cofinal property of the reflexive indecomposable Banach spaces. *Ann. Inst. Fourier*, 62:1–45, 2012.
15. Spiros A. Argyros and Andreas Tolias. Methods in the theory of hereditarily indecomposable Banach spaces. *Mem. Amer. Math. Soc.*, 170(806):vi+114, 2004.
16. A. V. Arkhangel′skiĭ. *Topological function spaces*, volume 78 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Translated from the Russian by R. A. M. Hoksbergen.
17. A. D. Arvanitakis. Some remarks on Radon-Nikodým compact spaces. *Fund. Math.*, 172(1):41–60, 2002.
18. Kari Astala and Hans-Olav Tylli. Seminorms related to weak compactness and to Tauberian operators. *Math. Proc. Cambridge Philos. Soc.*, 107(2):367–375, 1990.
19. A. Avilés and P. Koszmider. A continuous image of a Radon-Nikodým compact space which is not Radon-Nikodým. Preprint, 2012.
20. S. Banach. *Théorie des óperations linéaires.* Chelsea Publishing Co., New York, 1955.
21. Józef Banaś and Antonio Martinón. Measures of weak noncompactness in Banach sequence spaces. *Portugal. Math.*, 52(2):131–138, 1995.
22. Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2000.
23. C. Bessaga and A. Pelczyński. *Selected topics in infinite-dimensional topology.* PWN—Polish Scientific Publishers, Warsaw, 1975. Monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58].
24. F. S. De Blasi. On a property of the unit sphere in a banach space. *Colloq. Math.*, 65:333–343, 1992.

25. Benoît Bossard. *Thórie descriptive des ensembles en gómŕie des espaces de Banach.* PhD thesis, Univ. Paris VI, 1994.
26. Benoît Bossard. A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces. *Fund. Math.*, 172(2):117–152, 2002.
27. J. Bourgain. On separable Banach spaces, universal for all separable reflexive spaces. *Proc. Amer. Math. Soc.*, 79(2):241–246, 1980.
28. J. Bourgain and F. Delbaen. A class of special $\mathcal{L}_\infty$ spaces. *Acta Math.*, 145(3-4):155–176, 1980.
29. Richard D. Bourgin. *Geometric aspects of convex sets with the Radon-Nikodým property*, volume 993 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.
30. B. Cascales. On K-analytic locally convex spaces. *Arch. Math. (Basel)*, 49(3):232–244, 1987.
31. B. Cascales, V. P. Fonf, J. Orihuela, and S. Troyanski. Boundaries of Asplund spaces. *J. Funct. Anal.*, 259(6):1346–1368, 2010.
32. B. Cascales, O. F. K. Kalenda, and J. Spurný. A quantitative version of James's compactness theorem. To appear Proc. Edinburgh Math. Soc., 2012.
33. B. Cascales, W. Marciszesky, and M. Raja. Distance to spaces of continuous functions. *Topology Appl.*, 153(13):2303–2319, 2006.
34. B. Cascales, M. Muñoz, and J. Orihuela. James boundaries and σ-fragmented selectors. *Studia Math.*, 188(2):97–122, 2008.
35. B. Cascales, M. Muñoz, and J. Orihuela. The number of K-determination of topological spaces. *RACSAM*, 106(2):341–357, 2012.
36. B. Cascales and I. Namioka. The Lindelöf property and σ-fragmentability. *Fund. Math.*, 180(2):161–183, 2003.
37. B. Cascales, I. Namioka, and J. Orihuela. The Lindelöf property in Banach spaces. *Studia Math.*, 154(2):165–192, 2003.
38. B. Cascales, I. Namioka, J. Orihuela, and M. Raja. Banach spaces and topology I. Encyclopedia of General Topology. Elsevier Science B. B. Editors J. I. Nagata, J. E. Vaughan, K. P. Hart, 2002.
39. B. Cascales and J. Orihuela. Metrizability of precompact subsets in (LF)-spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 103(3-4):293–299, 1986.
40. B. Cascales and J. Orihuela. On compactness in locally convex spaces. *Math. Z.*, 195(3):365–381, 1987.
41. B. Cascales and J. Orihuela. A sequential property of set-valued maps. *J. Math. Anal. Appl.*, 156(1):86–100, 1991.
42. B. Cascales, J. Orihuela, and M. Ruiz. Compactness, optimality and risk. Preprint, 2012.
43. B. Cascales, J. Orihuela, and V. Tkachuk. Domination by second countable spaces and lindelöf σ-property. *Topology Appl.*, 158(2):204–214, 2011.
44. Bernardo Cascales, Gilles Godefroy, José Orihuela, and Robert Phelps. Editors: The interplay between measure theory, topology, and functional analysis. *J. Math. Anal. Appl.*, 350(2):425–426, 2009.
45. Bernardo Cascales and José Orihuela. Editors: Open problems in infinite dimensional geometry and topology. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):199–450, 2010.
46. Robert Cauty. Un espace métrique linéaire qui n'est pas un rétracte absolu. *Fund. Math.*, 146(1):85–99, 1994.
47. Robert Cauty. Solution du problème de point fixe de Schauder. *Fund. Math.*, 170(3):231–246, 2001.
48. G. Choquet. Ensembles $\mathcal{K}$-analytiques et $\mathcal{K}$-sousliniens. Cas général et cas métrique. *Ann. Inst. Fourier. Grenoble*, 9:75–81, 1959.
49. J. P. R. Christensen. *Topology and Borel structure.* North-Holland Publishing Co., Amsterdam, 1974. Descriptive topology and set theory with applications to functional analysis and measure theory, North-Holland Mathematics Studies, Vol. 10. (Notas de Matemática, No. 51).

50. J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.
51. H. H. Corson. Metrizability of compact convex sets. *Trans. Amer. Math. Soc.*, 151:589–596, 1970.
52. R. Deville. Geometrical implications of the existence of very smooth bump functions in Banach spaces. *Israel J. Math.*, 67(1):1–22, 1989.
53. R. Deville, G. Godefroy, and V. Zizler. *Smoothness and renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1993.
54. J. Diestel. *Sequences and series in Banach spaces*, volume 92 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984.
55. S. J. Dilworth, Denka Kutzarova, G. Lancien, and N. Lovasoa Randrianarivony. Asymptotic geometry of Banach spaces and uniform quotient maps. Preprint, 2012.
56. P. Dodos, J. Lopez-Abad, and S. Todorcevic. Banach spaces and Ramsey theory: some open problems. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):435–450, 2010.
57. Pandelis Dodos. On classes of Banach spaces admitting "small" universal spaces. *Trans. Amer. Math. Soc.*, 361(12):6407–6428, 2009.
58. Pandelis Dodos. *Banach spaces and descriptive set theory: selected topics*, volume 1993 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.
59. Pandelis Dodos and Valentin Ferenczi. Some strongly bounded classes of Banach spaces. *Fund. Math.*, 193(2):171–179, 2007.
60. Pandelis Dodos and Jordi Lopez-Abad. On unconditionally saturated Banach spaces. *Studia Math.*, 188(2):175–191, 2008.
61. A. Dow, H. Junnila, and J. Pelant. Coverings, networks and weak topologies. *Mathematika*, 53(2):287–320 (2007), 2006.
62. W. F. Eberlein. Weak compactness in Banach spaces. I. *Proc. Nat. Acad. Sci. U. S. A.*, 33:51–53, 1947.
63. P. Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973.
64. P. Enflo. On the invariant subspace problem for Banach spaces. *Acta Math.*, 158(3-4):213–313, 1987.
65. R. Engelking. *General topology*. PWN—Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60].
66. M. Fabian. *Gâteaux differentiability of convex functions and topology*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1997. Weak Asplund spaces, A Wiley-Interscience Publication.
67. M. Fabian, G. Godefroy, V. Montesinos, and V. Zizler. Inner characterizations of weakly compactly generated Banach spaces and their relatives. *J. Math. Anal. Appl.*, 297(2):419–455, 2004. Special issue dedicated to John Horváth.
68. M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, and V. Zizler. *Functional analysis and infinite-dimensional geometry*, volume 8 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.
69. M. Fabian, P. Hájek, V. Montesinos, and V. Zizler. A quantitative version of Krein's theorem. *Rev. Mat. Iberoamericana*, 21(1):237–248, 2005.
70. Valentin Ferenczi and Elói Medina Galego. Even infinite-dimensional real Banach spaces. *J. Funct. Anal.*, 253(2):534–549, 2007.
71. Valentin Ferenczi and Christian Rosendal. Banach spaces without minimal subspaces. *J. Funct. Anal.*, 257(1):149–193, 2009.
72. S. Ferrari, , L. Oncina J. Orihuela, and M. Raja. Metrization theory and the Kadec property. Preprint, 2012.
73. S. Ferrari, J. Orihuela, and M. Raja. Nonlinear transfer for strictly convex norms. Preprint, 2012.

74. K. Floret. Some aspects of the theory of locally convex inductive limits. In *Functional analysis: surveys and recent results, II (Proc. Second Conf. Functional Anal., Univ. Paderborn, Paderborn, 1979)*, volume 38 of *North-Holland Math. Stud.*, pages 205–237. North-Holland, Amsterdam, 1980.
75. K. Floret. *Weakly compact sets*, volume 801 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Lectures held at S.U.N.Y., Buffalo, in Spring 1978.
76. D. H. Fremlin, P. Holický, and J. Pelant. Lindelöf modifications and K-analytic spaces. *Mathematika*, 40(1):1–6, 1993.
77. F. García, L. Oncina, and J. Orihuela. Network characterization of Gul'ko compact spaces and their relatives. *J. Math. Anal. Appl.*, 297(2):791–811, 2004. Special issue dedicated to John Horváth.
78. F. García, L. Oncina, J. Orihuela, and S. Troyanski. Kuratowski's index of non-compactness and renorming in Banach spaces. *J. Convex Anal.*, 11(2):477–494, 2004.
79. G. Godefroy. Some applications of Simons' inequality. *Serdica Math. J.*, 26(1):59–78, 2000.
80. G. Godefroy, N. Kalton, and G. Lancien. Subspaces of $c_0(\mathbf{N})$ and Lipschitz isomorphisms. *Geom. Funct. Anal.*, 10(4):798–820, 2000.
81. Gilles Godefroy. Renormings of Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 781–835. North-Holland, Amsterdam, 2001.
82. Gilles Godefroy. Universal spaces for strictly convex Banach spaces. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 100(1-2):137–146, 2006.
83. Gilles Godefroy. Descriptive set theory and the geometry of Banach spaces. In *Perspectives in mathematical sciences. II*, volume 8 of *Stat. Sci. Interdiscip. Res.*, pages 63–82. World Sci. Publ., Hackensack, NJ, 2009.
84. Gilles Godefroy. Analytic sets of Banach spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):365–374, 2010.
85. W. T. Gowers. An infinite Ramsey theorem and some Banach-space dichotomies. *Ann. of Math. (2)*, 156(3):797–833, 2002.
86. W. T. Gowers. Ramsey methods in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1071–1097. North-Holland, Amsterdam, 2003.
87. W. T. Gowers and B. Maurey. The unconditional basic sequence problem. *J. Amer. Math. Soc.*, 6(4):851–874, 1993.
88. A. S. Granero, P. Hájek, and V. Montesinos Santalucía. Convexity and w^*-compactness in Banach spaces. *Math. Ann.*, 328(4):625–631, 2004.
89. A. S. Granero and M. Sánchez. The class of universally Krein-Šmulian Banach spaces. *Bull. Lond. Math. Soc.*, 39(4):529–540, 2007.
90. Antonio S. Granero. An extension of the Krein-Šmulian theorem. *Rev. Mat. Iberoam.*, 22(1):93–110, 2006.
91. Antonio S. Granero and Marcos Sánchez. Convexity, compactness and distances. In *Methods in Banach space theory*, volume 337 of *London Math. Soc. Lecture Note Ser.*, pages 215–237. Cambridge Univ. Press, Cambridge, 2006.
92. A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, 74:168–186, 1952.
93. Alexandre Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, 1955(16):140, 1955.
94. Gary Gruenhage. Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products. *Topology Appl.*, 17(3):287–304, 1984.
95. Gary Gruenhage. Generalized metric spaces. In *Handbook of set-theoretic topology*, pages 423–501. North-Holland, Amsterdam, 1984.
96. Gary Gruenhage. A note on Gul′ko compact spaces. *Proc. Amer. Math. Soc.*, 100(2):371–376, 1987.
97. Gary Gruenhage. Generalized metric spaces and metrization. In *Recent progress in general topology (Prague, 1991)*, pages 239–274. North-Holland, Amsterdam, 1992.
98. Gary Gruenhage. Metrizable spaces and generalizations. In *Recent progress in general topology, II*, pages 201–225. North-Holland, Amsterdam, 2002.

99. Gary Gruenhage. Spaces having small diagonal. *Topology Appl.*, 122(1-2):183–200, 2002.
100. A. Guirao and J. Orihuela. On $t-p$-locally uniformly rotund norms. Preprint, 2012.
101. Petr Hájek, Vicente Montesinos Santalucía, Jon Vanderwerff, and Václav Zizler. *Biorthogonal systems in Banach spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26. Springer, New York, 2008.
102. R. W. Hansell. Descriptive topology. In *Recent progress in general topology (Prague, 1991)*, pages 275–315. North-Holland, Amsterdam, 1992.
103. R. W. Hansell. Descriptive sets and the topology of nonseparable Banach spaces. *Serdica Math. J.*, 27(1):1–66, 2001.
104. R. Haydon. Trees in renorming theory. *Proc. London Math. Soc.*, 78(3):541–584, 1999.
105. R. Haydon, A. Moltó, and J. Orihuela. Spaces of functions with countably many discontinuities. *Israel J. Math.*, 158:19–39, 2007.
106. Richard Haydon. Locally uniformly convex norms in Banach spaces and their duals. *J. Funct. Anal.*, 254(8):2023–2039, 2008.
107. Richard Haydon, Mireille Levy, and Edward Odell. On sequences without weak* convergent convex block subsequences. *Proc. Amer. Math. Soc.*, 100(1):94–98, 1987.
108. S. Heinrich and P. Mankiewicz. Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces. *Studia Math.*, 73(3):225–251, 1982.
109. G. J. O. Jameson. *Topology and normed spaces.* Chapman and Hall, London, 1974.
110. J. E. Jayne, I. Namioka, and C. A. Rogers. σ-fragmentable Banach spaces. *Mathematika*, 39(2):197–215, 1992.
111. J. E. Jayne, I. Namioka, and C. A. Rogers. Topological properties of Banach spaces. *Proc. London Math. Soc. (3)*, 66(3):651–672, 1993.
112. J. E. Jayne, I. Namioka, and C. A. Rogers. σ-fragmented Banach spaces. II. *Studia Math.*, 111(1):69–80, 1994.
113. J. E. Jayne, J. Orihuela, A. J. Pallarés, and G. Vera. σ-fragmentability of multivalued maps and selection theorems. *J. Funct. Anal.*, 117(2):243–273, 1993.
114. J. E. Jayne and C. A. Rogers. *Selectors.* Princeton University Press, Princeton, NJ, 2002.
115. W. B. Johnson and J. Lindenstrauss. Basic concepts in the geometry of Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 1–84. North-Holland, Amsterdam, 2001.
116. W. B. Johnson, J. Lindenstrauss, and G. Schechtman. Banach spaces determined by their uniform structures. *Geom. Funct. Anal.*, 6(3):430–470, 1996.
117. J. Kąkol, W. Kubiś, and M. López-Pellicer. *Descriptive Topology in Selected Topics of Functional Analysis.* Developments in Mathematics 24, Springer Verlag, New York, 2011.
118. O. F. K. Kalenda. Valdivia compact spaces in topology and Banach space theory. *Extracta Math.*, 15(1):1–85, 2000.
119. J. L. Kelley. *General topology.* Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.
120. J. L. Kelley and I. Namioka. *Linear topological spaces*, volume 36 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing.
121. P. S. Kenderov and W. B. Moors. Fragmentability and sigma-fragmentability of Banach spaces. *J. London Math. Soc. (2)*, 60(1):203–223, 1999.
122. Piotr Koszmider. Banach spaces of continuous functions with few operators. *Math. Ann.*, 330(1):151–183, 2004.
123. Piotr Koszmider. A survey on Banach spaces $C(K)$ with few operators. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):309–326, 2010.
124. Piotr Koszmider. A $C(K)$ Banach space which doest not have tha Schroeder-Bernstein property. Preprint, 2012.

125. Piotr Koszmider, Miguel Martín, and Javier Merí. Extremely non-complex $C(K)$ spaces. *J. Math. Anal. Appl.*, 350(2):601–615, 2009.
126. Piotr Koszmider, Miguel Martín, and Javier Merí. Isometries on extremely non-complex Banach spaces. *J. Inst. Math. Jussieu*, 10(2):325–348, 2011.
127. G. Köthe. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.
128. A. Kryczka, S. Prus, and M. Szczepanik. Measure of weak noncompactness and real interpolation of operators. *Bull. Austral. Math. Soc.*, 62(3):389–401, 2000.
129. Andrzej Kryczka. Quantitative approach to weak noncompactness in the polygon interpolation method. *Bull. Austral. Math. Soc.*, 69(1):49–62, 2004.
130. Andrzej Kryczka and Stanisław Prus. Measure of weak noncompactness under complex interpolation. *Studia Math.*, 147(1):89–102, 2001.
131. Vegard Lima and N. Lovosoa Randrianarivony. Property (β) and uniform quotient maps. Israel J. Math., in press, 2012.
132. J. Lindenstrauss and L. Tzafriri. On Orlicz sequence spaces. *Israel J. Math.*, 10:379–390, 1971.
133. Joram Lindenstrauss. Some open problems in Banach space theory. In *Séminaire Choquet. Initiation à l' analyse tome 15, 1975-1976, Exp. No. 18*, pages 1–9. Secrétariat mathématique, Paris, 1975-76.
134. V. I. Lomonosov. Invariant subspaces of the family of operators that commute with a completely continuous operator. *Funkcional. Anal. i Priložen.*, 7(3):55–56, 1973.
135. Brenda MacGibbon. A criterion for the metrizability of a compact convex set in terms of the set of extreme points. *J. Functional Analysis*, 11:385–392, 1972.
136. J. F. Martínez, A. Moltó, J. Orihuela, and S. Troyanski. On locally uniformly rotund renormings in $C(K)$ spaces. *Canad. J. Math.*, 62(3):595–613, 2010.
137. Manor Mendel and Assaf Naor. Metric cotype. *Ann. of Math. (2)*, 168(1):247–298, 2008.
138. Ernest Michael. Continuous selections. I. *Ann. of Math. (2)*, 63:361–382, 1956.
139. A. Moltó, J. Orihuela, and S. Troyanski. Locally uniformly rotund renorming and fragmentability. *Proc. London Math. Soc. (3)*, 75(3):619–640, 1997.
140. A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia. On weakly locally uniformly rotund Banach spaces. *J. Funct. Anal.*, 163(2):252–271, 1999.
141. Aníbal Moltó, José Orihuela, Stanimir Troyanski, and Manuel Valdivia. *A nonlinear transfer technique for renorming*, volume 1951 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
142. Warren B. Moors. An elementary proof of James' characterization of weak compactness. *Bull. Aust. Math. Soc.*, 84(1):98–102, 2011.
143. Warren B. Moors and Evgenii A. Reznichenko. Separable subspaces of affine function spaces on convex compact sets. *Topology Appl.*, 155(12):1306–1322, 2008.
144. I. Namioka. Separate continuity and joint continuity. *Pacific J. Math.*, 51:515–531, 1974.
145. I. Namioka. Radon-Nikodým compacta. In *Encyclopedia of General Topology*, chapter 000, page 4 pages. Elsevier Science B.V. Editors: J.-I. Nagata, J. E. Vaughan and K. P. Hart, 2001.
146. I. Namioka. On generalizations of Radon-Nikodým compact spaces. In *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York)*, volume 26, 2, pages 741–750, 2001/02.
147. I. Namioka. Fragmentability in Banach spaces: interaction of topologies. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):283–308, 2010.
148. S. Negrepontis. Banach spaces and topology. In *Handbook of set-theoretic topology*, pages 1045–1142. North-Holland, Amsterdam, 1984.
149. E. Odell and Th. Schlumprecht. On the richness of the set of p's in Krivine's theorem. In *Geometric aspects of functional analysis (Israel, 1992–1994)*, volume 77 of *Oper. Theory Adv. Appl.*, pages 177–198. Birkhäuser, Basel, 1995.

150. E. Odell and Th. Schlumprecht. A Banach space block finitely universal for monotone bases. *Trans. Amer. Math. Soc.*, 352(4):1859–1888, 2000.
151. Luis Oncina. A new characterization of Eberlein compacta. *Studia Math.*, 146(1):69–81, 2001.
152. J. Orihuela. Pointwise compactness in spaces of continuous functions. *J. London Math. Soc. (2)*, 36(1):143–152, 1987.
153. J. Orihuela. On weakly Lindelöf Banach spaces. In *Progress in functional analysis (Peñíscola, 1990)*, volume 170 of *North-Holland Math. Stud.*, pages 279–291. North-Holland, Amsterdam, 1992.
154. J. Orihuela and M. Ruiz Galán. A coercive James's weak compactness theorem and nonlinear variational problems. *Nonlinear Anal.*, 75:598–611, 2012.
155. J. Orihuela and M. Ruiz Galán. Lebesgue property for convex risk measures in Orlicz spaces. *Mathematics and Financial*, 6(1):15–35, 2012.
156. J. Orihuela, W. Schachermayer, and M. Valdivia. Every Radon-Nikodým Corson compact space is Eberlein compact. *Studia Math.*, 98(2):157–174, 1991.
157. J. Orihuela and S. Troyanski. Deville's master lemma and Stone's discreteness in renorming theory. *J. Convex Anal.*, 16(3-4):959–972, 2009.
158. J. Orihuela and S. Troyanski. LUR renormings through Deville's master lemma. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 103(1):75–85, 2009.
159. José Orihuela, Richard J. Smith, and Stanimir Troyanski. Strictly convex norms and topology. *Proc. Lond. Math. Soc. (3)*, 104(1):197–222, 2012.
160. A. Pełczyński. On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in $C(S)$-spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 13:31–36, 1965.
161. Grzegorz Plebanek. A construction of a Banach space $C(K)$ with few operators. *Topology Appl.*, 143(1-3):217–239, 2004.
162. E. Odell R. Anisca, S. Dilworth and B. Sari. Banach space theory. Report of the meeting 4-9 March held at Banff Station, Canada, 2012.
163. M. Raja. Medibilidad en el espacios de Banach. Master's thesis, Universidad de Murcia (Dirigida: G. Vera), 1996.
164. M. Raja. Kadec norms and Borel sets in a Banach space. *Studia Math.*, 136(1):1–16, 1999.
165. M. Raja. Locally uniformly rotund norms. *Mathematika*, 46(2):343–358, 1999.
166. M. Raja. On dual locally uniformly rotund norms. *Israel J. Math.*, 129:77–91, 2002.
167. M. Raja. Weak* locally uniformly rotund norms and descriptive compact spaces. *J. Funct. Anal.*, 197(1):1–13, 2003.
168. M. Raja. Dentability indices with respect to measures of non-compactness. *J. Funct. Anal.*, 253(1):273–286, 2007.
169. C. J. Read. A solution to the invariant subspace problem. *Bull. London Math. Soc.*, 16(4):337–401, 1984.
170. Evgenii A. Reznichenko. Convex compact spaces and their maps. *Topology Appl.*, 36(2):117–141, 1990. Seminar on General Topology and Topological Algebra (Moscow, 1988/1989).
171. H. P. Rosenthal. A characterization of Banach spaces containing l^1. *Proc. Nat. Acad. Sci. U.S.A.*, 71:2411–2413, 1974.
172. J. Saint-Raymond. Characterizing convex functions on a reflexive banach space. Preprint, 2012.
173. T. Schlumprecht. An arbitrarily distortable Banach space. *Israel J. Math.*, 76(1-2):81–95, 1991.
174. Laurent Schwartz. Sur le théorème du graphe fermé. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A602–A605, 1966.
175. Laurent Schwartz. *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

176. Dmitri B. Shakhmatov. Compact spaces and their generalizations. In *Recent progress in general topology (Prague, 1991)*, pages 571–640. North-Holland, Amsterdam, 1992.
177. Richard J. Smith. Strictly convex norms, G_δ-diagonals and non-Gruenhage spaces. *Proc. Amer. Math. Soc.*, 140(9):3117–3125, 2012.
178. Richard J. Smith and Stanimir Troyanski. Renormings of $C(K)$ spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):375–412, 2010.
179. V. Šmulian. Über lineare topologische Räume. *Rec. Math. [Mat. Sbornik] N. S.*, 7 (49):425–448, 1940.
180. V. V. Srivatsa. Baire class 1 selectors for upper semicontinuous set-valued maps. *Trans. Amer. Math. Soc.*, 337(2):609–624, 1993.
181. W. Szlenk. The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces. *Studia Math.*, 30:53–61, 1968.
182. M. Talagrand. Esembles K-analytiques et fonctions croisantes de compacts. *Séminaire Choquet. Initiation à l'analyse*, 17:C1–C2, 1977.
183. M. Talagrand. Espaces de Banach faiblement $\mathcal{K}$-analytiques. *Ann. of Math. (2)*, 110(3):407–438, 1979.
184. Michel Talagrand. Sur une conjecture de H. H. Corson. *Bull. Sci. Math. (2)*, 99(4):211–212, 1975.
185. M. Tarbard. Hereditarily indecomposable, separable $\mathcal{L}^\infty$ Banach spaces with l^1 dual having few but not very few operators. To appear in J. London Math. Soc. (2012) doi: 10.1112/jlms/jdr066, 2012.
186. V. V. Tkachuk. A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic. *Acta Math. Hungar.*, 107(4):253–265, 2005.
187. N. Tomczak-Jaegermann. Banach spaces of type p have arbitrarily distortable subspaces. *Geom. Funct. Anal.*, 6(6):1074–1082, 1996.
188. B. Tsirelson. Not every banach space contains l^p or c_0. *Func. Anal Appl.*, 8:138–141, 1974.
189. M. Valdivia. *Topics in locally convex spaces*, volume 67 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982. Notas de Matemática [Mathematical Notes], 85.
190. V. Zizler. Nonseparable Banach spaces. In *Handbook of the Geometry of Banach spaces vol. 2.*, pages 1745–1816. Elsevier Science B. V., 2003.

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