Topology, measure theory and Banach spaces

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Universidad de Murcia

Second Meeting on Vector Measures and Integration.
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The co-authors


1. Bourgain property and compactness with respect to boundaries

2. Bourgain property and Birkhoff integrability

3. Aumann & Debreu & Pettis integrals multifunctions
The boundary problem

Throughout the lecture...

- $X$ is a Banach space equipped with its norm $\| \|$;
- $K$ is a Hausdorff compact and $C(K)$ is equipped with its supremum norm.
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### Diagram

```
\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (0,4); \node at (0,4) [right] {$e_3$};
\draw[->] (0,0) -- (4,0); \node at (4,0) [below] {$e_1$};
\draw[->] (0,0) -- (-1,1); \node at (-1,1) [left] {$e_2$};
\draw (0,0) .. controls (1,2) and (3,2) .. (4,0);
\draw [dashed] (0,0) .. controls (1,1) and (3,1) .. (4,0);
\end{tikzpicture}
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The boundary problem (Godefroy)…extremal test for compactness

Let $X$ Banach space, $B \subset B_{X^*}$ boundary and denote by $\tau_p(B)$ the topology defined on $X$ by the pointwise convergence on $B$. Let $H$ be a norm bounded and $\tau_p(B)$-compact subset of $X$.

Is $H$ weakly compact?
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Lemma

Let $K$ be a compact space and $B \subset \mathcal{B}_C(K)^*$ a boundary. Given a sequence $(f_n)$ in $C(K)$ and $x \in K$, then there is $\mu \in B$ such that

$$f_n(x) = \int_K f_n \, d\mu \quad \text{for every } n \in \mathbb{N}.$$
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**Proof.**

✓ $g(t) := 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(t) - f_n(x)|}{1 + |f_n(t) - f_n(x)|}$, $t \in K$, $0 \leq g \leq 1$. 
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- Then $F = \bigcap_{n=1}^{\infty} \{y \in K; f_n(y) = f_n(x)\} = \{y \in K : g(y) = 1 = \|g\|_\infty\}.$
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✓ Then for every $n \in \mathbb{N}$,

$$\int_K f_n d\mu = \int_F f_n d\mu = \int_F f_n(x) d\mu = f_n(x)$$

because $\mu$ is a probability itself.
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Key point...de Wilde’s result

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✓ **Bad news:** $\tau_p(B)$ is not compatible with $\langle X, X^* \rangle$.

✓ **Good news:** we can overcome the difficulties for many Banach spaces.
VECTOR MEASURES

By J. DIESTEL and J. J. UHL, Jr.

MATHEMATICAL SURVEYS
NUMBER 15

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Bourgain property & compactness
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Looking for inspiration...
**Theorem 11 (Kreĭn-Smulian).** The closed convex hull of a weakly compact subset of a Banach space is weakly compact.

**Proof.** Let $W$ be a weakly compact set in a Banach space $X$. To show that the closed convex hull of $W$ is weakly compact, it suffices by the Eberlein-Smulian theorem to show that the convex hull of $W$ is relatively weakly sequentially compact. Since any sequence in the convex hull of $W$ is in a separable subspace of $X$, it follows from the Hahn-Banach theorem that $W$ itself may be assumed to be norm separable.

Thus suppose $W$ is a norm separable weakly compact set in $X$ and let $g$ be the identity function on $W$. Evidently $g$ is separably valued and $x^*g$ is continuous on $W$ equipped with the weak topology for all $x^* \in X^*$. From the Pettis Measurability Theorem 1.2, it follows that $g$ is $\mu$-measurable for every regular measure $\mu$ defined on the weakly Borel sets of $W$.

Now $W$ is a compact Hausdorff space in its weak topology. Thus for $\mu \in C(W)^*$, the Bochner integral $\int_W g d\mu$ exists since $g$ is $\mu$-measurable and bounded. Define $T:C(W)^* \to X$ by $T(\mu) = \int_W g d\mu$ for $\mu \in C(W)^*$. Then if $(\mu_\alpha)$ is a net in $C(W)^*$ that converges to $\mu \in C(W)^*$ in the weak*-topology and $x^* \in X^*$, then

$$\lim_{\alpha} x^* T(\mu_\alpha) = \lim_{\alpha} x^* \int_W g d\mu_\alpha = \lim_{\alpha} \int_W x^* g d\mu_\alpha = x^* T(\mu)$$

since $x^* g \in C(W)$ for every $x^* \in X^*$. Hence $T$ is continuous for the weak*-topology of $C(W)^*$ and weak topology of $X$; accordingly $T$ is a weakly compact operator. Thus $T^*$ is weakly compact with $\tau_{(C(W)^*)^*, X^*}$; hence $T$ is a weakly compact and weakly measurable function on $X$. Let $S^*$ be a weakly compact set in $X^*$; then $T(S^*)$ is a closed convex subset of $W$. Since $T(S^*)$ is weakly compact, Theorem 11 implies that the closed convex hull of a norm compact subset of $X$ is weakly compact.

**Geometric Interpretation:** This proof is closely related to the proof of Theorem 11. This type of argument is applicable to the Bochner integral on a Banach space $X$. If $W$ is separable and the identity function on $W$. Evidently $g$ is separably valued and $x^*g$ is continuous on $W$ equipped with the weak topology for all $x^* \in X^*$. From the Pettis Measurability Theorem 1.2, it follows that $g$ is $\mu$-measurable for every regular measure $\mu$ defined on the weakly Borel sets of $W$.

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since $x^* g \in C(W)$ for every $x^* \in X^*$. Hence $T$ is continuous for the weak*-topology of $C(W)^*$ and weak topology of $X$; accordingly $T$ is a weakly compact operator. Thus if $S^*$ is the closed unit ball of $C(W)^*$, then $T(S^*)$ is a weakly compact and convex subset of $X$. Moreover the point mass measures on $W$ are mapped onto $W$ by $T$. Hence $T \subseteq T(S^*)$ and the closed convex hull of $W$ is a subset of the weakly compact set $T(S^*)$. This completes the proof.
Krein-Smulyan type result

Wish…

Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\overline{\text{co}(H)}_{\tau_p(B)}$ is $\tau_p(B)$-compact.

Proof.- Fix $\mu$ a Radon probability on $(H, \tau_p(B))$, find a barycenter for $\mu$?

$$\text{find } x_\mu \in X \text{ with } x^*(x_\mu) = \int_H x^*|_H d\mu, \text{ for every } x^* \in X^?$$
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 ✓ if $B$ is convex and 1-norming then Hahn-Banach implies $\overline{B}^{w^*} = B_{X^*}$.

 ✓ $B|_H := \{x^*|_H : x^* \in B\} \subset C(H, \tau_p(B))$ and $B_{X^*}|_H = \overline{B|_H}^{\tau_p(H)}$;
Krein-Smulyan type result

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Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\overline{\text{co}(H)}_{\tau_p(B)}$ is $\tau_p(B)$-compact.

Proof.- Fix $\mu$ a Radon probability on $(H, \tau_p(B))$, find a barycenter for $\mu$?

$$\text{find } x_\mu \in X \text{ with } x^*(x_\mu) = \int_H x^*|_H d\mu, \text{ for every } x^* \in X^*?$$

✓ **Difficulty:** $x^*|_H$ is measurable only for $x^* \in B$;
✓ since $\tau_p(B) = \tau_p(\text{co}(B))$, we can assume $B$ convex;
✓ if $B$ is convex and 1-norming then Hahn-Banach implies $\overline{B}_{w^*} = B_{X^*}$.
✓ $B|_H := \{x^*|_H : x^* \in B\} \subset C(H, \tau_p(B))$ and $B_{X^*}|_H = \overline{B|_H}_{\tau_p(H)}$;
✓ if $B|_H$ does not have independent sequences (Rosenthal) then $\overline{B|_H}_{\tau_p(H)}$ is made up of $\mu$-measurable functions for each $\mu;$
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$$\text{find } x_\mu \in X \text{ with } x^*(x_\mu) = \int_H x^*|_H d\mu, \text{ for every } x^* \in X^*?$$

$(b^*_n)_n$ in $B$ is independent on $H$ if there are $s < t$ such that

$$\left( \bigcap_{n \in P} \{w \in H : b^*_n(w) < s\} \right) \cap \left( \bigcap_{n \in Q} \{w \in H : b^*_n(w) > t\} \right)$$

for every disjoint finite sets $P, Q \subset \mathbb{N}$.

✓ if $B|_H$ does not have independent sequences (Rosenthal) then $\overline{B|_H}^{\tau_p(H)}$ is made up of $\mu$-measurable functions for each $\mu$;
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✓ $B|_H := \{x^*|_H : x^* \in B\} \subset C(H, \tau_p(B))$ and $B_{X^*}|_H = \overline{B|_H}_{\tau_p(H)}$;
✓ if $B|_H$ does not have independent sequences (Rosenthal) then $\overline{B|_H}_{\tau_p(H)}$ is made up of $\mu$-measurable functions for each $\mu$;
✓ indeed, $B|_H$ as above has Bourgain property with respect to $\mu$. 
Bourgain property...a bit of history

Definition

We say that a family $\mathcal{F} \subset \mathbb{R}^\Omega$ has **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \ldots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathcal{F}$

$$\inf_{1 \leq i \leq n} \cdot |\text{diam}(f(B_i))| < \varepsilon.$$
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\]

The property of Bourgain

- The notion wasn’t published by Bourgain.
Bourgain property & compactness

Bourgain property & Birkhoff integrability

Aumann&Debreu&Pettis integrals multifunctions

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- The notion wasn’t published by Bourgain.
- It appears in a paper by [RS85] and refers to handwritten notes by Bourgain.
Remarkable facts about Bourgain property

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Properties

- If $\mathcal{F} = \{f\}$, TFAE:
  1. (Bourgain property) For every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there is $B \in \Sigma$, $B \subset A$ with $\mu(B) > 0$ and $|f| \cdot \text{diam} f(B) < \varepsilon$.
  2. $f$ is measurable.
Remarkable facts about Bourgain property

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- If $\mathcal{F}$ has Bourgain property, then $\mathcal{F}$ is made up of measurable functions.
Remarkable facts about Bourgain property

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We say that a family $\mathcal{F} \subset \mathbb{R}^{\Omega}$ has **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \ldots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathcal{F}$

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**Properties**

- If $\mathcal{F} = \{f\}$, TFAE:
  
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- $\mathcal{F}$ has Bourgain property $\Rightarrow \overline{\mathcal{F}}_{\tau_p(\Omega)}$ has too.

- $\mathcal{F}$ has Bourgain property and $f \in \overline{\mathcal{F}}_{\tau_p(\Omega)}$, then there is a sequence $(f_n)$ in $\mathcal{F}$ that converges to $f$, $\mu$-almost everywhere.
... back to Krein-Smulyan type result

Wish...

Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup \{ x^*(x) : x^* \in B \}$). For every norm bounded $\tau_p(B)$-relatively compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\text{co}(H)_{\tau_p(B)}$ is $\tau_p(B)$-compact.

✓ if $B|_H$ does not have independent sequences (Rosenthal), then $B|_H$ has Bourgain property with respect to $\mu$. 
Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-relatively compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\overline{\text{co}(H)}^{\tau_p(B)}$ is $\tau_p(B)$-compact.

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- $B_{X^*}\vert_H = \overline{B\vert_H}_{\tau_p(H)}$ has Bourgain property;
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- then for each $A \subset B_{X^*}$, $T_\mu(\overline{A^w}) \subset \overline{T_\mu(A)}$;
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✓ if $B|_H$ has an independent sequence on $H$ $\Rightarrow$ $\beta N \subset (B_{X^*}, w^*)$. 
Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-relatively compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\text{co}(H)_{\tau_p(B)}$ is $\tau_p(B)$-compact, assuming $\beta N \not\subset (B_{X^*}, w^*)$.

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Theorem: Manjabacas, Vera and B.C., 1997

Take $X$ Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup \{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-relatively compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\overline{\text{co}(H)_{\tau_p(B)}}$ is $\tau_p(B)$-compact, assuming $\ell^1(c) \not\subset X$.

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... back to Krein-Smulian type result
What we know about the boundary problem for $X$

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Take $X$ Banach space and $B \subseteq B_{X^*}$ norming (i.e. $\|x\| = \sup \{ x^*(x) : x^* \in B \}$). For every norm bounded $\tau_p(B)$-relatively compact subset $H$ of $X$ its $\tau_p(B)$-closed convex hull $\text{co}(H)_{\tau_p(B)}$ is $\tau_p(B)$-compact, assuming $\ell^1(c) \not\subset X$.

**Corollary: Manjabacas, Vera and B.C., 1997**

Let $X$ be a Banach space such that $\ell^1(c) \not\subset X$ and $B$ any boundary for $B_{X^*}$. If $H \subseteq X$ is norm bounded and $\tau_p(B)$-compact then $H$ is weakly compact.
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**Is the Theorem cheap?**

A. S. Granero 2006

Take $X$ Banach space. TFAE:

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What we know about the boundary problem for $X$

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**Is the Corollary cheap?**
Given $H \subset X$, $\tau_p(B)$ compact and $\mu$ Radon probability we have studied (Pettis) integrability of $id : H \hookrightarrow X$ using Bourgain property of $Z_{id} = \{x^* \circ id : x^* \in B_{X^*}\} \subset \mathbb{R}^H$. 

In general if $(\Omega, \Sigma, \mu)$ is a complete probability space and $f : \Omega \rightarrow X$ is bounded and such that $Z_f = \{x^* \circ f : x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega$ has Bourgain property what can you say about $f$? Using techniques of Pettis integration the known answer is:

...but in this case the outcome is in fact better.
1. Given $H \subset X$, $\tau_p(B)$ compact and $\mu$ Radon probability we have studied (Pettis) integrability of $id : H \hookrightarrow X$ using Bourgain property of $Z_{id} = \{ x^* \circ id : x^* \in B_{X^*} \} \subset \mathbb{R}^H$.

2. In general if $(\Omega, \Sigma, \mu)$ is a complete probability space and $f : \Omega \rightarrow X$ is bounded and such that $Z_f = \{ x^* \circ f : x^* \in B_{X^*} \} \subset \mathbb{R}^\Omega$, has Bourgain property what can you say about $f$?
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has Bourgain property what can you say about $f$?

Using techniques of Pettis integration the known answer is: $f$ is Pettis integrable... but in this case the outcome is in fact better.
Birkhoff definition

Let $f : \Omega \longrightarrow X$ be a function. If $\Gamma$ is a partition of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$, the function $f$ is called **summable** with respect to $\Gamma$ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f, \Gamma) = \left\{ \sum_n f(t_n) \mu(A_n) : t_n \in A_n \right\}$$

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The function $f$ is said to be **Birkhoff integrable** if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ for which $f$ is summable and

$$\| - \text{diam}(J(f, \Gamma)) < \varepsilon.$$
Let $f : \Omega \rightarrow X$ be a function. If $\Gamma$ is a partition of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$, the function $f$ is called **summable** with respect to $\Gamma$ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f, \Gamma) = \left\{ \sum_{n} f(t_n)\mu(A_n) : t_n \in A_n \right\}$$

is made up of unconditionally convergent series.

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In this case, the **Birkhoff integral** $(B)\int_{\Omega} f \, d\mu$ of $f$ is the only point in the intersection

$$\bigcap\{\overline{\text{co}}(J(f, \Gamma)) : f \text{ is summable with respect to } \Gamma\}. $$
Birkhoff integrability: properties

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5. Birkhoff integrability has been historically ignored.

Our basic result

We characterize Birkhoff integrability via the property of Bourgain.
Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \rightarrow X$ be a bounded function. TFAE:

(i) $f$ is Birkhoff integrable;
(ii) $Z_f = \{\langle x^*, f \rangle : x^* \in B_{X^*} \}$ has Bourgain property.

Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \rightarrow X$ be a function. TFAE:

(i) $f$ is Birkhoff integrable;
(ii) $Z_f$ is uniformly integrable, $Z_f$ has Bourgain property.
Applications to URL integrable functions

Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \rightarrow X$ be a function. TFAE:

(i) $f$ is Birkhoff integrable;

(ii) there is $x \in X$ satisfying: for every $\varepsilon > 0$ there is a countable partition $\Gamma$ of $\Omega$ in $\Sigma$ for which $f$ is summable and

$$\|S(f, \Gamma, T) - x\| < \varepsilon \text{ for every choice } T \text{ in } \Gamma;$$

(iii) there is $y \in X$ satisfying: for every $\varepsilon > 0$ there is a countable partition $\Gamma$ of $\Omega$ in $\Sigma$ such that $f$ is summable with respect to each countable partition $\Gamma'$ finer than $\Gamma$ and

$$\|S(f, \Gamma', T') - y\| < \varepsilon \text{ for every choice } T' \text{ in } \Gamma'.$$

In this case, $x = y = \int_{\Omega} f \, d\mu$. 
Zbl 0974.28007

Kadets, V.M.; Tseytlin, L.M.

On “integration” of non-integrable vector-valued functions.

Let $\mu$ be the Lebesgue measure on $[0,1]$ and $X$ be a Banach space. A function $f : [0,1] \to X$ is called absolutely Riemann-Lebesgue integrable over a measurable set $A \subset [0,1]$ if there is $x \in X$ such that for every $\varepsilon > 0$ there exists a measurable partition $\langle \Delta_i \rangle_{i=1}^\infty$ of $A$ such that for every finer measurable partition $\langle \Gamma_j \rangle_{j=1}^\infty$ of $A$ and arbitrary points $s_j \in \Gamma_j$ one has $\| \sum_j f(s_j) \mu(\Gamma_j) - x \| < \varepsilon$ and $\sum_j f(s_j) \mu(\Gamma_j)$ is absolutely convergent ($\langle \Gamma_j \rangle_{j=1}^\infty$ is finer than $\langle \Delta_i \rangle_{i=1}^\infty$ if each $\Delta_i$ is a union of some $\Gamma_j$’s). In case of unconditional convergence one gets a definition of unconditionally Riemann-Lebesgue integrable function.

There are no results placing ARL and URL integrals among other known types of integrals such as Birkhoff’s integral or generalized McShane’s integral which have similar definitions (and it is relatively easy to see that URL integrable functions are also Birkhoff integrable).

The rest of the paper is devoted to the study.

Kazimierz Musiał (Wrocław)
Applications to dual spaces with WRNP

Definition

1. $X^*$ has the weak Radon-Nikodým property;

2. for every complete probability space $(\Omega, \Sigma, \mu)$ and for every $\mu$-continuous countably additive vector measure $\nu : \Sigma \rightarrow X^*$ of $\sigma$-finite variation there is a Pettis integrable function $f : \Omega \rightarrow X^*$ such that

$$\nu(E) = \int_E f \ d\mu$$

for every $E \in \Sigma$. 
Applications to dual spaces with WRNP

Theorem: Musiał, Ryll-Nardzewski, Janicka and Bourgain

Let $X$ be a Banach space. TFAE:

1. $X^*$ has the weak Radon-Nikodým property;
2. $X$ does not contain a copy of $\ell^1$;
3. for every complete probability space $(\Omega, \Sigma, \mu)$ and for every $\mu$-continuous countably additive vector measure $\nu : \Sigma \rightarrow X^*$ of $\sigma$-finite variation there is a Pettis integrable function $f : \Omega \rightarrow X^*$ such that

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Rodriguez-B.C. 2005
The integral for a multifunction

\[ F : \Omega \rightarrow \text{cwk}(X) \]

There are several standard ways of dealing with integration for \( F \):

1. Take a reasonable embedding \( j \) from \( \text{cwk}(X) \) into a Banach space \( Y = \ell_{\infty}(B^X) \) and deal with the integrability of \( j \circ F \);
2. Take all integrable selectors \( f \) of \( F \) and consider \( \int F \, d\mu = \{ \int f \, d\mu : f \text{ integra. sel. } F \} \);
3. Debreu, [Deb67], used the embedding technique dealing with Bochner integrability;
4. Aumann, [Aum65], used the selector technique;
5. We used the embedding technique with Birkhoff integrability: Rodriguez-B.C., 2004.
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From the notion of Pettis integrability for such an $F$ studied in the literature one readily infers that if we embed $j : \text{cwk}(X) \to \ell_\infty(B_X^*)$ then $j \circ F$ is integrable with respect to a norming subset of $B_{\ell_\infty}(B_{X^*})^*$. 

A natural question arises: When is $j \circ F$ Pettis integrable?

Pettis integrability of any $\text{cwk}(X)$-valued function $F$ is equivalent to the Pettis integrability of $j \circ F$ if and only if $X$ has the Schur property. . . if and only if equivalent to the fact that $\text{cwk}(X)$ is separable when endowed with the Hausdorff distance.
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Theorem: Orihuela, Muñoz, B.C., to appear

Let $J : X \to 2^{B_{X^*}}$ be the duality mapping

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

TFAE:

(i) $X$ is Asplund, i.e., $X^*$ has RNP;
(ii) for some fixed $0 < \varepsilon < 1$, $J$ has an $\varepsilon$-selector $f$ that sends norm separable subsets of $X$ into norm separable subsets of $X^*$;
(iii) for some fixed $0 < \varepsilon < 1$, dual unit ball $B_{X^*}$ is norm $\varepsilon$-fragmented.

$\varepsilon$-selector: $d(f(x), J(x)) < \varepsilon$ for every $x \in X$
Two... three nice problems

1. The boundary problem in full generality (Godefroy).
2. Characterize Banach spaces $X$ for which $(B_{X^*}, w^*)$ is sequentially compact (Diestel).
3. Characterize Banach spaces $X$ for which $(B_{X^*}, w^*)$ is angelic.
References


