Universidad de Murcia | Departamento Matemáticas

The Bishop-Phelps-Bollobás Property and Asplund Operators

B. Cascales

http://webs.um.es/beca

4th Workshop on Optimization and Variational Analysis
In honor of Prof. Marco A. López on his 60th birthday
Elche, June 16th 2010

Stay focused

1. Introduction
   - Our Functional Analysis Group
   - A few samples... connection with optimization?
2. Bishop-Phelps-Bollobás theorem and Asplund operators
   - Bishop-Phelps theorem and Bollobás observation
   - The Bishop-Phelps property for operators
   - The Bishop-Phelps-Bollobás property for operators
   - Our main result: applications

Notation

- X, Y Banach spaces;
- 2^X subsets; $cwk(X)$ convex weakly compact sets;
- B_X **closed** unit ball; S_X unit sphere;
- $L(X,Y)$ bounded linear operators from X to Y;
- $C_0(L)$ space of continuous functions, vanishing at ∞.

  $$\|f\| = \sup_{s \in L} |f(s)|,$$

  where L is a **locally compact** Hausdorff space.
- (Ω, Σ, μ) complete probability space.

Optimization and Variational Analysis is a very important and active part of the Mathematical Sciences, with a strongly developed and still challenging mathematical theory and numerous applications to Economics, Engineering, Applied Sciences, and many other areas of human activity. This field of Mathematics provides an excellent example of how difficult and in fact senseless it is to split Mathematics into "pure" and "applied" areas.

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Boris S. Mordukhovich
Wayne State University
Detroit, MI. USA University.

What we do:

http://www.um.es/beca/papers.php

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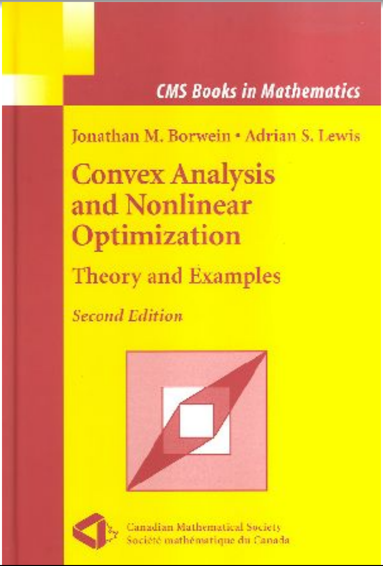
http://www.um.es/beca/papers.php

Research lines:

- Closed graph theorems;
- Topology; descriptive set theory; usco maps;
- Banach spaces; renorming;
- RNP and Asplundness in Banach spaces;
- vector and multi-valued integration;
- topological and measurable selectors;
- Krein-Smulyan, Krein-Milman, and Bishop-Phelps theorems;
- James theorem and application to financial mathematics;
- etc.

. . . connection with optimization?



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Optimization studies properties of minimizers and maximizers of functions. . . A (global) maximizer of a function $f : D \to \mathbb{R}$ is a point $\bar{x}$ in D at which f attains its maximum $\sup_D f$. In this case we refer to $\bar{x}$ as an optimal solution of the optimization problem $\sup_D f$.

Integration. . . some selected papers

B. Cascales and **J. Rodríguez,** *The Birkhoff integral and the property of Bourgain*, **Math. Ann. 331** (2005), no. 2, 259–279.

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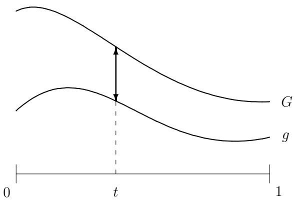
B. Cascales, **V. Kadets, and J. Rodríguez,** *Measurability and selections of multi-functions in Banach spaces*, **J. Convex Anal. 17** (2010), No. 1, 229-240.

$F : \Omega \longrightarrow cwk(X)$ –convex w-compact



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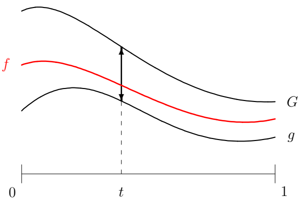

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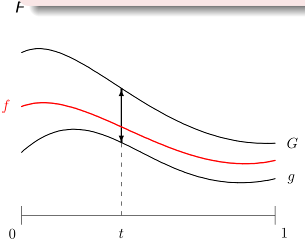
$$\int F\,d\mu = \left\{ \int f\,d\mu : f \text{ integra. sel.} F \right\}.$$

Integration... some selected papers

NEW THINGS: the theory was stuck in the separable case

1. Characterization of multi-functions admitting strong selectors;
2. scalarly measurable selectors for scalarly measurable multi-functions;
3. Pettis integration; the theory was stuck in the separable case;
4. existence of w^*-scalarly measurable selectors;
5. Gelfand integration; relationship with the previous notions.



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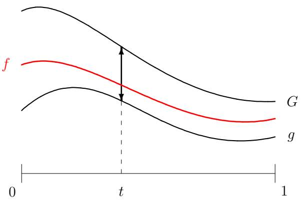

The techniques used

1. (Rådström embedding) if $F : \Omega \to cwk(X)$, we study the real-valued map

$$t \mapsto \delta^*(x^*, F(t)) := \sup\{\langle x^*, x\rangle : \ x \in F(t)\}$$

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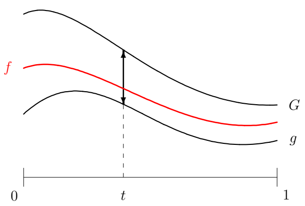

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$$t\mapsto \delta^*(x^*,F(t)):=\sup\{\langle x^*,x\rangle:\ x\in F(t)\}$$
2. (Exposed points: Lindenstrauss-Troyanski) If $H\in cwk(X)$, we can find a point $x_0\in H$ and some $x_0^*\in X^*$ such that $x_0^*(x_0)>x_0^*(x)$ for every $x\in H\setminus\{x_0\}$.

Boundaries...some selected papers

B. Cascales and G. Godefroy, Angelicity and the Boundary Problem, **Mathematika** 45 (1998), no. 1, 105-11.

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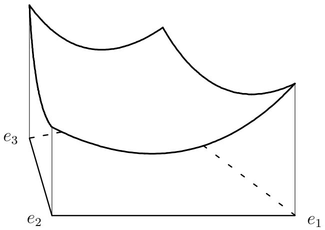

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for some $\bar{x}^* \in B$.

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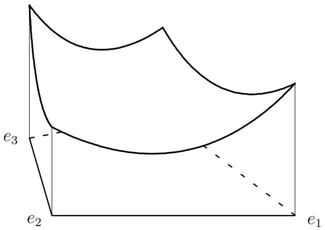

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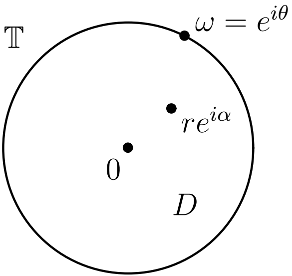

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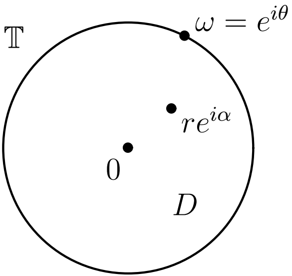

A representative result

Let $J : X \to 2^{B_{X^*}}$ be the duality mapping: defined at each $x \in X$ by

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

There is a *reasonable* selector $f : X \to X^*$ for J iff X is Asplund (in this case $\overline{f(X)}^{\|\cdot\|} = B_{X^*}$).

The Bishop-Phelps-Bollobás Property and Asplund Operators

with R. M. Aron and O. Kozhushkina
Kent State University

Bishop-Phelps theorem

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is subreflexive. The theorem mentioned in the title will be proved for *real* Banach spaces; the result for complex spaces follows from this by considering the spaces over the real field and using the known isometry between complex functionals and the real functionals defined by their real parts.

Bollobás observation

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' the unit spheres in a Banach space B and its dual space B', respectively.

THEOREM 1. *Suppose* $x \in S$, $f \in S'$ *and* $|f(x)-1| \leqslant \varepsilon^2/2$ $(0 < \varepsilon < \frac{1}{2})$. *Then there exist* $y \in S$ *and* $g \in S'$ *such that* $g(y) = 1$, $\|f-g\| \leqslant \varepsilon$ *and* $\|x-y\| < \varepsilon+\varepsilon^2$.

The Bishop-Phelps property for operators

Definition

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

Definition (Lindenstrauss)

(X, Y) has the Bishop-Phelps Property (BPP) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

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4. there are spaces X, Y and Z such that $(X, C([0,1]))$, (Y, ℓ^p) $(1 < p < \infty)$ and $(Z, L^1([0,1]))$ fail BPP, Schachermayer (1983), Gowers (1990) and Acosta (1999);

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6. $(L^1([0,1]), L^\infty([0,1]))$ has BPP, Finet-Payá (1998),

Bishop-Phelps-Bollobás Property for operators

Definition (Acosta, Aron, García and Maestre, 2008)

(X,Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon>0$ there are $\eta(\varepsilon)>0$ such that for all $T\in S_{L(X,Y)}$, if $x_0\in S_X$ is such that

$$\|T(x_0)\|>1-\eta(\varepsilon),$$

then there are $u_0\in S_X$, $S\in S_{L(X,Y)}$ with

$$\|S(u_0)\|=1$$

and

$$\|x_0-u_0\|<\varepsilon \text{ and } \|T-S\|<\varepsilon.$$

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Acosta, Aron, García and Maestre, 2008

1. Y has *certain* biorthogonal system (X, Y) has BPBP for any X;
2. (l_1, Y) Y finite dimensional, or uniformly convex or $Y = L_1(\mu)$ for a σ-finite measure or $Y = C(K)$;
3. the pair (ℓ^1, c_0) fails BPBP, c_0 with a strictly convex renorming;
4. (ℓ_∞^n, Y) has BPBP for Y uniformly convex.

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The techniques don't work for c_0

No Y infinite dimensional is known s.t. (c_0, Y) has BPBP

Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2010)

Let $T : X \to C_0(L)$ be an **Asplund operator** with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X,C_0(L))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \le 3\varepsilon.$$

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

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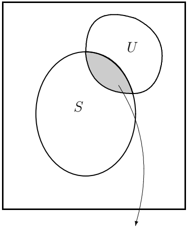

$$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon$$

Definition

B_{X^*} is fragmented if for every $\varepsilon > 0$ and every non empty subset $S \subset B_{X^*}$ there exists a w^*-open subset $U \subset X$ such that $U \cap S \neq \emptyset$ and

$$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon.$$

Stegall, 1975

An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "T_1"'] & & Y \\ & Z \arrow[ur, "T_2"'] & \end{tikzcd}$$

Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

An idea of the proof

Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2010)

Let $T : X \to C_0(L)$ be an **Asplund operator** with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X,C_0(L))}$ satisfying

$$\|S(u_0)\| = 1$$

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$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

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5. define $S : X \to C_0(L)$ by
   $$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s).$$

Operator Ideals

Approximating operator $S : X \to C_0(L)$,:

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Observe:

$$S = \text{RANK 1 OPERATOR} + T_f \circ T,$$

Consequence:
If $\mathscr{I} \subset \mathscr{A} = \mathscr{A}(X, C_0(L))$ is a sub-ideal of Asplund operators then

$$T \in \mathscr{I} \Rightarrow S \in \mathscr{I}.$$

The above applies to:

- Finite rank operators $\mathscr{F}$;
- Compact operators $\mathscr{K}$;
- Weakly compact operators $\mathscr{W}$;
- p-summing operators Π_p.

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Corollary

Let $T \in L(X, C_0(L))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Corollary

$(X, C_0(L))$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$**, for instance**).

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THANK YOU!