Brondsted-Rockafellar variational principle, Asplundness and operators attaining their norm

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Stay focused

- Kind of problems studied. Credit to co-authors.
- Bishop-Phelps property.
- Bishop-Phelps-Bollobás property.
- Our results: ingredients, proofs and applications.
What kind of problem are we going to talk about?

A BISHOP-PHELPS-BOLLOBÁS TYPE THEOREM FOR UNIFORM ALGEBRAS

B. CASCALES, A. J. GUIRAO AND V. KADETS

1. INTRODUCTION

Mathematical optimization is associated to maximizing or minimizing real functions. James’s compactness theorem [17] and Bishop-Phelps’s theorem [5] are two landmark results along this line in functional analysis. The former characterizes reflexive Banach spaces $X$ as those for which continuous linear functionals $x^* \in X^*$ attain their norm in the unit sphere $S_X$. The latter establishes that for any Banach space $X$ every continuous linear functional $x^* \in X^*$ can be approximated (in norm) by linear functionals that attain the norm in $S_X$. This paper is concerned with the study of a strengthening of Bishop-Phelps’s theorem that mixes ideas of Bollobás [6] –see Theorem 3.1 here– and Lindenstrauss [21] –who initiated the study of the Bishop-Phelps property for bounded operators between Banach spaces.
The problem for $x^* : X \to \mathbb{R}$ form and $T : X \to Y$ operator

$$\|x^*\| = \sup\{|x^*(x)| : \|x\| = 1\} \not= \max\{|x^*(x)| : \|x\| = 1\}$$

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\} \not= \max\{\|T(x)\| : \|x\| = 1\}$$
The problem for \( x^*: X \to \mathbb{R} \) form and \( T: X \to Y \) operator

\[
\|x^*\| = \sup \{|x^*(x)| : \|x\| = 1\} \quad \text{not always} \quad = \quad \max \{|x^*(x)| : \|x\| = 1\}
\]

\[
\|T\| = \sup \{\|T(x)\| : \|x\| = 1\} \quad \text{not always} \quad = \quad \max \{\|T(x)\| : \|x\| = 1\}
\]

A first glance to our result

Our paper is devoted to showing that Asplund operators with range in a uniform Banach algebra have the Bishop-Phelps-Bollobás property, i.e., they are approximated by norm attaining Asplund operators at the same time that a point where the approximated operator almost attains its norm is approximated by a point at which the approximating operator attains it. To prove this result we establish a Uryshon type lemma producing peak complex-valued functions in uniform algebras that are small outside a given open set and whose image is inside a symmetric rhombus with main diagonal \([0,1]\) and small height.
Credit to co-authors


Theorem (Bishop-Phelps, 1961)

If $X$ is a Banach, then $\overline{NAX^*} = X^*$.

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS
Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is subreflexive if those functionals which attain their supremum on the unit sphere $S$ of $E$ are norm-dense in $E^*$, i.e., if for each $f$ in $E^*$ and each $\epsilon > 0$ there exist $g$ in $E^*$ and $x$ in $S$ such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive $[1]^1$ as well as incomplete spaces which are subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-
The Bishop-Phelps property for operators

**Definition**

An operator \( T : X \to Y \) is **norm attaining** if there exists \( x_0 \in X, \|x_0\| = 1 \), such that \( \|T(x_0)\| = \|T\| \).

**Definition (Lindenstrauss)**

\((X, Y)\) has the Bishop-Phelps Property (BPp) if every operator \( T : X \to Y \) can be uniformly approximated by norm attaining operators.

1. \((X, K)\) has BPp for every \( X \), Bishop-Phelps (1961);
2. \( \{ T \in L(X; Y) : T^{**} \in NA(X^{**}; Y^{**}) \} = L(X; Y) \) for every pair of Banach spaces \( X \) and \( Y \), Lindenstrauss (1963);
3. \( X \) with RNP, then \((X, Y)\) has BPp for every \( Y \), Bourgain (1977);
4. there are spaces \( X, Y \) and \( Z \) such that \((X, C([0,1])), (Y, \ell^p) \ (1 < p < \infty) \) and \((Z, L^1([0,1]))\) fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. \((C(K), C(S))\) has BPp for all compact spaces \( K, S \), Johnson and Wolfe, (1979).
6. \((L^1([0,1]), L^\infty([0,1]))\) has BPp, Finet-Payá (1998),
Bishop and Phelps proved in [1] that every real or complex Banach space is subreflexive, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by $S$ and $S'$ the unit spheres in a Banach space $B$ and its dual space $B'$, respectively.

**THEOREM 1.** Suppose $x \in S$, $f \in S'$ and $|f(x)| < 1 - \varepsilon^2/2$ ($0 < \varepsilon < 1$). Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| < \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$.

**Proof.** Our first proof was rather complicated but we discovered later that a slight improvement of the proof in [1] gives this stronger result. This "proof" is presented here.

Naturally it is sufficient to verify the theorem for real Banach spaces and real functionals. It is actually proved, only not explicitly stated, in [1] that if $z \in S$, $f \in S'$ and $f(z) > 0$ then there exists $g \in S'$ which attains its supremum on the unit sphere at some point $x_0 \in S$, $\|f - g\| < \varepsilon$ and $\|x_0 - z\| < \varepsilon$. Naturally here $0 \leq f(x_0 - z) \leq 1 - f(z)$.

So putting $x = z$, $y = x_0$ we know that there are $y \in S$, $g \in S'$ such that $g(y) = 1$, $\|f - g\| < \varepsilon$ and $\|x_0 - z\| < \varepsilon(1 - \varepsilon^2/2) + \varepsilon^2$.

**Remark.** Theorem 1 is best possible in the following sense. For any $0 < \varepsilon < 1$ there exist a Banach space $B$, point $x \in S$, and functional $f \in S'$ such that $f(x) = 1 - \varepsilon^2/2$ but if $y \in S$, $g \in S'$ and $g(y) = 1$ then either $\|f - g\| < \varepsilon$ or $\|x - y\| < \varepsilon$.

**Proof.** Turn $\mathbb{R}^2$ into a real Banach space by taking the following unit ball:

$$
\{(a, b): -1 \leq a + (1 - \varepsilon)b \leq 1, -1 \leq b \leq 1\}
$$

Let $f(a, b) = (\varepsilon/2)a + (1 - \varepsilon^2/2)b$ and take $x = (0, 1)$. Then $\|f\| = 1$, $f(x) = 1 - \varepsilon^2/2$ and it is immediate that if $g \in S'$, $\|f - g\| < \varepsilon$ then $g$ must attain its supremum at the same point as $f$, at $(\varepsilon, 1)$, which is of distance $\varepsilon$ from $x$.

**Corollary.**... the way it is oftentimes presented

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that $|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}$, then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that $|y^*(u_0)| = 1$, $\|x_0 - u_0\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$. 
A variational principle implying BPB

Theorem 3.17 (Brøndsted-Rockafellar). Suppose that $f$ is a convex proper lower semicontinuous function on the Banach space $E$. Then given any point $x_0 \in \text{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x^* \in \partial_\epsilon f(x_0)$, there exist $x \in \text{dom}(f)$ and $x^* \in E^*$ such that

$$x^* \in \partial f(x), \quad \|x - x_0\| \leq \epsilon/\lambda \text{ and } \|x^* - x^*_0\| \leq \lambda.$$ 

In particular, the domain of $\partial f$ is dense in $\text{dom}(f)$.

Corollary...the constants are better

Given $1 > \epsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\epsilon^2}{2},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|x^* - y^*\| < \epsilon.$$
Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

\((X, Y)\) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any \(\varepsilon > 0\) there are \(\eta(\varepsilon) > 0\) such that for all \(T \in S_L(X, Y)\), if \(x_0 \in S_X\) is such that

\[\| T(x_0) \| > 1 - \eta(\varepsilon),\]

then there are \(u_0 \in S_X, S \in S_L(X, Y)\) with

\[\| S(u_0) \| = 1\]

and

\[\| x_0 - u_0 \| < \varepsilon \text{ and } \| T - S \| < \varepsilon.\]

1. \(Y\) has certain almost-biorthogonal system \((X, Y)\) has BPBP any \(X\);
2. \((\ell^1, Y)\) BPBP is characterized through a condition called AHSP: it holds for \(Y\) finite dimensional, uniformly convex, \(Y = L^1(\mu)\) for a \(\sigma\)-finite measure or \(Y = C(K)\);
3. there is pair \((\ell^1, X)\) failing BPBP, but having BPP;
4. \((\ell^\infty, Y)\) has BPBP \(Y\) uniformly convex no hope for \(c_0\):
   \[\eta(\varepsilon) = \eta(n, \varepsilon) \to 1 \text{ with } n \to \infty.\]
**Definition:** Acosta, Aron, García and Maestre, 2008

\((X, Y)\) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any \(\varepsilon > 0\) there are \(\eta(\varepsilon) > 0\) such that for all \(T \in S(L(X, Y))\), if \(x_0 \in S_X\) is such that

\[
\|T(x_0)\| > 1 - \eta(\varepsilon),
\]

then there are \(u_0 \in S_X, S \in S(L(X, Y))\) with

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1. \(Y\) has *certain* almost-biorthogonal system \((X, Y)\) has BPBP any \(X\);
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\[
\eta(\varepsilon) = \eta(n, \varepsilon) \to 1 \text{ with } n \to \infty.
\]

**PROBLEM?**

No \(Y\) infinite dimensional was known s.t. \((c_0, Y)\) has BPBP.
Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathcal{A} \subset C(K)$ be a uniform algebra and $T : X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathcal{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$
An **operator** \( T \in L(X, Y) \) is **Asplund**, if it factors through an Asplund space:

\[
\begin{align*}
X & \xrightarrow{T} Y \\
& \downarrow T_1 \quad \downarrow T_2 \\
& \quad \rightarrow Z
\end{align*}
\]

\( Z \) is Asplund; \( T_1 \in L(X, Z) \) and \( T_2 \in L(Z, Y) \).

\( T \) Asplund operator \( \iff \) \( T^*(B_{Y^*}) \) is fragmented by the norm of \( X^* \).
Asplund spaces: Namioka, Phelps and Stegall

Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ is an Asplund space, i.e., whenever $f$ is a convex continuous function defined on an open convex subset $U$ of $X$, the set of all points of $U$ where $f$ is Fréchet differentiable is a dense $G_δ$-subset of $U$.

(ii) every $w^*$-compact subset of $(X^*, w^*)$ is fragmented by the norm;

(iii) each separable subspace of $X$ has separable dual;

(iv) $X^*$ has the Radon-Nikodým property.
An idea of the proof for $\mathcal{A} = C(K)$

**Theorem (R. M. Aron, O. Kozhushkina and B. C. 2011)**

Let $T : X \to C(K)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X, C(K))}$ satisfying that

$$\|\tilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \tilde{T}\| < 2\varepsilon.$$

1. **Black box** provides a suitable open set $U \subset K$, $y^* \in S_{X^*}$ and $\rho < 2\varepsilon$ with

$$1 = |y^*(u_0)| = \|u_0\| \quad \text{and} \quad \|x_0 - u_0\| < \varepsilon \quad \text{and} \quad \|T^*(\delta_t) - y^*\| < \rho \quad \forall t \in U$$

2. **Uryshon’s lemma** that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying

$$f(t_0) = \|f\|_\infty = 1, \quad f(K) \subset [0, 1] \quad \text{and} \quad \text{supp}(f) \subset U.$$

3. $\tilde{T}$ is explicitly defined by

$$\tilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - f(t)) \cdot T(x)(t), \quad x \in X, \quad t \in K,$$

4. **The suitability** of $U$ is used to prove that $\|T - \tilde{T}\| < 2\varepsilon.$
An idea of the proof for $\mathcal{A} = A(\overline{D})$

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $T : X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X,A(\overline{D}))}$ satisfying that

$$\|\tilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \tilde{T}\| < 2\varepsilon.$$

1. Black box gives an open set $U \subset \overline{D}$ $y^* \in S_{X^*}$ & $\rho < 2\varepsilon$ with

$$1 = |y^*(u_0)| = \|u_0\| \quad \text{and} \quad \|x_0 - u_0\| < \varepsilon \quad \text{&} \quad \|T^*(\delta_t) - y^*\| < \rho \quad \forall t \in U.$$

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Asplund operators. Bishop-Phelps-Bollobás
Our result

New ingredient to face these problems: fragmentability

Our main result

About the proofs

Applications

An idea of the proof for $\mathcal{A} = A(\overline{D})$

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $T : X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,A(\overline{D}))}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Black box** gives an open set $U \cap T \neq \emptyset$, $y^* \in S_{X^*}$ & $\rho < 2\varepsilon$ with $1 = |y^*(u_0)| = \|u_0\|$ and $\|x_0 - u_0\| < \varepsilon$ & $\|T^*(\delta_t) - y^*\| < \rho \ \forall t \in U$.

2. Uryshon’s lemma that applied to an arbitrary $t_0 \in U \cap T$ produces a function $f \in A(\overline{D})$ satisfying $f(t_0) = \|f\|_{\infty} = 1$, $f(\overline{D}) \subset R_\varepsilon$ and $f$ small in $\overline{D} \setminus U$.

3. $\widetilde{T}$ is explicitly defined by $\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - \varepsilon')(1 - f(t)) \cdot T(x)(t)$

4. The suitability of $U$ is used to prove that $\|T - \widetilde{T}\| < 2\varepsilon$. 

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Asplund operators. Bishop-Phelps-Bollobás
An idea of the proof for $\mathcal{A} = A(D)$

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1. **Black box** gives an open set $U \cap T \neq \emptyset$, $y^* \in S_{X^*}$ & $\rho < 2\varepsilon$ with

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   f(t_0) = \| f \|_{\infty} = 1, \quad f(D) \subset R_\varepsilon \quad \text{and} \quad f \text{ small in } D \setminus U.
   \]

   explicitly defined by

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   \tilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - \varepsilon')(1 - f(t)) \cdot T(x)(t)
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   suitability of $U$ is used to prove that $\| T - \tilde{T} \| < 2\varepsilon$. 
Our key Uryshon type lemma for $A(D)$

**Lemma 2.8.** Let $\mathcal{A} \subset C(K)$ be a unital uniform algebra and $\Gamma_0$ its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathcal{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_{\infty} = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset R_{\varepsilon}$. In particular,

$$|f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K. \quad (2.8)$$
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for all $t \in K$. \hspace{1cm} (2.8)
Lemma 2.8. Let \( \mathcal{A} \subset C(K) \) be a unital uniform algebra and \( \Gamma_0 \) its Choquet boundary. Then, for every open set \( U \subset K \) with \( U \cap \Gamma_0 \neq \emptyset \) and \( 0 < \varepsilon < 1 \), there exist \( f \in \mathcal{A} \) and \( t_0 \in U \cap \Gamma_0 \) such that \( f(t_0) = \|f\|_\infty = 1 \), \( |f(t)| < \varepsilon \) for every \( t \in K \setminus U \) and \( f(K) \subset R_\varepsilon \). In particular,

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**Lemma 2.8.** Let $\mathcal{A} \subset C(K)$ be a unital uniform algebra and $\Gamma_0$ its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathcal{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_{\infty} = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset R_\varepsilon$. In particular,

$$|f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1,$$

for all $t \in K$. (2.8)

Our Uryshon type lemma is suited for calculations with a computer.
Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X, \mathfrak{A})}$ satisfying that

$$\|\tilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon$$

and

$$\|T - \tilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X, C_0(L))$ weakly compact with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ weakly compact with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Corollary

$(X, C_0(L))$ has the BPBP for any Asplund space $X$ and any locally compact Hausdorff topological space $L$ (e.g., $X = c_0(\Gamma)$, for instance).

Corollary

$(X, C_0(L))$ has the BPBP for any $X$ and any scattered locally compact Hausdorff topological space $L$. 
Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathcal{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that

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Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X, \mathcal{A})}$ satisfying that

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Corollary

Let $T \in L(X, A(D))$ weakly compact with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$  

Then there are $u_0 \in S_X$ and $S \in L(X, A(D))$ weakly compact with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Remark

The theorem applies in particular to the ideals of finite rank operators $\mathcal{F}$, compact operators $\mathcal{K}$, $p$-summing operators $\Pi_p$ and of course to the weakly compact operators $\mathcal{W}$ themselves. To the best of our knowledge even in the case $\mathcal{W}(X, \mathcal{A})$ the Bishop-Phelps property that follows is a brand new result.
Our main result

New ingredient to face these problems: fragmentability

About the proofs

Applications

GRACIAS!

B. Cascales