The Lindelöf property and $\sigma$-fragmentability

B. Cascales and I. Namioka

Universidad de Murcia/University of Washington.

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Let $(X, \tau)$ be a Tychonoff (completely regular and $T_1$) space, and let $C(X,I)$ be the space of all continuous functions $f : X \to I = [0,1]$. Then the map $\Phi : X \to I^{C(X,I)}$, given by $\Phi(x)(f) = f(x)$ for $x \in X, f \in C(X,I)$, embeds $X$ topologically in $(I^{C(X,I)}, \tau_p)$ (see e.g. [Kel75]).
Let \((X, \tau)\) be a Tychonoff (completely regular and \(T_1\)) space, and let \(C(X, I)\) be the space of all continuous functions \(f : X \to I = [0, 1]\). Then the map \(\Phi : X \to I^{C(X, I)}\), given by \(\Phi(x)(f) = f(x)\) for \(x \in X, f \in C(X, I)\), embeds \(X\) topologically in \((I^{C(X, I)}, \tau_p)\) (see e.g. [Kel75]).

Let \((M, \rho)\) be a metric space with the metric \(\rho\) bounded, and let \(D\) be an index set. We consider various topologies, pseudometrics, metrics, etc. on the product space \(M^D\) and study their relationship between them in subspaces \(X \subset M^D\), namely,

- the product (= pointwise) topology \(\tau_p\)
- the topology \(\gamma(D)\) of uniform convergence on the family of all countable subsets \(\mathcal{C}\) of \(D\).
- the metric \(d\) of uniform convergence on \(D\).
Let $X$ be a $K$-analytic subspace of $M^D$ where $(M, \rho)$ is a metric space with $\rho$ bounded. Then the following statements are equivalent.

(a) The space $(X, \tau_\rho)$ is $\sigma$-fragmented by $d$.

(b) For each compact set $K \subset X$, $(K, \tau_\rho)$ is fragmented by $d$.

(c) For each $A \in \mathcal{C}$, the pseudo-metric space $(X, d_A)$ is separable.

(d) $(X, \gamma(D))$ is Lindelöf.

(e) $(X, \gamma(D))^\mathbb{N}$ is Lindelöf.
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Known and easy parts: (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) [JNR93] (a simpler proof [NP96]) and [CNO03].

(a) $\implies$ (c) needs the following simple lemma.
Theorem

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(a) $\implies$ (c) needs the following simple lemma.

Lemma

Let $(T, \tau)$ be metrizable and separable and let $\delta$ be a metric on $T$. Then $(T, \tau)$ is $\sigma$-fragmented by $\delta$ if and only if $(T, \delta)$ is separable.
Difficult part: \((c) \Rightarrow (d)\) (by contradiction)

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Useful facts about Baire sets:

- A subset $Z$ of $T$ is called a zero-set (in $T$) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.
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- A subset $Z$ of $T$ is called a zero-set (in $T$) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.

- Let $\mathcal{L}$ (or $\mathcal{L}(T)$) denote the family of all zero-sets in $T$. Then $\mathcal{L}$ is closed under finite unions and countable intersections.
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- A subset $Z$ of $T$ is called a zero-set (in $T$) if $Z = f^{-1}(0)$ for some continuous function $f : T \rightarrow \mathbb{R}$.
- Let $\mathcal{Z}$ (or $\mathcal{Z}(T)$) denote the family of all zero-sets in $T$. Then $\mathcal{Z}$ is closed under finite unions and countable intersections.
- The $\sigma$-algebra generated by $\mathcal{Z}$ is denoted by Baire($T$) (Baire sets in $T$).
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- Baire($T$) ⊂ Souslin($\mathcal{Z}$).
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- The $\sigma$-algebra generated by $\mathcal{Z}$ is denoted by Baire($T$) ($Baire \ sets \ in \ T$).

- Baire($T$) $\subset$ Souslin($\mathcal{Z}$).

- If $X$ is $K$-analytic subset of $M^D$, then each zero-set in $X$, being closed, is $K$-analytic and therefore each member of Souslin($\mathcal{Z}$) is $K$-analytic. Since Baire($T$) $\subset$ Souslin($\mathcal{Z}$), each Baire set in $X$ is $K$-analytic hence Lindelöf relative to $\tau_p$. 
Preparatory things

**Notation:** $x \in X$, $S \subset D$ and $\varepsilon > 0$.
- $U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\}$. 
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- \( U(x,S,\varepsilon) := \{ y \in X : d_S(y,x) < \varepsilon \} \).
- \( \mathcal{U} = \{ U_j : j \in J \} \) is a family of \( \gamma(D) \)-open sets in \( X \) that covers \( X \) without a countable subcover. We may assume that each \( U_j \) is of the form

\[
U_j = U(x_j,A_j,\varepsilon_j) = \{ y \in X : d_{A_j}(y,x_j) < \varepsilon_j \},
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where \( x_j \in X, \ A_j \in \mathcal{C}, \ \varepsilon_j > 0 \) for each \( j \in J \).
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- For each $A \in \mathcal{C}$, let $U(A) = \bigcup\{U_j : j \in J, A_j \subset A\}$. 
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- For each \( A \in \mathscr{C} \), let \( U(A) = \bigcup \{ U_j : j \in J, A_j \subset A \} \).

**Remark**

If \( A \subset A' \), then \( U(A) \subset U(A') \) and \( X = \bigcup \{ U(A) : A \in \mathscr{C} \} \).
Assume (c) holds and (d) doesn’t

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \text{Baire}(X)$ whenever $x \in X, A \in \mathcal{C}, \varepsilon > 0$.
(ii) $U(A) \in \text{Baire}(X)$: $U(A)$ is $K$-analytic and Lindelöf, $A \in \mathcal{C}$.
(iii) $S \subset X$ is covered by a count. subfamily of $\mathcal{U}$ iff $S \subset U(A)$.
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Let \( \mathcal{Y} \) be the family of all K-analytic subsets \( Y \) of \( (X,\tau_p) \) such that there is no countable subfamily of \( \mathcal{U} \) that covers \( Y \).
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Two possibilities:
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Let \( \mathcal{Y} \) be the family of all \( K \)-analytic subsets \( Y \) of \( (X,\tau_p) \) such that there is no countable subfamily of \( \mathcal{U} \) that covers \( Y \).

**Two possibilities:**

- For each \( Y \in \mathcal{Y} \) and each \( \varepsilon > 0 \), there is a \( Z \in \mathcal{Y} \) such that \( Z \subset Y \) and \( d\text{-diam} (Z) \leq \varepsilon \).
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Let \( \mathcal{Y} \) be the family of all \( K \)-analytic subsets \( Y \) of \( (X, \tau_p) \) such that there is no countable subfamily of \( \mathcal{U} \) that covers \( Y \).

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- The negation of the previous case.
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Let \( \mathcal{Y} \) be the family of all \( K\text{-analytic subsets } Y \text{ of } (X,\tau_p) \) such that there is no countable subfamily of \( \mathcal{U} \) that covers \( Y \).

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- The negation of the previous case.

The proof

We show that each case leads to a contradiction.
Corollary

Let $X$ be a $K$-analytic subspace of $M^D$ where $(M, \rho)$ is a metric space with $\rho$ bounded. If $(X, \gamma(D))$ is Lindelöf, then $(X, \gamma(D))^\mathbb{N}$ is Lindelöf.
Corollary

Let $X$ be a $K$-analytic subspace of $M^D$ where $(M, \rho)$ is a metric space with $\rho$ bounded. If $(X, \gamma(D))$ is Lindelöf, then $(X, \gamma(D))^\mathbb{N}$ is Lindelöf.

- $\phi : (M^D)^\mathbb{N} \to (M^\mathbb{N})^D$ is $\tau_\rho$ and $\gamma$-homeomorphism.
  $$\phi(\xi)(p)(j) = \xi(j)(p) \text{ for all } \xi \in (M^D)^\mathbb{N}, p \in D, j \in \mathbb{N}$$
- $M^\mathbb{N}$ is metrizable ($\rho_\infty(m, m')$). Consider
  $$d_\infty(x, x') = \sup\{\rho_\infty(x(p), x'(p)) : p \in D\} \text{ for } x, x' \in (M^\mathbb{N})^D.$$
Corollary

Let $X$ be a $K$-analytic subspace of $M^D$ where $(M, \rho)$ is a metric space with $\rho$ bounded. If $(X, \gamma(D))$ is Lindelöf, then $(X, \gamma(D))^N$ is Lindelöf.

- $\phi : (M^D)^N \rightarrow (M^N)^D$ is $\tau_\rho$ and $\gamma$-homeomorphism.
  
  $\phi(\xi)(p)(j) = \xi(j)(p)$ for all $\xi \in (M^D)^N, p \in D, j \in \mathbb{N}$

- $M^N$ is metrizable $(\rho_\infty(m, m'))$. Consider
  
  $d_\infty(x, x') = \sup\{\rho_\infty(x(p), x'(p)) : p \in D\}$ for $x, x' \in (M^N)^D$.

- $X^N$ is $K$-analytic, hence so is $\phi(X^N)$ and each compact subset of $\phi(X^N)$ is fragmented by $d_\infty$. 
Corollary

Let $X$ be a $K$-analytic subspace of $M^D$ where $(M,\rho)$ is a metric space with $\rho$ bounded. If $(X,\gamma(D))$ is Lindelöf, then $(X,\gamma(D))^N$ is Lindelöf.

- $\phi : (M^D)^N \to (M^N)^D$ is $\tau_\rho$ and $\gamma$-homeomorphism.
  
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  $d_\infty(x,x') = \sup\{\rho_\infty(x(p),x'(p)) : p \in D\}$ for $x,x' \in (M^N)^D$.

- $X^N$ is $K$-analytic, hence so is $\phi(X^N)$ and each compact subset of $\phi(X^N)$ is fragmented by $d_\infty$.

- Hence by (a)$\Leftrightarrow$(b), $(\phi(X^N),\gamma(D)) = (X,\gamma(D))^N$ is Lindelöf.
Let $I = [-1,1]$ and let $\Gamma$ be an arbitrary index set. For $x \in I^\Gamma$ we write $\text{supp}(x) = \{ \gamma \in \Gamma : x(\gamma) \neq 0 \}$. Consider

$$F(\Gamma) = \{ x \in [-1,1]^\Gamma : \text{supp}(x) \text{ is finite} \}$$

and

$$\Sigma(\Gamma) = \{ x \in [-1,1]^\Gamma : \text{supp}(x) \text{ is countable} \}.$$
Let $I = [-1,1]$ and let $\Gamma$ be an arbitrary index set. For $x \in I^\Gamma$ we write $\text{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Consider

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**Definition**

A compact Hausdorff space $K$ is said to be *Corson compact* if $K$ is homeomorphic to a $\tau_p$-compact subset of $\Sigma(\Gamma)$.
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A compact Hausdorff space $K$ is said to be *Corson compact* if $K$ is homeomorphic to a $\tau_p$-compact subset of $\Sigma(\Gamma)$.

**Properties:**

- if $A$ is a countable subset of a Corson compact space $K$, then the closure of $A$ is compact and metrizable.
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A compact Hausdorff space $K$ is said to be **Corson compact** if $K$ is homeomorphic to a $\tau_p$-compact subset of $\Sigma(\Gamma)$.

**Properties:**

- if $A$ is a countable subset of a Corson compact space $K$, then the closure of $A$ is compact and metrizable.
- $(\Sigma(\Gamma), \tau_p)$ is countably tight. Hence the Corson compact spaces are countable tight.
Application 1

If $K$ is Corson compact then $(C(K), \gamma(K))^\mathbb{N}$ is Lindelöf.
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In general for $K$ Corson the space $(C(K), \tau_p)$ IS NOT $K$-analytic.
Application 1

If $K$ is Corson compact then $(C(K),\gamma(K))^\mathbb{N}$ is Lindelöf.

In general for $K$ Corson the space $(C(K),\tau_p)$ IS NOT $K$-analytic.

Lemma

Let $\Gamma$ be an index set and let $H$ be a norm bounded subset of $\ell^\infty(\Gamma) \subset \mathbb{R}^\Gamma$. If

$$\overline{\text{aco}(H)}^{\tau_p} = \overline{\text{aco}(H)}^\parallel \parallel,$$

then $X := \overline{\text{span}H}^\parallel \parallel$ is $K$-analytic with respect to the pointwise topology $\tau_p$ of $\mathbb{R}^\Gamma$. In particular, if $H$ is a norm bounded $\tau_p$-compact subset of $\ell^\infty(\Gamma)$ that is norm-fragmented, then $\overline{\text{span}H}^\parallel \parallel$ is $K$-analytic relative to $\tau_p$. 
Theorem

Let \((X, \tau)\) be a \(K\)-analytic Tychonoff space. TFAE:

(a) The space \(X\) is \(\sigma\)-scattered.

(b) The space \(X\) does not contain a compact perfect subset.

(c) The space \((X, \tau_\delta)\) is Lindelöf.

(d) The space \((X, \tau_\delta)^\mathbb{N}\) is Lindelöf.
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(d) The space \((X, \tau_\delta)^N\) is Lindelöf.

(i) For any countable set \(A \subset C(X), \overline{A^R}^X\) is \(\tau_p\)-metrizable.

(ii) \((B_1(X), \tau_p)\) is Fréchet-Urysohn.

(iii) \((C(X), \tau_p)\) is Fréchet-Urysohn.

(iv) \((C(X), \tau_p)\) is sequential.

(v) \((C(X), \tau_p)\) is a \(k\)-space.

(vi) \((C(X), \tau_p)\) is a \(k_R\)-space.
Theorem

A dual Banach space $X^*$ has the Radon-Nikodym property (RNP) iff $X^*$ is Lindelöf with the topology $\gamma(X)$ of uniform convergence on bounded sequences of $X$, [Ori92]
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If a dual Banach space $X^*$ is weakly Lindelöf then, $(X^*,w)^\mathbb{N}$ is Lindelöf, [Ori92].
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Theorem

If a dual Banach space $X^*$ is weakly Lindelöf then, $(X^*,w)^\mathbb{N}$ is Lindelöf, [Ori92].

Theorem

If $X$ is a Banach space and $H$ is a weak*-compact subset of $X^*$ which is weak-Lindelöf, then $\overline{\text{co}(H)}^{w^*} = \overline{\text{co}(H)}^\|\|$ and this closed convex hull is weakly Lindelöf again; furthermore $Y = \overline{\text{span}(H)}^\|$ is weakly Lindelöf (in fact $B_{Y^*}$ is Corson compact), [CNO03].
Theorem

Let \((X,\|\|)\) be a Banach space such that, for some norming subset \(B\) of \(B_{X^*}\), \((X,\sigma(X,B))\) is \(K\)-analytic. TFAE:

(i) \(X\) has property \((C)\) and \((X,\sigma(X,B))\) is \(\sigma\)-fragmented by \(\|\|\).

(ii) \((X,w)\) is Lindelöf.

(iii) \((B_{X^*},w^*)\) countably tight, \(w^*\)-separable subsets are metrizable.
Theorem

Let \((X, \| \|)\) be a Banach space such that, for some norming subset \(B \subset B_{X^*}\), \((X, \sigma(X,B))\) is \(K\)-analytic. TFAE:

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Lemma

Let \(X\) be a Banach space and \(B \subset X^*\) a norming subset. If \(X\) has property \((C)\), then \(\gamma(B)\) is stronger than the weak topology of \(X\).
How wide is the class of Banach spaces $X$ for which there is $B \subset B_{X^*}$ norming and $(X, \sigma(X,B))$ is $K$-analytic?
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Includes

- Weakly $K$-analytic Banach spaces.
How wide is the class of Banach spaces $X$ for which there is $B \subset B_{X^*}$ norming and $(X, \sigma(X,B))$ is $K$-analytic?

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- Weakly $K$-analytic Banach spaces.
- Dual Banach spaces.
How wide is the class of Banach spaces $X$ for which there is $B \subset B_{X^*}$ norming and $(X,\sigma(X,B))$ is $K$-analytic?

Includes

- Weakly $K$-analytic Banach spaces.
- Dual Banach spaces.
- Representable Banach spaces, [GT82].
How wide is the class of Banach spaces $X$ for which there is $B \subset B_{X^*}$ norming and $(X, \sigma(X,B))$ is $K$-analytic?

Includes

- Weakly $K$-analytic Banach spaces.
- Dual Banach spaces.
- Representable Banach spaces, [GT82].
- Banach spaces generated by a $RN$-compact, [CNO03].
The central results are sharp

Under CH, there is a Čech-analytic Lindelöf Tychonoff space $Y$ that is $\sigma$-scattered and such that $(Y, \tau_\delta)$ is not Lindelöf.

There is a countably $K$-determined uncountable subspace $Y \subset \mathbb{R}$ such that the compact subsets of $Y$ are countable. $Y$ does not contain perfect compact subsets, $Y$ isn't $\sigma$-scattered and $(Y, \tau_\delta)$ is not Lindelöf.

There is a compact space $K$ such that $(C(K), \gamma(K))$ is Lindelöf and $K$ is not Corson.
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