The Bourgain property and Birkhoff integrability

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Contemporary Ramifications of Banach Space Theory
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Our basic result

We characterize Birkhoff integrability via the property of Bourgain.
Some consequences

- Extremal tests can be proved.
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Boundary problem

If $X$ is a Banach space not containing $\ell^1(\mathbb{R})$ and $B \subset B_{X^*}$ a boundary (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \|x\|$) then the norm bounded $\sigma(X, B)$- relatively compact subsets of $X$ are relatively weakly compact.
Some consequences

- Extremal tests can be proved.
- New characterization of the WRNP.
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Banach spaces without copies of $\ell^1$

For dual Banach spaces WRNP is characterized via Birkhoff integrable Radon-Nikodým derivatives instead of Pettis integrable ones.
Some consequences

- Extremal tests can be proved.
- New characterization of the WRNP.
- Riemann-Lebesgue unconditional integrability.
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Birkhoff integrability is rediscovered

Kadets et al. in 2000-2002, [KSS+02, KT00] introduced and studied a notion of integrability that is equivalent to Birkhoff integrability introduced in 1935.
Some consequences

- Extremal tests can be proved.
- New characterization of the WRNP.
- Riemann-Lebesgue unconditional integrability.
- Integrals for multifunctions.
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Aumann and Debreu

Certain integrals for multifuctions used for models in Mathematics for Economy can be computed as limits in the Hausdorff distance of Riemann (Minkowski) sums of sets.
The property of Bourgain

The notion wasn’t published by Bourgain.
for each $x$ in $E$. Since the set $\{\langle f, x \rangle : \|x\| \leq 1\}$ contains no copy of the $l_1$-basis in $L_\infty(\Sigma, \mu)$ and the conditional expectation operator $\xi$ is a contraction from $L_\infty(\Sigma, \mu)$ into $L_\infty(\Gamma, \mu)$, we may conclude that $T(B_E)$ contains no copy of the $l_1$-basis in $L_\infty(\Gamma, \mu)$. Consequently $T(B_E)$ is weakly precompact in $L_\infty(\Gamma, \mu)$ and there is a Pettis integrable kernel $g : (\Omega, \Gamma, \mu) \to E^*$ for the operator

$$T^* : L_1(\Gamma, \mu) \to E^*.$$ 

Then $\langle g, x \rangle = Tx = \xi(\langle f, x \rangle | \Gamma)$ a.e. for every $x$ in $E$. Therefore

$$\int_B \langle g, x \rangle \, d\mu = \int_B \xi(\langle f, x \rangle | \Gamma) \, d\mu = \int_B \langle f, x \rangle \, d\mu$$

for every set $B$ in $\Gamma$ and hence $\int_B g \, d\mu = \int_B f \, d\mu$ for every set $B$ in $\Gamma$. This shows that $g$ is a Pettis conditional expectation of $f$ for the $\sigma$-algebra $\Gamma$.

In view of Theorems 5 and 9, one can ask the following.

**Question.** If, in Theorem 9, we suppose that the set

$$\{\langle f, x \rangle : \|x\| \leq 1\}$$

is almost weakly precompact in $L_\infty(\mu)$, does $f$ have a Pettis conditional expectation with respect to all sub-$\sigma$-algebras of $\Sigma$?

If the above were true, then any function satisfying the conditions of Theorem 5 would have a Pettis conditional expectation with respect to all Radon measures on all sub-$\sigma$-algebras of the Borel $\sigma$-algebra of $K$.

**IV. The Bourgain property**

So far we have seen that the family $\{\langle f, x \rangle : \|x\| \leq 1\}$ plays a strong role in determining Pettis integrability for a bounded scalarly measurable function $f$ from $\Omega$ into a dual space $E^*$. We continue this approach in this part, but, rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on $\Omega$. A property of real-valued functions formulated by J. Bourgain [2] is the cornerstone of our discussion.

**Definition 10.** Let $(\Omega, \Sigma, \mu)$ be a measure space. A family $\Psi$ of real-valued functions on $\Omega$ is said to have the **Bourgain property** if the following condition is satisfied: For each set $A$ of positive measure and for each $\alpha > 0$, there is a finite collection $F$ of subsets of positive measure of $A$ such that for each function $f$ in $\Psi$, the inequality $\sup_{B} f(B) - \inf_{B} f(B) < \alpha$ holds for some member $B$ of $F$.
A first glimpse to some consequences
The definitions and a bit of their history
The results and their applications
References

The property of Bourgain
Birkhoff integral: a vector approach to Fréchet scalar ideas

The property of Bourgain

- The notion wasn't published by Bourgain.
- It appears in a paper by [RS85] and refers to handwritten notes by Bourgain.

Definition

We say that a family $\mathcal{F} \subset R^\Omega$ has Bourgain property if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \ldots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathcal{F}$

$$\inf_{1 \leq i \leq n} |g| - \text{diam} (f(B_i)) < \varepsilon.$$
**Bourgain Property**

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\[
\inf_{1 \leq i \leq n} |\cdot| \text{-diam } (f(B_i)) < \varepsilon.
\]

**Remarkable facts**

- **If** \( \mathcal{F} = \{f\}, \) **TFAE:**
  - (i) (Bourgain property) For every \( \varepsilon > 0 \) and every \( A \in \Sigma \) with \( \mu(A) > 0 \) there is \( B \in \Sigma, B \subset A \) with \( \mu(B) > 0 \) and \( |\cdot| \text{-diam } f(B) < \varepsilon. \)
  - (ii) \( f \) is measurable.
Bourgain Property

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  (ii) \( f \) is measurable.

- If \( \mathcal{F} \) has Bourgain property, then \( \mathcal{F} \) is made up of measurable functions.
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- $\mathcal{F}$ has Bourgain property $\Rightarrow$ $\overline{\mathcal{F}}$ has too.
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- $\mathcal{F}$ has Bourgain property and $f \in \overline{\mathcal{F}}$, then there is a sequence $(f_n)$ in $\mathcal{F}$ that converges to $f$, $\mu$-almost everywhere.

- $\mathcal{F} \subset \mathbb{R}^\Omega$ has Bourgain property $\Rightarrow \mathcal{F}$ is stable, [Tal84, 9-5-4].
Fréchet interpretation of Lebesgue integral

Given \( f : \Omega \rightarrow \mathbb{R} \), for each partition \( \Gamma \) of \( \Omega \) into countably many sets \( (A_n) \) of \( \Sigma \) consider the relative upper and lower sums:

\[
J^*(f, \Gamma) = \sum_n \sup_{A_n} f \mu(A_n) \quad \text{and} \quad J_*(f, \Gamma) = \sum_n \inf_{A_n} f \mu(A_n),
\]

(assuming both series are well defined and absolutely convergent).
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(assuming both series are well defined and absolutely convergent).

We have:

- $J_*(f, \Gamma) \leq J^*(f, \Gamma')$ whenever $J_*(f, \Gamma)$ and $J^*(f, \Gamma')$ are defined.
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- The intersection of the “relative integral ranges” \( J_*(f, \Gamma) \leq x \leq J^*(f, \Gamma) \), for variable \( \Gamma \) is not empty.
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- \( J_*(f, \Gamma) \leq J^*(f, \Gamma') \) whenever \( J_*(f, \Gamma) \) and \( J^*(f, \Gamma') \) are defined.
- The intersection of the “relative integral ranges” \( J_*(f, \Gamma) \leq x \leq J^*(f, \Gamma) \), for variable \( \Gamma \) is not empty.
- This intersection is a single point \( x \) if, and only if, \( f \) is Lebesgue integrable and \( x = \int_{\Omega} f \, d\mu \).
Fréchet interpretation of Lebesgue integral

This way of presenting the theory of integration due to M. Lebesgue has the advantage, over the way M. Lebesgue presented his theory himself, that is very much close to the views of Riemann-Darboux to which many students are familiar with.
Let $f : \Omega \rightarrow X$ be a function. If $\Gamma$ is a partition of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$, the function $f$ is called **summable** with respect to $\Gamma$ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f, \Gamma) = \left\{ \sum_n f(t_n)\mu(A_n) : t_n \in A_n \right\}$$

is made up of unconditionally convergent series.
Birkhoff views

Let \( f : \Omega \rightarrow X \) be a function. If \( \Gamma \) is a partition of \( \Omega \) into countably many sets \( (A_n) \) of \( \Sigma \), the function \( f \) is called \textit{summable} with respect to \( \Gamma \) if the restriction \( f|_{A_n} \) is bounded whenever \( \mu(A_n) > 0 \) and the set of sums

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The function \( f \) is said to be \textbf{Birkhoff integrable} if for every \( \varepsilon > 0 \) there is a countable partition \( \Gamma = (A_n) \) of \( \Omega \) in \( \Sigma \) for which \( f \) is summable and

\[
\| \| - \text{diam} (J(f, \Gamma)) < \varepsilon.
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Let $f : \Omega \longrightarrow X$ be a function. If $\Gamma$ is a partition of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$, the function $f$ is called **summable** with respect to $\Gamma$ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

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The function $f$ is said to be **Birkhoff integrable** if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ for which $f$ is summable and

$$\| -\text{diam } (J(f, \Gamma)) < \varepsilon.$$

In this case, the **Birkhoff integral** $(B) \int_\Omega f \, d\mu$ of $f$ is the only point in the intersection

$$\bigcap \{ \text{co}(J(f, \Gamma)) : f \text{ is summable with respect to } \Gamma \}.$$
Theorem

Let \( f : \Omega \rightarrow X \) be a bounded function. The following statements are equivalent:

(i) \( f \) is Birkhoff integrable;

(ii) for every \( \varepsilon > 0 \) there is a countable partition \( \Gamma = (A_n) \) of \( \Omega \) in \( \Sigma \) such that for each \( t_k, t'_k \in A_k, k \in \mathbb{N} \), we have

\[
\left| \sum_{k=1}^{m} \langle x^*, f \rangle(t_k)\mu(A_k) - \sum_{k=1}^{m} \langle x^*, f \rangle(t'_k)\mu(A_k) \right| < \varepsilon
\]

for every \( m \in \mathbb{N} \) and every \( x^* \in B_{X^*} \);

(iii) \( Z_f = \{ \langle x^*, f \rangle : x^* \in B_{X^*} \} \) has Bourgain property.
Let \( f : \Omega \to X \) be a bounded function. The following statements are equivalent:

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• (i) \( \Leftrightarrow \) (ii) pretty easy since (ii) is a reformulation of (i).
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Let $f: \Omega \to X$ be a bounded function. The following statements are equivalent:

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$$\left| \sum_{k=1}^{m} \langle x^*, f \rangle(t_k) \mu(A_k) - \sum_{k=1}^{m} \langle x^*, f \rangle(t_k') \mu(A_k) \right| < \varepsilon$$

for every $m \in \mathbb{N}$ and every $x^* \in B_{X^*}$;

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- (i) $\Leftrightarrow$ (ii) pretty easy since (ii) is a reformulation of (i).
- (ii) $\Rightarrow$ (iii) straightforward.
- (iii) $\Rightarrow$ (ii) mimic a result by Talagrand.
Let $f : \Omega \to X$ be a function. TFAE:

(i) $f$ is Birkhoff integrable;
(ii) $Z_f$ is uniformly integrable, $Z_f$ has Bourgain property.

Proof. 

(i) $\Rightarrow$ (ii) $Z_f$ has Bourgain property is easy; $Z_f$ is uniformly integrable because $f$ is Pettis integrable.

(ii) $\Rightarrow$ (i) by proving (ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i) requires some extra work.
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(iii) $Z_f$ is uniformly integrable, $Z_f$ has Bourgain property and there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

(iii) $\Rightarrow$ (i)
Our basic result

Birkhoff integral as a limit of a net. WRNP and the Birkhoff integral

Birkhoff integral for multifunctions

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The Bourgain property and Birkhoff integrability

• \( f \) is Pettis integrable in \( \Omega \).

(iii) \( \Rightarrow \) (i)

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(iii) \( Z_f \) is uniformly integrable, \( Z_f \) has Bourgain property and there is a countable partition \( \Gamma = (A_n) \) of \( \Omega \) in \( \Sigma \) such that \( f(A_n) \) is bounded whenever \( \mu(A_n) > 0 \).
• $f$ is Pettis integrable in $\Omega$.
• Each $f|_{A_n}$ is Birkhoff integrable in $A_n$.

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Let \( f : \Omega \to X \) be a function. TFAE:

(i) \( f \) is Birkhoff integrable;

(ii) \( Z_f \) is uniformly integrable, \( Z_f \) has Bourgain property.

(iii) \( Z_f \) is uniformly integrable, \( Z_f \) has Bourgain property and there is a countable partition \( \Gamma = (A_n)_n \) of \( \Omega \) in \( \Sigma \) such that \( f(A_n) \) is bounded whenever \( \mu(A_n) > 0 \).

\[
(iii) \Rightarrow (i)
\]

\[
\left\| \sum_k f(t_{n,k}) \mu(A_{n,k}) - \sum_k f'(t'_{n,k}) \mu(A_{n,k}) \right\| < \frac{\varepsilon}{2^n}
\]

for arbitrary choices \( (t_{n,k}) \) and \( (t'_{n,k}) \) in \( \Gamma_n \).
(iii) \Rightarrow (i)

**Theorem**

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- \( f \) is Pettis integrable in \( \Omega \).
- Each \( f|_{A_n} \) is Birkhoff integrable in \( A_n \).
- Fix \( \varepsilon > 0 \). For each \( n \in \mathbb{N} \) take a partition \( \Gamma_n = (A_{n,k})_k \) of \( A_n \) in \( \Sigma \) such that
  \[
  \left\| \sum_k f(t_{n,k})\mu(A_{n,k}) - \sum_k f(t'_{n,k})\mu(A_{n,k}) \right\| < \frac{\varepsilon}{2^n}
  \]
  for arbitrary choices \( (t_{n,k}) \) and \( (t'_{n,k}) \) in \( \Gamma_n \).
- For \( \Gamma := (A_{n,k})_{n,k} \), the series \( \sum_{n,k} f(t_{n,k})\mu(A_{n,k}) \) c. u. for every choice \( T = (t_{n,k}) \) in \( \Gamma \) because:
(iii) ⇒ (i)

Theorem

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for arbitrary choices \( (t_{n,k}) \) and \( (t'_{n,k}) \) in \( \Gamma_n \).
- For \( \Gamma := (A_{n,k})_n \), the series \( \sum_{n,k} f(t_{n,k})\mu(A_{n,k}) \) c. u. for every choice \( T = (t_{n,k}) \) in \( \Gamma \) because:

- \( \sum_{n,k} \int_{A_{n,k}} f \ d\mu \) c. u. (\( f \) is Pettis integrable)
Theorem

Let \( f : \Omega \rightarrow X \) be a function. TFAE:

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\[ (iii) \Rightarrow (i) \]

- \( f \) is Pettis integrable in \( \Omega \).
- Each \( f|_{A_n} \) is Birkhoff integrable in \( A_n \).
- Fix \( \varepsilon > 0 \). For each \( n \in \mathbb{N} \) take a partition \( \Gamma_n = (A_n, k) \) of \( A_n \) in \( \Sigma \) such that
  \[ \left\| \sum_k f(t_{n,k})\mu(A_n, k) - \sum_k f(t'_{n,k})\mu(A_n, k) \right\| < \frac{\varepsilon}{2^n} \]
  for arbitrary choices \( (t_{n,k}) \) and \( (t'_{n,k}) \) in \( \Gamma_n \).
- For \( \Gamma := (A_n, k) \), the series \( \sum_{n,k} f(t_{n,k})\mu(A_n, k) \) c. u. for every choice \( T = (t_{n,k}) \) in \( \Gamma \) because:
  - \( \sum_{n,k} \int_{A_n, k} f \ d\mu \) c. u. (\( f \) is Pettis integrable)
  - for each finite set \( Q \subset \mathbb{N} \)
    \[ \left\| \sum_{k \in Q} (f(t_{n,k})\mu(A_n, k) - \int_{A_n, k} f d\mu) \right\| \leq \frac{\varepsilon}{2^n}. \]
Let $f : \Omega \to X$ be a function. TFAE:

(i) $f$ is Birkhoff integrable;
(ii) $Z_f$ is uniformly integrable, $Z_f$ has Bourgain property.
(iii) $Z_f$ is uniformly integrable, $Z_f$ has Bourgain property and there is a countable partition $\Gamma = (A_n, k)_k$ of $\Omega$ in $\Sigma$ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

(iii) $\Rightarrow$ (i)

- $f$ is Pettis integrable in $\Omega$.
- Each $f|_{A_n}$ is Birkhoff integrable in $A_n$.
- Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$ take a partition $\Gamma_n = (A_{n,k})_k$ of $A_n$ in $\Sigma$ such that
  \[ \left\| \sum_k f(t_{n,k})\mu(A_{n,k}) - \sum_k f(t'_{n,k})\mu(A_{n,k}) \right\| < \frac{\varepsilon}{2^n} \]
  for arbitrary choices $(t_{n,k})$ and $(t'_{n,k})$ in $\Gamma_n$.
- For $\Gamma := (A_{n,k})_{n,k}$, the series $\sum_{n,k} f(t_{n,k})\mu(A_{n,k})$ c. u. for every choice $T = (t_{n,k})$ in $\Gamma$ because:
  - $\sum_{n,k} \int_{A_{n,k}} f \ d\mu$ c. u. ($f$ is Pettis integrable)
  - for each finite set $Q \subset \mathbb{N}$
    \[ \left\| \sum_{k \in Q} (f(t_{n,k})\mu(A_{n,k}) - \int_{A_{n,k}} f \ d\mu) \right\| \leq \frac{\varepsilon}{2^n} . \]

- $\|\|-\text{diam} (\left\{ \sum_{n,k} f(t_{n,k})\mu(A_{n,k}) : t_{n,k} \in A_{n,k} \right\}) \leq \varepsilon$. 

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Theorem

Let \( f : \Omega \rightarrow X \) be a function. TFAE:

(i) \( f \) is Birkhoff integrable;

(ii) there is \( x \in X \) satisfying: for every \( \varepsilon > 0 \) there is a countable partition \( \Gamma \) of \( \Omega \) in \( \Sigma \) for which \( f \) is summable and

\[
\| S(f, \Gamma, T) - x \| < \varepsilon \quad \text{for every choice } T \text{ in } \Gamma;
\]

(iii) there is \( y \in X \) satisfying: for every \( \varepsilon > 0 \) there is a countable partition \( \Gamma \) of \( \Omega \) in \( \Sigma \) such that \( f \) is summable with respect to each countable partition \( \Gamma' \) finer than \( \Gamma \) and

\[
\| S(f, \Gamma', T') - y \| < \varepsilon \quad \text{for every choice } T' \text{ in } \Gamma'.
\]

In this case, \( x = y = \int_{\Omega} f \ d\mu \).
Theorem

Let $X$ be a Banach space. The following statements are equivalent:

1. $X^*$ has the weak Radon-Nikodým property;
2. $X$ does not contain a copy of $\ell^1$;
3. for every complete probability space $(\Omega, \Sigma, \mu)$ and for every $\mu$-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of $\sigma$-finite variation there is a Birkhoff integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \Sigma$. 
The integral as limits of Riemann-Lebesgue sums

Let $F : \Omega \rightarrow cwk(\mathbb{R}^n)$ be a multi-valued function. The following conditions are equivalent:

1. $F$ is Debreu integrable;

2. there is $B \in cwk(X)$ with the following property: for every $\varepsilon > 0$ there is a countable partition $\Gamma_0$ of $\Omega$ in $\Sigma$ such that for every countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ finer than $\Gamma_0$ and any choice $T = (t_n)$ in $\Gamma$, the series $\sum_n \mu(A_n)F(t_n)$ is unconditionally convergent and

$$h\left(\sum_n \mu(A_n)F(t_n), B\right) \leq \varepsilon.$$ 

In this case, $B = (B) \int_\Omega F \, d\mu$. 
Boundary problem

If $X$ is a Banach space not containing $\ell^1(\mathbb{R})$ and $B \subseteq B_{X^*}$ a boundary (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \|x\|$) then the norm bounded $\sigma(X, B)$-relatively compact subsets $H$ of $X$ are relatively weakly compact.

**Proof.**

- It is enough to prove that $\overline{\text{co}(H)}^{\sigma(X, B)}$ is $\sigma(X, B)$-compact.
- We prove that for each Radon probability measure $\mu$ on $H$ the identity $\text{id} : H \to X$ is $\mu$-Pettis (Birkhoff) integrable.
References


