Bishop-Phelps-Bollobás theorem and Asplund operators

B. Cascales

Websites: http://webs.um.es/beca (Personal)
          http://www.um.es/beca (Group)

Prague Topological Symposium, August 2011
Introduction: Bishop Phelps theorem
- Credit to co-authors and a few papers by others
- Bishop-Phelps theorem
- The Bishop-Phelps property for operators

Bishop-Phelps-Bollobás theorem and Asplund operators
- Bollobás observation and BPBp for operators
- Our main result: applications
- Remarks and further development

Final comments: other applications of fragmentability
- Fragmentability, topology and boundaries
- Fragmentability and measure theory
Notation

- $X, Y, E, B$ Banach spaces;
- $B_X$ closed unit ball; $S_X$ unit sphere;
- $L(X, Y)$ bounded linear operators from $X$ to $Y$;
- $C_0(L)$ space of continuous functions, vanishing at $\infty$.

$$\|f\| = \sup_{s \in L} |f(s)|,$$

where $L$ is a **locally compact** Hausdorff space.

- $(\Omega, \Sigma, \mu)$ complete probability space.
Introduction: Bishop Phelps theorem

Credit to co-authors and a few papers by others

Bishop-Phelps theorem

The Bishop-Phelps property for operators

... Bishop-Phelps property
Credit to co-authors and previous work


Bishop-Phelps theorem

Theorem (Bishop-Phelps, 1961)

If $X$ is a Banach, then $\overline{NAX^*} = X^*$. 
The Bishop-Phelps theorem

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---

**A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE**

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere $S$ of $E$ are norm-dense in $E^*$, i.e., if for each $f$ in $E^*$ and each $\epsilon > 0$ there exist $g$ in $E^*$ and $x$ in $S$ such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1] as well as incomplete spaces which are subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-
WEAKLY COMPACT SETS

BY

ROBERT C. JAMES(1)

It has been conjectured that a closed convex subset C of a Banach space B is weakly compact if and only if each continuous linear functional on B attains a maximum on C [5]. This reduces easily to the case in which C is bounded, and will be answered in the affirmative [Theorem 4] after some preliminary results are established. Following suggestions by Namioka and Peck, the result is then generalized, first to weakly closed subsets
A QUANTITATIVE VERSION OF JAMES’ COMPACTNESS THEOREM

BERNARDO CASCALES, ONDŘEJ F.K. KALENDA AND JIŘÍ SPURNÝ

Abstract. We introduce two measures of weak non-compactness $J_{\alpha E}$ and $J_{\alpha}$ that quantify, via distances, the idea of boundary behind James’ compactness theorem. These measures tell us, for a bounded subset $C$ of a Banach space $E$ and for given $x^* \in E^*$, how far from $E$ or $C$ one needs to go to find $x^{**} \in \overline{C}^{w^*} \subset E^{**}$ with $x^{**}(x^*) = \sup x^*(C)$. A quantitative version of James’ theorem concerning relatively weakly compact sets is obtained, and in particular, it yields that

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2010 Mathematics Subject Classification. 46B50.

Key words and phrases. Banach space, measure of weak non-compactness, James’ compactness theorem.

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A possible generalization of this theorem remains open: Suppose $E$ and $F$ are Banach spaces, and let $\mathfrak{L}(E, F)$ be the Banach space of all continuous linear transformations from $E$ into $F$, with the usual norm. For which $E$ and $F$ are those $T$ such that $\|T\| = \|Tx\|$ (for some $x$ in $E$, $\|x\| = 1$) dense in $\mathfrak{L}(E, F)$? This is true for arbitrary $E$ if $F$ is an ideal in $m(A)$ (the space of bounded functions on the set $A$, with the supremum norm).
Bishop-Phelps property

Question (Bishop-Phelps)

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Theorem (Lindenstrauss, 1963)

Let $Y$ be a strictly convex Banach space, isomorphic to $c_0$, and let $X = Y \oplus c_0$ where $c_0$ has the usual norm and consider the supremum norm on the direct sum. Then $NAL(X; X)$ is NOT dense in $L(X; X)$. 
Introduction: Bishop Phelps theorem

The Bishop-Phelps property for operators

Definition

An operator \( T : X \to Y \) is **norm attaining** if there exists \( x_0 \in X, \| x_0 \| = 1 \), such that \( \| T(x_0) \| = \| T \| \).
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$(X, Y)$ has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.
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3. $X$ with RNP, then $(X, Y)$ has BPP for every $Y$, Bourgain (1977);
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6. $(L^1([0,1]), L^\infty([0,1]))$ has BPp, Finet-Payá (1998),
... Bishop-Phelps-Bollobás property
AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in [1] that every real or complex Banach space is subreflexive, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by $S$ and $S'$ the unit spheres in a Banach space $B$ and its dual space $B'$, respectively.

**Theorem 1.** Suppose $x \in S$, $f \in S'$ and $|f(x) - 1| \leq \varepsilon^2/2$ ($0 < \varepsilon < \frac{1}{2}$). Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| \leq \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$. 
A different way of writing BPB

**Theorem 1.** Suppose $x \in S$, $f \in S'$ and $|f(x) - 1| \leq \frac{\varepsilon^2}{2}$ ($0 < \varepsilon < \frac{1}{2}$). Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| \leq \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$.

**Corollary.** The way is oftentimes presented

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$
Theorem 3.17 (Brøndsted-Rockafellar). Suppose that $f$ is a convex proper lower semicontinuous function on the Banach space $E$. Then given any point $x_0 \in \text{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial f(x_0)$, there exist $x \in \text{dom}(f)$ and $x^* \in E^*$ such that

$$x^* \in \partial f(x), \quad \|x - x_0\| \leq \epsilon/\lambda \text{ and } \|x^* - x_0^*\| \leq \lambda.$$ 

In particular, the domain of $\partial f$ is dense in $\text{dom}(f)$. 

Convex Functions, Monotone Operators and Differentiability

Robert R. Phelps

Lecture Notes in Mathematics

1364

2nd edition

Springer
A variational principle implying BPB

Theorem 3.17 (Brøndsted-Rockafellar). Suppose that \( f \) is a convex proper lower semicontinuous function on the Banach space \( E \). Then given any point \( x_0 \in \text{dom}(f) \), \( \epsilon > 0 \), \( \lambda > 0 \) and any \( x^*_0 \in \partial f(x_0) \), there exist \( x \in \text{dom}(f) \) and \( x^* \in E^* \) such that

\[
x^* \in \partial f(x), \quad \|x - x_0\| \leq \epsilon/\lambda \text{ and } \|x^* - x^*_0\| \leq \lambda.
\]

In particular, the domain of \( \partial f \) is dense in \( \text{dom}(f) \).

1. Take \( f : E \to [0, +\infty] \) 0 at \( C \) and \( +\infty \) at \( E \setminus C \);
2. \( \epsilon^2/2 \) instead of \( \epsilon \), \( \lambda = \epsilon/2 \);
3. replace \( x^* \in E^* \) in the corollary above by \( x^*/\|x^*\| \)

Corollary...the constants are better

Given \( 1 > \epsilon > 0 \), if \( x_0 \in S_X \) and \( x^* \in S_{X^*} \) are such that

\[
|x^*(x_0)| > 1 - \frac{\epsilon^2}{2},
\]

then there are \( u_0 \in S_X \) and \( y^* \in S_{X^*} \) such that

\[
|y^*(u_0)| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|x^* - y^*\| < \epsilon.
\]
Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

\((X, Y)\) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any \(\varepsilon > 0\) there are \(\eta(\varepsilon) > 0\) such that for all \(T \in S_{L(X,Y)}\), if \(x_0 \in S_X\) is such that

\[\|T(x_0)\| > 1 - \eta(\varepsilon),\]

then there are \(u_0 \in S_X, \, S \in S_{L(X,Y)}\) with

\[\|S(u_0)\| = 1\]

and

\[\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.\]
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1. $Y$ has *certain* almost-biorthogonal system $(X, Y)$ has BPBP any $X$;

2. $(\ell^1, Y)$ BPBP is characterized through a condition called AHSP: it holds for $Y$ finite dimensional, uniformly convex, $Y = L1(\mu)$ for a $\sigma$-finite measure or $Y = C(K)$;
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\[\eta(\varepsilon) = \eta(n, \varepsilon) \to 1 \text{ with } n \to \infty.\]
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3. there is pair \((\ell^1, X)\) failing BPBP, but having BPp;
4. \((\ell^\infty_n, Y)\) has BPBP \(Y\) uniformly convex no hope for \(c_0\):
   \[\eta(\varepsilon) = \eta(n, \varepsilon) \to 1\] with \(n \to \infty\).
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4. \((\ell^\infty_n, Y)\) has BPBP \(Y\) uniformly convex no hope for \(c_0\):

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\eta(\varepsilon) = \eta(n, \varepsilon) \to 1 \text{ with } n \to \infty.
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**PROBLEM?**

No \(Y\) infinite dimensional is known s.t. \((c_0, Y)\) has BPBP.

---

B. Cascales
Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2011)

Let \( T : X \to C_0(L) \) be an Asplund operator with \( \| T \| = 1 \).
Suppose that \( \frac{1}{2} > \epsilon > 0 \) and \( x_0 \in S_X \) are such that
\[
\| T(x_0) \| > 1 - \frac{\epsilon^2}{4}.
\]

Then there are \( u_0 \in S_X \) and an Asplund operator \( S \in S_{L(X,C_0(L))} \)
satisfying
\[
\| S(u_0) \| = 1, \| x_0 - u_0 \| < \epsilon \text{ and } \| T - S \| \leq 3\epsilon.
\]
An operator $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
&T_1\downarrow & \\
\downarrow & T_2 & \\
Z & \xleftarrow{\cdot} & \\
\end{array}
$$

$Z$ is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$. 

Stegall, 1975
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An operator $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:

\[ X \xrightarrow{T_1} Z \xrightarrow{T} Y \]

$Z$ is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

$T$ Asplund operator $\iff T^*(B_{Y^*})$ is fragmented by the norm of $X^*$. 
Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ is an Asplund space, i.e., whenever $f$ is a convex continuous function defined on an open convex subset $U$ of $X$, the set of all points of $U$ where $f$ is Fréchet differentiable is a dense $G_δ$-subset of $U$.

(ii) every $w^*$-compact subset of $(X^*, w^*)$ is fragmented by the norm;

(iii) each separable subspace of $X$ has separable dual;

(iv) $X^*$ has the Radon-Nikodým property.
Asplund spaces: Namioka, Phelps and Stegall

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(iv) $X^*$ has the Radon-Nikodým property.

\[ \| \cdot \| - \text{diam}(U \cap S) \leq \varepsilon \]

**Definition**

$B_{X^*}$ is fragmented if for every $\varepsilon > 0$ and every non empty subset $S \subset B_{X^*}$ there exists a $w^*$-open subset $U \subset X$ such that $U \cap S \neq \emptyset$ and

\[ \| \cdot \| - \text{diam}(U \cap S) \leq \varepsilon. \]
Let $T : X \rightarrow C_0(L)$ be an Asplund operator with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$ 

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X, C_0(L))}$ weakly compact with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$ 

(X, $C_0(L)$) has the BPBP for any Asplund space $X$ and any locally compact Hausdorff topological space $L$ ($X = c_0(\Gamma)$, for instance).

(X, $C_0(L)$) has the BPBP for any $X$ and any scattered locally compact Hausdorff topological space $L$. 

**Corollary**

Let $T \in L(X, C_0(L))$ weakly compact with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$ 

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ weakly compact with $\|S\| = 1$ satisfying

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An idea of the proof

**Theorem**

Let $T : X \to C_0(L)$ be an **Asplund operator** with $\| T \| = 1$. Suppose that \( \frac{1}{2} > \varepsilon > 0 \) and $x_0 \in S_X$ are such that

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Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X, C_0(L))}$ satisfying

\[
\| S(u_0) \| = 1
\]

and

\[
\| x_0 - u_0 \| < \varepsilon \text{ and } \| T - S \| \leq 3\varepsilon.
\]
Let $\phi : L \rightarrow X^*$ given by $\phi(s) = \delta_s \circ T$;

**Theorem**

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$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$  

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X, C_0(L))}$ satisfying

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Theorem

Let \( T : X \to C_0(L) \) be an Asplund operator with \( \| T \| = 1 \). Suppose that \( \frac{1}{2} > \varepsilon > 0 \) and \( x_0 \in S_X \) are such that
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$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$ Then there are $u_0 \in S_X$ and an Asplund operator $S \in S(L, C_0(L))$ satisfying

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Lemma

Let $T : X \to C_0(L)$ be an Asplund operator with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

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Then there exist:

(a) a $w^*$-open set $U \subset X^*$ with $U \cap \phi(L) \neq \emptyset$;

(b) $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$,

$$\|x_0 - u_0\| < \varepsilon, \|z^* - y^*\| < 3\varepsilon$$

for every $z^* \in U \cap \phi(L)$.

1. Let $\phi : L \to X^*$ given by $\phi(s) = \delta_s \circ T$;
2. take $s_0 \in L$ such that $|\phi(s_0)(x_0)| = |T(x_0)(s_0)| > 1 - \frac{\varepsilon^2}{4}$;
3. $U_1 = \{x^* \in X^* : |x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}\}$,
4. $\phi(s_0) \in U_1 \cap \phi(L)$;
5. $\phi(L) \subset B_{X^*}$ is fragmented;
6. $U_2 \subset X^*$ such that $(U_1 \cap \phi(L)) \cap U_2 \neq \emptyset$ and

$$\|\cdot\|\text{-diam}((U_1 \cap \phi(L)) \cap U_2) \leq \varepsilon;$$

7. Let $U := U_1 \cap U_2$;
8. Pick a point, $x_0^* \in U \cap \phi(L)$ normalize it $\frac{x_0^*}{\|x_0^*\|}$ and use...

BPB in the scalar case

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that $|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}$, then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon$$

and $\|x^* - y^*\| < \varepsilon$. 
An idea of the proof

1. Let $\phi : L \to X^*$ given by $\phi(s) = \delta_s \circ T$;
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   $$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s).$$

---

**Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2011)**

Let $T : X \to C_0(L)$ be an Asplund operator with $\|T\| = 1$. Suppose that
\[ \frac{1}{2} > \varepsilon > 0 \text{ and } x_0 \in S_X \text{ are such that} \]
\[ \|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}. \]

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X,C_0(L))}$ satisfying
\[ \|S(u_0)\| = 1 \]
and
\[ \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon. \]
Approximating operator $S : X \to C_0(L),:\$

$$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s)$$

Observe:

$$S = \text{RANK 1 OPERATOR} + T_f \circ T,$$
Approximating operator $S : X \to C_0(L),:

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Observe:

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Consequence:

If $\mathcal{I} \subset \mathcal{A} = \mathcal{A}(X, C_0(L))$ is a sub-ideal of Asplund operators then

$$T \in \mathcal{I} \Rightarrow S \in \mathcal{I}.$$

The above applies to:

- Finite rank operators $\mathcal{F}$;
- Compact operators $\mathcal{K}$;
- $p$-summing operators $\Pi_p$;
- Weakly compact operators $\mathcal{W}$. 
Corollary

Let \( T \in L(X, C_0(L)) \) weakly compact with \( \|T\| = 1 \), \( \frac{1}{2} > \varepsilon > 0 \), and \( x_0 \in S_X \) be such that

\[
\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.
\]

Then there are \( u_0 \in S_X \) and \( S \in L(X, C_0(L)) \) weakly compact with \( \|S\| = 1 \) satisfying

\[
\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.
\]

Corollary

\((X, C_0(L))\) has the BPBP for any Asplund space \( X \) and any locally compact Hausdorff topological space \( L \) (\( X = c_0(\Gamma) \), for instance).

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\((X, C_0(L))\) has the BPBP for any \( X \) and any scattered locally compact Hausdorff topological space \( L \).
The results are true for the complex case and the constants $\frac{\varepsilon^2}{4}$ can be improved to $\frac{\varepsilon^2}{2}$;
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The technicality that leads to our results is really better:

Lemma: Aron, Cascales and Kozhushkina, 2011

Let $T : X \rightarrow Y$ be an Asplund operator with $\|T\| = 1$, let $\frac{1}{2} > \varepsilon > 0$ and choose $x_0 \in S_X$ such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$ 

For any given 1-norming set $B \subset B_{Y^*}$ if we write $M := T^*(B)$ then there are:

(a) a $w^*$-open set $U \subset X^*$ with $U \cap M \neq \emptyset$ and

(b) points $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$ such that

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|z^* - y^*\| < 3\varepsilon \text{ for every } z^* \in U \cap M.$$
The previous lemma has been used already as it is to establish the BPBp for Asplund operators $T : X \to C(K, Y)$, for some $Y$’s (Acosta, Maestre and Garcia; to be published);
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Our expectation is to use the lemma for the disk algebra $A(\mathbb{T})$ (or other uniform algebras), the reason being, the construction

$$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s).$$

needs algebra struct. with a boundary with many peak points:
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needs algebra struct. with a boundary with many peak points:

$$z \to \left| \frac{z + 1}{2} \right|$$
$$z \to \left| \left( \frac{z + 1}{2} \right)^{50} \right|$$
$$z \to \left| \left( \frac{z + 1}{2} \right)^{1000} \right|$$
Very hot references


THANK YOU!
Final comments: other applications of fragmentability
### Fragmentability ⇒ topology and boundaries


Final comments: other applications of fragmentability

**Fragmentability **⇒ topology and boundaries


**Lindelöf Property**

If \((X^*, w)\) is Lindelöf, then \((X^*, w)^2\), is Lindelöf. (For \((X, w)\) the problem remains open 40 years later, Corson).
Final comments: other applications of fragmentability

Fragmentability $\Rightarrow$ topology and boundaries


**Lindelöf Property**

If $(X^*, w)$ is Lindelöf, then $(X^*, w)^2$, is Lindelöf. (For $(X, w)$ the problem remains open 40 years later, Corson).

**Boundaries and selectors**

Let $J : X \rightarrow 2^{B_{X^*}}$ be the duality mapping: defined at each $x \in X$ by

$$J(x) := \{ x^* \in B_{X^*} : x^*(x) = \|x\| \}.$$  

There is a *reasonable* selector $f : X \rightarrow X^*$ for $J$ iff $X$ is Asplund (in this case $f(X) \|\cdot\| = B_{X^*}$).


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(Debreu Nobel prize in 1983) to take a reasonable embedding $j$ from $cwk(X)$ into the Banach space $Y(=\ell_\infty(B_X^*))$ and then study the integrability of $j \circ F$;


1. (Debreu Nobel prize in 1983) to take a reasonable embedding \( j \) from \( cwk(X) \) into the Banach space \( Y(=\ell_{\infty}(B_{X^*})) \) and then study the integrability of \( j \circ F \);

2. (Aumann Nobel prize in 2005) to take all integrable selectors \( f \) of \( F \) and consider

\[
\int F \, d\mu = \left\{ \int f \, d\mu : f \text{ integra. sel. } F \right\}.
\]
Fragmentability and measure theory

\[ f : \Omega \to E \]

For every \( \varepsilon > 0 \) \( A \in \Sigma^+ \) there is \( B \in \Sigma_A^+ \) such that

\[ \| \cdot \| - \text{diam} f (B) < \varepsilon. \]

Is there a reasonable extension of the above for multi-functions?
For every $\varepsilon > 0$ $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that

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Is there a reasonable extension of the above for multi-functions?

**Definition**

$F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\text{diam}(D) < \varepsilon$ such that

$$F(t) \cap D \neq \emptyset$$

for every $t \in B$. 
**Property (P)**

$F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\text{diam}(D) < \varepsilon$ such that $F(t) \cap D \neq \emptyset$ for every $t \in B$.

1. Fix $n = 0$;
2. take $\varepsilon := (1/2)^n$;
3. apply (P) for $A = \Omega$, $\varepsilon$ and $F$;
4. a maximality argument produces a partition of $B'$s;
5. enumerate $B'$s as $\{B_n\}$ and choose any $x_n \in D_n$;
6. define $f_\varepsilon := \sum_n \chi_{B_n} x_n$;
7. $f_\varepsilon$ is $\mu$-measurable and $d(f_\varepsilon(t), F(t)) < \varepsilon$ $\mu$-a.e.;
8. define $F_\varepsilon(t) := F(t) \cap B(f_\varepsilon(t), \varepsilon)$;
9. IF $F_\varepsilon$ satisfies (P) GOTO 11;
10. STOP;
11. $n := n + 1$;
12. GOTO 2.

**Conclusion**

We produce a sequence $(f_n) : \Omega \to E$ of $\mu$-measurable functions such that $(f_n(t))$ is Cauchy $\mu$-a.e., hence it is convergent.
Corollary (Kuratowski-Ryll Nardzewski, 1965)

Let $F : \Omega \to 2^E$ be a multi-function with closed non empty values of $E$. If $E$ is separable and $F$ satisfies that

$$\{ t \in \Omega : F(t) \cap O \neq \emptyset \} \in \Sigma$$

for each open set $O \subset E$.\hspace{1cm} (E)

Then $F$ admits a $\mu$-measurable selector $f$. 
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\]

Then \( F \) admits a \( \mu \)-measurable selector \( f \).

Very little is known in the non separable case

Theorem (Kadets, Rodríguez and B. C. -2009)

For a multi-function \( F : \Omega \to \text{wk}(E) \) TFAE:

(i) \( F \) admits a strongly measurable selector.

(ii) There exist a set of measure zero \( \Omega_0 \in \Sigma \), a separable subspace \( Y \subset E \) and a multi-function \( G : \Omega \setminus \Omega_0 \to \text{wk}(Y) \) that is Effros measurable and such that \( G(t) \subset F(t) \) for every \( t \in \Omega \setminus \Omega_0 \);

(iii) \( F \) satisfies property (P).
Consequences

NEW THINGS: the theory was stuck in the separable case

1. Characterization of multi-functions admitting strong selectors;
2. Scalarly measurable selectors for scalarly measurable multi-functions;
3. Pettis integration; the theory was stuck in the separable case;
4. Existence of $w^*$-scalarly measurable selectors;
5. Gelfand integration; relationship with the previous notions.
6. RNP for multi-functions;
7. Set selectors.
GRACIAS!