Distance to spaces of continuous functions

B. Cascales, W. Marciszewski and M. Raja
(Top. Appl. 2005)

Universidad de Murcia/University of Warsaw.

Kent, February 2005

The starting point...

List   Details

- M. Fabian, P. Hájek, V. Montesinos, and V. Zizler.
  A quantitative version of Krein's Theorem..
  Rev. Mat. Iberoamericana, 2005.
- A. S. Granero.
  An extension of Krein-Šmulian theorem.
  Revista Iberoamericana, 2005.
- A. S. Granero, P. Hájek, and V. Montesinos Santalucía.
  Convexity and w^*-compactness in Banach spaces.
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Main result

Let X be a Banach space and let $H \subset X$ be a bounded subset of X. Then

$$\widehat{\mathrm{d}}(\overline{\mathrm{conv}(H)}, X) \leq 2\widehat{\mathrm{d}}(\overline{H}, X),$$

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- closures are taken in the bidual X^{**};
- $\widehat{\mathrm{d}}(A, X) := \sup\{d(a, X) : a \in A\}$ for $A \subset X^{**}$;
- $\widehat{\mathrm{d}}(A, X) = 0$ iff $A \subset X$, hence the inequality implies Krein's theorem.

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- Let X be a Banach space and let $H \subset X^{**}$ be a bounded subset of X^{**}. Then

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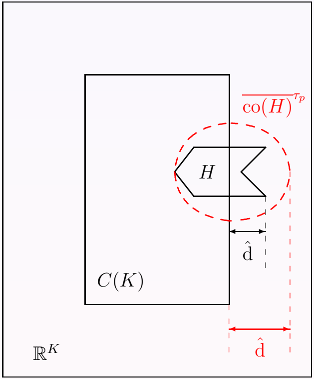
- Some of the constant involved are sharp.

...our goal

...goals

- To take the results where *(I think!)* they belongs *i.e.* to the context of $C(K)$ and $\mathbb{R}^K$ spaces endowed with τ_p;

...our goal



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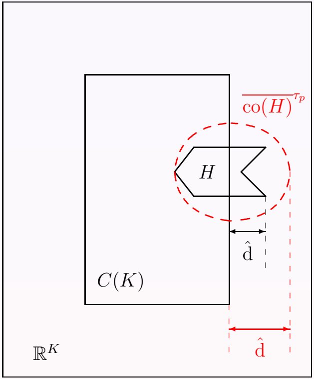

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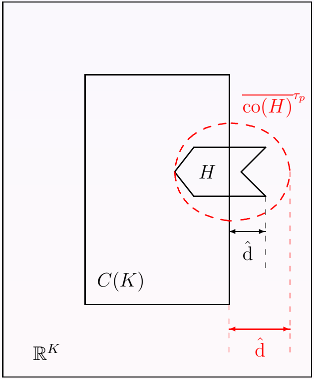

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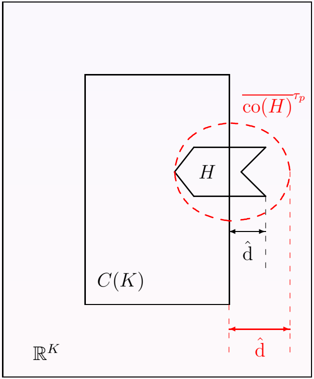

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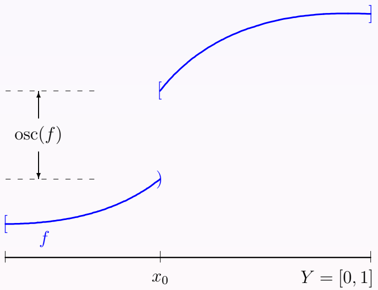
tools

- new reading of the *classical*;
- *double limits* used by Grothendieck.

Distances vs. oscillations



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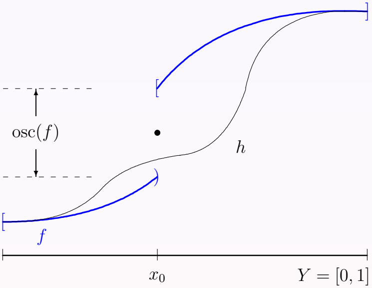

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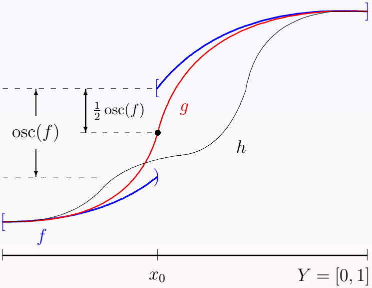

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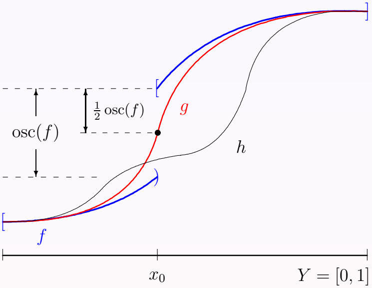

Theorem

Let Y be a normal space[a]. If $f \in \mathbb{R}^Y$ is bounded, then

$$d(f, C^*(Y)) = \frac{1}{2}\operatorname{osc}(f)\,.$$

[a] $[\operatorname{osc}(f) = \sup_{x\in Y} \operatorname{osc}(f,x)]$

Distances vs.oscillations

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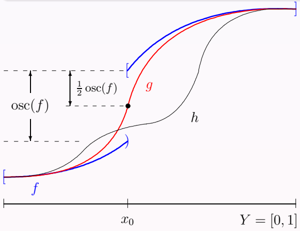

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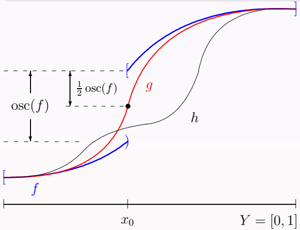

1. It is easy to check that $d(f, C^*(Y)) \geq \operatorname{osc}(f)/2$.

Distances vs.oscillations

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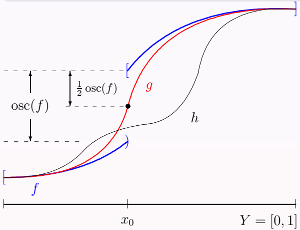

1. It is easy to check that $d(f, C^*(Y)) \geq \mathrm{osc}(f)/2$.
2. For $x \in Y$, $\mathcal{U}_x$ family of neighb.

$$\mathrm{osc}(f) \geq \inf_{U \in \mathcal{U}_x} \sup_{y,z \in U} \big(f(y) - f(z)\big)$$

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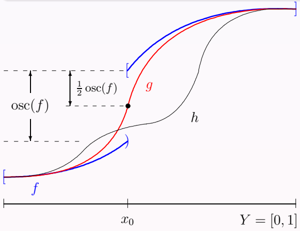

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$$f_2(x) := \sup_{U \in \mathcal{U}_x} \inf_{z \in U} f(z) + \frac{\operatorname{osc}(f)}{2}$$

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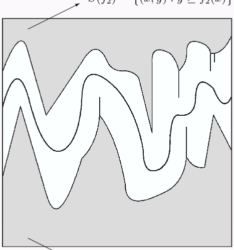

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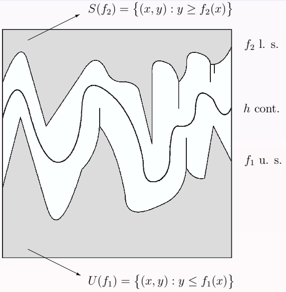

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4. Squeeze h between f_2 and f_1 and $d(f, C^*(Y)) = \|f - h\|_\infty = \mathrm{osc}(f)/2$.

Oscillations vs. iterated limits.

Definition

$H \subset Z^X$ ε-interchanges limits with X if

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq \varepsilon$$

whenever (x_n) in X and (f_m) in H and all limits involved do exist.

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- For the notion of H ε-interch. limits with X *sequences* can be replaced by *nets*.

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First properties... K compact

- For the notion of H ε-interch. limits with X *sequences* can be replaced by *nets*.
- $H \subset C(K)$ unif. bdd. then H ε-interchanges limits with K iff

$$\operatorname{osc}^*(f, x) = \inf_{U \in \mathcal{U}_x} \sup_{y \in U} d(f(y), f(x)) \leq \varepsilon$$

for each $x \in K$ and $f \in \overline{H}^{\tau_p}$.

Iterated limits vs. distances

Corollary

For $H \subset C(K)$ unif. bdd. the following properties hold:

Iterated limits vs. distances

Corollary

For $H \subset C(K)$ unif. bdd. the following properties hold:

1. *If H ε-interchanges limits with K then $\operatorname{osc}(f) \leq 2\varepsilon$ for every $f \in \overline{H}^{\tau_p}$.*

Iterated limits vs. distances

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To bear in mind

- To study distances is equiv. to study iterated limits;

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To bear in mind

- To study distances is equiv. to study iterated limits;
- The above estimates are sharp.

ε-interchanging limit property and convex hulls

Theorem

Let Z be a compact convex subset of a normed space E, let K be a set, and let $H \subset Z^K$. Then, for each $\varepsilon \geq 0$, H ε-interchanges limits with K if, and only if, $\mathrm{conv}(H)$ ε-interchanges limits with K.

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Theorem

- *If $H \subset C(K)$ is uniformly bounded then:*

$$\hat{\mathrm{d}}(\overline{\mathrm{conv}(H)}^{\tau_p}, C(K)) \leq 2\hat{\mathrm{d}}(\overline{H}^{\tau_p}, C(K)).$$

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- *If $H \subset \mathbb{R}^K$ is uniformly bounded then:*

$$\hat{\mathrm{d}}(\overline{\operatorname{conv}(H)}^{\tau_p}, C(K)) \leq 5\hat{\mathrm{d}}(\overline{H}^{\tau_p}, C(K)).$$

$$\boxed{5 = 2 \times 2 + 1}$$

Distances to spaces of affine continuous functions

Theorem

If K is compact convex subset of a l.c.s. and $f \in \mathcal{A}(K)$ then

$$d(f, C(K)) = d(f, \mathcal{A}^C(K)).$$

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$$\begin{aligned}f_2(x) &:= \sup_{U \in \mathcal{U}_x} \inf_{z \in U} f(z) + \frac{\delta}{2} \\ &\geq \inf_{U \in \mathcal{U}_x} \sup_{y \in U} - \frac{\delta}{2} =: f_1(x)\end{aligned}$$

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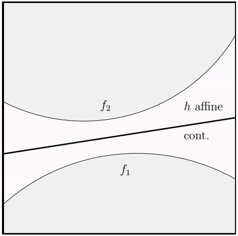

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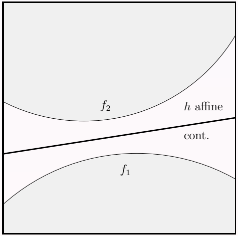
$$d(f, C(K)) = d(f, \mathcal{A}^C(K)).$$

f_2 l. s. convex



f_1 u. s. concave

1. It is easy to check that $d(f, \mathcal{A}^C(K)) \geq \operatorname{osc}(f)/2$.
2. For $x \in Y$, $\mathcal{U}_x$ family of neighb.

$$\delta > \operatorname{osc}(f) \geq \inf_{U \in \mathcal{U}_x} \sup_{y,z \in U} \big(f(y) - f(z)\big)$$
$$\geq \inf_{U \in \mathcal{U}_x} \sup_{y \in U} f(y) - \sup_{U \in \mathcal{U}_x} \inf_{z \in U} f(z)$$

3.

$$f_2(x) := \sup_{U \in \mathcal{U}_x} \inf_{z \in U} f(z) + \frac{\delta}{2}$$
$$\geq \inf_{U \in \mathcal{U}_x} \sup_{y \in U} - \frac{\delta}{2} =: f_1(x)$$

4. Squeeze h between f_2 and f_1 and $\|f - h\|_\infty \leq \delta/2$.

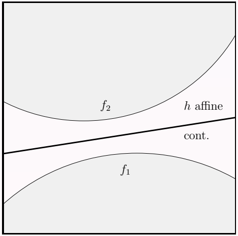
Distances to spaces of affine continuous functions

Theorem

If K is compact convex subset of a l.c.s. and $f \in \mathcal{A}(K)$ then

$$d(f, C(K)) = d(f, \mathcal{A}^C(K)).$$

f_2 l. s. convex



f_1 u. s. concave

Corollary

Let X be a Banach space and let B_{X^*} be the closed unit ball in the dual X^* endowed with the w^*-topology. Let $i : X \to X^{**}$ and $j : X^{**} \to \ell_\infty(B_{X^*})$ be the canonical embedding. Then, for every $x^{**} \in X^{**}$ we have:

$$d(x^{**}, i(X)) = d(j(x^{**}), C(B_{X^*})).$$

Quantitative Krein's theorem

Corollary, [FHMZ05, Theorem 2]

Let X be a Banach space and let $H \subset X$ be bdd. Then

$$\hat{\mathrm{d}}(\overline{\mathrm{conv}(H)}^{\mathrm{w}^*}, X) \leq 2\hat{\mathrm{d}}(\overline{H}^{\mathrm{w}^*}, X).$$

Corollary, [Gra05, Theorem 5]

Let X be a Banach space and let $H \subset X^{**}$ be bdd. Then

$$\hat{\mathrm{d}}(\overline{\mathrm{conv}(H)}^{\mathrm{w}^*}, X) \leq 5\hat{\mathrm{d}}(\overline{H}^{\mathrm{w}^*}, X).$$

Theorem (Quantitative version of Grothendieck's theorem)

For a compact space K, $B_{C(K)^*}$ endowed with the w^* topology and $H \subset C(K)$ uniformly bounded we have

$$\frac{1}{2}\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^{B_{C(K)^*}}}, C(B_{C(K)^*})) \leq 4\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K))$$

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- $\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) = 0 \Leftrightarrow H$ is τ_p-relatively compact in $C(K)$.
- $\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^{B_{C(K)^*}}}, C(B_{C(K)^*})) = 0 \Leftrightarrow H$ is weakly relatively compact in $C(K)$.

Theorem (Quantitative version of Gantmacher's theorem)

Let X and Y be Banach spaces, $T : X \to Y$ an operator and $T^* : Y^* \to X^*$ its adjoint operator. Then

$$\frac{1}{2}\hat{\mathrm{d}}(\overline{T(B_X)}^{w^*}, Y) \leq \hat{\mathrm{d}}(\overline{T^*(B_{Y^*})}^{w^*}, X^*) \leq 4\hat{\mathrm{d}}(\overline{T(B_X)}^{w^*}, Y).$$

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