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Descomposiciones y encajes en espacios de funciones
Decompositions and Embeddings in Function Spaces

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DECOMPOSITIONS AND EMBEDDINGS IN FUNCTION SPACES

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DECOMPOSITIONS AND EMBEDDINGS IN FUNCTION SPACES

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Introducción

Uno de los objetivos principales de la topología general es el de caracterizar y clasificar los espacios topológicos. Con tal propósito, los especialistas han utilizado diversos métodos, herramientas y teorías desarrolladas no sólo dentro del ámbito de la topología general, sino incluso algunos métodos provenientes de otras áreas de las matemáticas tales como el análisis funcional, el álgebra topológica, la teoría de conjuntos y la teoría de juegos, por mencionar algunas.

Dentro del contexto antes descrito surge la teoría de espacios de funciones dotados de la topología de la convergencia puntual sobre los espacios topológicos de Tychonoff, también conocida como C_p-teoría. Estos espacios tienen ricas estructuras topológicas y algebraicas que son tan importantes que determinan en gran medida la topología de los espacios de Tychonoff, tal y como lo muestra Nagata al probar que dos espacios de funciones $C_p(X)$ y $C_p(Y)$ son topológicamente isomorfos si, y sólo si, X y Y son homeomorfos.

Más aún, hacia la mitad del siglo pasado, especialistas en análisis funcional propusieron problemas que son puramente topológicos y se presentan de manera natural en sus líneas de investigación. De modo que, el desarrollo de las técnicas resultantes contribuyó a mejorar el conocimiento dentro del análisis funcional a la vez que generó nuevas técnicas en la topología general. De este proceso de enriqueciminto mutuo nos llega el estudio de las clases de los espacios compactos como los compactos de Eberlein, Talagrand, Gul'ko, Corson y Valdivia por ejemplo (ver [Eb], [Ta], [Gu], [Co], [AMN], [Sc]). Estas clases de espacios compactos constituyen una fuente extensa de ejemplos en topología general.

Uno de los primeros resultados que evidenció la estrecha relación entre el análisis funcional y la topología general es la caracterización de los subconjuntos compactos de los espacios de Banach con la topología débil (posteriormente llamados compactos de

Eberlein) en términos puramente topológicos. Por una parte Amir y Lindenstrauss probaron que X es un compacto de Eberlein si, y sólo si, X está inmerso en un Σ_*-producto de rectas reales (ver [AL]). Mientras que Rosenthal caracterizó a los espacios compactos de Eberlein por medio de familias separadoras σ-punto-finitas de conjuntos cozero (ver [Ro]).

Los espacios compactos de Corson son una clase más extensa que la de los espacios compactos de Eberlein, sin embargo, aún tienen características categóricas interesantes. Por ejemplo, si X es un espacio compacto de Corson entonces $C_p(X)$ es un espacio Lindelöf y $d(X) = w(X)$ (ver [Ar2]) En particular, si un compacto de Corson es separable entonces es metrizable. Es un hecho sorprendente que los espacios compactos de Corson, que parecían ser una clase estudiada exclusivamente en análisis funcional, resultó ser especialmente útil para establecer caraterizaciones de propiedades topológicas de espacios compactos arbitrarios. Para ilustrar lo anterior podemos mencionar un teorema clásico de Shapirovsky donde se afirma que un espacio compacto K tiene estrechez numerable si, y sólo si, existe un mapeo continuo e irreducible de X sobre un espacio compacto de Corson (ver [Sh]).

Numerosos estudios en topología general están dedicados a los espacios compactos K para los cuales $C_p(K)$ es un espacio Lindelöf Σ, llamados compactos de Gul'ko. La investigación de estos espacios desde un punto de vista puramente topológico dio como resultado un teorema profundo de Gul'ko en el cual se demuestra que si para un compacto K sucede que $C_p(K)$ es Lindelöf Σ entonces K es un compacto de Corson (ver [Gu]). En otras palabras, sucede que una clase que emergió del terreno de la topología general es una parte importante de la jerarquía de las clases de espacios compactos que se estudian en el análisis funcional.

Los topólogos también obtuvieron resultados importantes sobre los espacios compactos de Eberlein. Arhangel'skii probó que si K es un espacio compacto de Eberlein entonces $C_p(X)$ es un espacio $K_{\sigma\delta}$ y, en particular, es K-analítico ([ver [Ar] y [Ta0]). En esta línea, desde el punto de vista del estudio de los espacios de Banach, Talagrand demostró que los espacios de Banach débilmente, compactamente generados, son débilmente K-analíticos (Ver [Ta0]). Benyamini, Rudin y Wage demostraron que cualquier imagen continua de un compacto de Eberlein sigue siendo un espacio compacto de Eberlein (ver [BRW]).

Los espacios con la propiedad Lindelöf Σ eran estudiados de manera paralela tanto en topología general como en la teoría descriptiva de conjuntos (bajo el nombre de espacios numerablemente K-determinados). Durante los pasados 30 años los topólogos y los especialistas en la teoría descriptiva de conjuntos se percataron de la estrecha relación entre sus respectivos resultados. Esta interacción originó lo que hoy se conoce como la teoría descriptiva de conjuntos en espacios de funciones.

Dentro del contexto descrito anteriormente, el doctorando ha conducido su investigación a lo largo de diversas líneas. La naturaleza, los objetivos, los resultados conocidos y problemas abiertos de cada una de éstas se describen en las siguientes subsecciones de esta introducción. Igualmente, se mencionan los logros alcanzados por el autor del presente texto dentro de cada una de tales áreas de investigación durante el curso de sus estudios de doctorado.

Algunos autores consideran que una descomposición de un espacio topológico es una familia de subconjuntos del espacio, a menudo una partición. Este no es el caso que nos ocupa. En este texto recuperamos el espíritu de [Tk2] y [Tk3] de manera que una descomposición de un espacio topológico es un proceso por medio del cual se obtienen familias de subconjuntos del espacio. Así describimos nuestra idea de descomposición, que es la que guía toda la investigación reportada en la presente tesis. En ningún momento definimos una descomposición y mucho menos como una familia de subespacios de un espacio dado. Sí definimos las familias que nos interesa obtener y describimos la manera de descomponer los espacios que nos permite obtener tales familias.

Adoptamos aquí la filosofía que acabamos de describir porque una estrategia que ha probado su eficacia para estudiar y caracterizar los espacios topológicos y los espacios de funciones consiste en descomponerlos en sus subespacios y estudiar la naturaleza de tales descomposiciones. Hay un gran número de ejemplos de cómo tales descomposiciones determinan en gran medida la topología de los espacios que las admiten. Como muestra, observemos un espacio X que puede descomponerse para obtener un cubrimiento conservativo o M-ordenado de sus subespacios compactos. En cada caso, la existencia de una tal descomposición tiene fuertes implicaciones sobre las propiedades topológicas del espacio X. Más aún, si el espacio que tiene un cubrimiento compacto ya sea M-ordenado o conservativo es de la forma $C_p(Y)$ entonces su estructura nos permite concluir fuertes condiciones sobre su topología.

Los resultados incluidos en esta tesis se presentan con las correspondientes referencias a dónde han sido publicados previamente. Si no aparece alguna referencia, el resultado en cuestión aparece en esta tesis por primera vez.

Esta tesis tiene tres capítulos principales.

En las tres primeras secciones del Capítulo 1 se presentan los resultados previos que establecen el contexto dentro del cual se llevó a cabo la investigación del tema de ese capítulo. La Sección 1.4 contiene resultados originales ya publicados por el autor de la tesis además de resultados aún no publicados.

El Capítulo 2 comienza con una sección de resultados preliminares y las tres secciones siguientes recogen los resultados originales que sobre el tema se obtuvieron en colaboración con A. Avilés.

Las primeras dos secciones del Capítulo 3 incluyen resultados publicados por el autor de la tesis, algunos de ellos en colaboración con V. Tkachuk. El contenido de tales secciones es parte de los preliminares teóricos aplicados para obtener los resultados originales aún por publicar que aparecen en la Sección 3.3. Finalmente la Sección 3.4 incluye los resultados relevantes al tercer capítulo obtenidos en colaboración con A. Avilés.

Los problemas relevantes que permanecen abiertos están listados en la sección correspondiente. Finalmente, la tesis incluye un capítulo con las conclusiones.

A continuación describimos en detalle el contenido de la tesis.

Dominación en espacios de funciones

Dado un espacio Z, la familia $\mathscr{K}(Z)$ es el retículo de los subconjuntos compactos de Z. Una familia $\mathscr{A}$ se llama fundamental si para cada $K \in \mathscr{K}(Z)$ existe $A \in \mathscr{A}$ tal que $K \subset A$. Si todos los elementos de un cubrimiento $\mathscr{C}$ de X son compactos, entonces la familia $\mathscr{C}$ se llama compacta. Por otra parte, una familia $\mathscr{B}$ es M-ordenada por algún espacio M, si $\mathscr{B} = \{B_K : K \in \mathscr{K}(M)\}$ y además $K \subset L$ implica $B_K \subset B_L$. Un espacio X es dominado por un espacio M si tiene un cubrimiento compacto M-ordenado. Diremos que X es fuertemente M-dominado si tiene un cubrimiento $\mathscr{C}$ compacto fundamental M-ordenado.

Dado un espacio X sea M el espacio obtenido al dotar de la topología discreta al conjunto $\mathscr{K}(X)$ de todos los subespacios compactos de X. Es claro que X es fuertemente dominado por M. El hecho de que cada espacio topológico sea dominado por un espacio métrico motiva la siguiente definición.

Definición 1.4.1. *Dado un espacio X el índice de dominación por un espacio métrico de X, denotado por $dm(X)$, es el cardinal definido por:*

$$dm(X) = min\{w(M) : M \textit{ es un espacio métrico tal que } X \textit{ es dominado por } M\}.$$

Análogamente, definimos el índice de dominación fuerte.

Definición 1.4.2. *Dado un espacio X el índice de dominación fuerte por un espacio métrico de X, denotado por $sdm(X)$, es el cardinal definido por:*

$$sdm(X) = min\{w(M) : M \text{ es métrico y } X \text{ es fuertemente dominado por } M\}.$$

Recordemos que una multifunción compacto-valuada superiormente semicontinua se denota *usco* en [CMO], en ese mismo artículo se presenta el índice de K-determinación de un espacio X que se denota por $\ell\Sigma(X)$ y está definido como:

$$\ell\Sigma(X) = min\{w(M) : M \text{ es métrico y } \exists \text{ una usco sobreyectiva } \varphi : M \to \exp(X)\}.$$

Las propiedades generales de estabilidad de dm se presentan a continuación:

Proposición 1.4.3. *Para todo espacio X sucede lo siguiente:*

(I) $dm(X) \leq \ell\Sigma(X)$.

(II) *Si $dm(X) \leq \kappa$ entonces el índice de dominación por un métrico de cualquier imagen continua de X es, a su vez, no mayor que κ.*

(III) *Si $dm(X) \leq \kappa$ y Y es cerrado en X entonces $dm(Y) \leq \kappa$.*

(IV) *Si $X = \bigcup_{i\in\omega} X_i$ y $dm(X_i) \leq \kappa$ entonces $dm(X) \leq \kappa$.*

(V) $\mathrm{ext}(X) \leq dm(X)$.

En el caso de los espacios de funciones, el estudio del cardinal dm y especialmente sdm puede aportar condiciones para la metrizabilidad de dichos espacios como se ha demostrado bajo la hipótesis del continuo en [COT, Theorem 3.10]. Los autores del citado artículo demuestran bajo CH que si X es compacto y $C_p(X)$ es fuertemente dominado por un espacio segundo numerable entonces X es numerable y por ende $C_p(X)$ es metrizable. El doctorando ha resuelto el problema [COT, Problem 4.11], en otras palabras, ha logrado probar en ZFC que para cualquier espacio compacto X el espacio $C_p(X)$ es fuertemente dominado por un espacio segundo numerable si, y sólo si, es metrizable y por lo tanto X es numerable. Los autores de [COT] preguntan también si la misma conclusión se puede alcanzar si no se supone que X es compacto. El doctorando ha obtenido algunas soluciones parciales a ese y otros problemas abiertos que aparecen en [COT].

El siguiente resultado está demostrado en el Capítulo 1, en él se sintetiza lo que se sabe de los espacios para los cuales el índice *sdm* es numerable. Nótese que la afirmación (IV) resuelve el problema [COT, Problem 4.11].

Teorema 1.4.24. *Para cualquier espacio X tal que $C_p(X)$ es fuertemente dominado por un espacio segundo numerable, se cumple lo siguiente:*

(I) *Si X es separable entonces es numerable.*

(II) *Si X es disperso entonces es numerable.*

(III) *Toda imagen continua segundo numerable de X es numerable.*

(IV) *Si X es compacto entonces es numerable.*

(V) *Si X es pseudocompacto entonces es numerable.*

(VI) *Si $K \subset X$ es compacto entonces K es disperso.*

(VII) *Si X es Lindelöf-p entonces X es igual a la unión de una familia numerable de sus subconjuntos compactos dispersos.*

Los autores de [COT] demostraron que para cualquier espacio X de Tychonoff tenemos que $\ell\Sigma(C_p(X)) = \omega$ si, y sólo si, $dm(C_p(X)) = \omega$. En [CMO] los autores prueban implícitamente que si X es un espacio angélico entonces $\ell\Sigma(X) = dm(X)$. Esto implica, en particular, que para cada espacio compacto o segundo numerable X tenemos que $\ell\Sigma(C_p(X)) = dm(C_p(X))$. De modo que la siguiente pregunta surge de manera natural.

> ¿Es cierto que $\ell\Sigma(C_p(X)) = dm(C_p(X))$ para todo espacio de Tychonoff X?
> Ver Problema 5.

Es evidente que la fuente más importante de dominación son los mapeos compacto-valuados superiormente semicontinuos sobreyectivos y la clase más importante de tales mapeos son los mapeos continuos sobreyectivos. Por lo tanto, decidimos imitar el estudio del cardinal $\ell\Sigma$ realizado en [CMO] y estudiar el índice cósmico definido por:

Definición 1.4.31. *Dado un espacio topológico X denotamos el índice cósmico de X por $mi(X)$ y lo definimos por:*

$$mi(X) = min\{w(Y) : M \text{ es un espacio métrico que se condensa sobre } X\}.$$

En este contexto probamos que si K es un espacio compacto de Corson entonces $mi(K) = w(K)$, ver el Corolario 1.4.40.

Espacios Eberlein-Grothendieck dispersos

Según la definición de Arhangel'skii los espacios de Eberlein-Grothendieck son aquellos homeomorfos a un subespacio de $C_p(K)$ para algún espacio compacto K. Notemos que si X es un subconjunto de un espacio de Banach E dotado de la topología débil entonces X está inmerso en $C_p(\mathbf{B}_{E^*})$ y por ende X es un espacio de Eberlein-Grothendieck. El principal propósito del Capítulo 2 es presentar y estudiar el siguiente problema:

Problema 2.3.1 *Dado un espacio X de Eberlein-Grothendieck disperso. ¿Es cierto que X es σ-discreto?*

Un caso particular de este problema ha sido publicado en [Hay] donde Haydon pregunta si para cada compacto K el espacio $C_p(K,\{0,1\})$ es σ-discreto siempre que es σ-disperso. Esta clase de preguntas está relacionada con las siguientes nociones dadas a conocer por J.E. Jayne, I. Namioka y C.A. Rogers en [JNR].

Definición 2.2.2. *Dado un conjunto X, una métrica ρ en X y $\varepsilon > 0$, una familia $\mathscr{A}$ de subconjuntos de X es ε-pequeña si $diam_\rho(A) < \varepsilon$ para cada $A \in \mathscr{A}$. Un espacio topológico (X,τ) tiene la propiedad SLD con respecto a una métrica ρ en X si para cada $\varepsilon > 0$ existe un cubrimiento numerable $\{X_n : n \in \omega\}$ de X tal que para cada $n \in \omega$ el subespacio X_n admite un cubrimiento τ-abierto ε-pequeño. Por otra parte, un espacio topológico (X,τ) es σ-fragmentable por una métrica ρ en X si para todo $\varepsilon > 0$ existe un cubrimiento numerable $\{X_n : n \in \omega\}$ de X tal que para cada $n \in \omega$ y cada $Y \subset X_n$ existe un subconjunto U no vacío τ-abierto de Y con $diam_\rho(U) < \varepsilon$.*

Es claro que si un espacio topológico tiene la propiedad SLD con respecto a alguna métrica entonces es σ-fragmentable con respecto a la misma métrica, sin embargo la siguiente pregunta sigue abierta:

Problema 2.2.3. *¿Son equivalentes las propiedades de σ-fragmentabilidad y la propiedad SLD en los espacios de Banach con la topología débil y la métrica de la norma, o bien en los espacios de la forma $C_p(K)$ donde K es compacto y se considera la métrica de la convergencia uniforme?*

Este problema tiene su origen en la teoría de renormamiento de espacios de Banach. Se cree que estas propiedades pueden dar una caracterización interna de los espacios de Banach que admiten una norma con la propiedad de Kadets-Klee. Para mayor información acerca de este tema referimos al lector a [MOTV, Section 3.2, p.54]. Es fácil ver que si un espacio con la métrica discreta es SLD entonces es σ-discreto y si es σ-fragmentable entonces es σ-disperso. La relación entre los dos problemas citados se muestra por medio de la siguiente pregunta que sigue abierta:

Problema 2.2.4. *¿Si $C_p(K,\{0,1\})$ es σ-disperso (respectivamente σ-discreto), esto implica que $C_p(K)$ es σ-fragmentable (respectivamente SLD)?*

Una versión de este problema considera la topología débil en $C(K)$ en lugar de la topología de la convergencia puntual. Sabemos que la respuesta es afirmativa cuando K es disperso (ver [Hay] y [Mtz]). Puesto que la restricción de la métrica uniforme de $C_p(K)$ a $C_p(K,\{0,1\})$ es la métrica discreta, responder afirmativamente a los dos problemas citados aportaría una respuesta afirmativa al tercer problema en el caso de espacios de funciones.

Observemos que, a partir de resultados conocidos, se sigue que nuestro problema principal tiene una respuesta afirmativa cuando X es compacto: Alster probó en [Al] que un espacio compacto disperso de Eberlein K es fuertemente Eberlein lo cual implica que K está inmerso en $\{0,1\}^\Gamma$ para algún Γ y $|supp(x)| < \omega$ para cada $x \in K$. Dado $n \in \omega$ podemos definir $X_n = \{x \in K : |supp(x)| = n\}$ de modo que $K = \bigcup_{n\in\omega} X_n$ donde cada X_n es discreto. En el Capítulo 2 probaremos algunas generalizaciones de este hecho tales como:

Teorema 2.2.5. *Si X es un espacio Eberlein-Grothendieck localmente compacto y disperso de cardinalidad ω_1, entonces X es σ-discreto.*

Además, en el Capítulo 2 demostramos lo siguiente:

Teorema 2.2.6. *Si X es un espacio Eberlein-Grothendieck localmente numerable y disperso de cardinalidad ω_1, entonces X es σ-discreto.*

Recordemos que una sucesión transfinita $\{x_\alpha : \alpha < \lambda\}$ de elementos de un espacio topológico es derecha si para cada $\mu < \lambda$ existe un abierto U para el cual tenemos que $U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}$. Un espacio topológico es disperso si, y sólo si, se puede escribir como una sucesión derecha $X = \{x_\alpha : \alpha < \lambda\}$. Desde este punto de vista el teorema anterior implica que nuestro problema principal tiene solución positiva en el primer caso no trivial, cuando $\lambda = \omega_1$.

Corolario 2.2.8. *Si $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$ es una ω_1-sucesión derecha, entonces X es σ-discreto.*

En ambos resultados mencionados anteriormente, X es homeomorfo a algún espacio $X' \subset C_p(K)$ donde K tiene peso ω_1. Por [DJP, Theorem 1.2] el espacio X es hereditariamente meta-Lindelöf. Esta es la hipótesis que de hecho se usó para probar los citados resultados, de modo que el teorema sobre los espacios dispersos localmente compactos de Eberlein-Grothendieck se ha demostrado aplicando las ideas desarrolladas en [Al] para mostrar que cada cubrimiento abierto de un espacio localmente compacto disperso hereditariamente meta-Lindelöf de cardinalidad no mayor a ω_1 tiene un refinamiento punto-finito; mientras que el resultado sobre los espacios dispersos localmente numerables de Eberlein-Grothendieck se obtiene mostrando que los espacios hereditariamente meta-Lindelöf localmente numerables dispersos son σ-discretos. El resultado anterior está publicado en [HP] sin demostración, sin embargo, el argumento que se sugiere no parece ser correcto. Una aplicación a los espacios de funciones se ha obtenido por el doctorando a través de un corolario que muestra que, al menos cuando K es disperso, la propiedad SLD de $C_p(K)$ puede ser caracterizada como una especie de ω_1-σ-fragmentabilidad.

Adicionalmente se ha considerado otra generalización de los espacios compactos de Eberlein dispersos: la clase de los espacios Eberlein-Grothendieck dispersos, con la propiedad Lindelöf Σ. En este caso es posible mostrar que estos espacios son σ-discretos como una consecuencia sencilla de los resultados en [Ha] y [Ny]. Más aún, para el subcaso de los espacios Eberlein-Grothendieck Lindelöf Čech-completos dispersos, se recupera el hecho conocido de que tales espacios son σ-compactos aplicando los métodos de juegos topológicos desarrollados por R. Telgarsky en [Te].

Los resultados mencionados dependen fuertemente de que los espacios considerados son hereditariamente meta-Lindelöf. Por lo tanto, parece natural preguntarnos si tal propiedad es suficiente para conseguir que un espacio disperso sea σ-discreto. En otras palabras:

> ¿Es cierto que cada espacio Eberlein-Grothendieck hereditariamente meta-Lindelöf disperso es σ-discreto? Ver Problema 9.

En el Capítulo 2 se muestra que este no es el caso en general, para tal efecto se construye un ejemplo de un espacio X hereditariamente meta-lindelöf disperso Hausdorff X que no es σ-discreto. Sin embargo, aún no es claro si tal espacio X es o no Eberlein-Grothendieck.

El Problema 2.3.1, presentado al inicio de este apartado, aún no ha sido resuelto, y todavía hay una gran variedad de subcasos a considerar. El más importante de los subcasos es el de los espacios de Lindelöf que nos lleva a la siguiente pregunta natural:

> ¿Es cierto que todo espacio disperso Eberlein-Grothendieck Lindelöf es σ-discreto? Ver Problema 6.

Cubrimientos conservativos de espacios de funciones

Recordemos que dado un espacio X, una familia $\mathscr{F}$ de subconjuntos de X es conservativa si para cada $\mathscr{G} \subset \mathscr{F}$ sucede que $\overline{\bigcup G} = \bigcup\{\overline{G} : G \in \mathscr{G}\}$. Es claro que si D es denso en X entonces $\{D \cup \{x\} : x \in X\}$ es un cubrimiento conservativo de X. A su vez, si $X = \bigcup\{F_n : n \in \omega\}$ donde F_n es compacto para cada $n \in \omega$ entonces la familia $\{G_n : n \in \omega\}$ donde $G_n = F_0 \cup \ldots \cup F_n$ es un cubrimiento compacto conservativo de X.

Un estudio de los espacios que se pueden representar como unión conservativa de subespacios buenos se ha realizado inicialmente en el contexto del estudio de las propiedades de cubrimiento. Potozny y Junnila, e independientemente Katuta, demostraron, en [PJ] y [Ka] respectivamente, que un espacio de Hausdorff debe ser metacompacto si admite un cubrimiento conservativo por subespacios compactos. Potozny construyó en [Pot] un ejemplo de un espacio que no es normal y tiene un cubrimiento así. Smith y Telgarsky establecieron en [ST] que la propiedad de tener un cubrimiento conservativo de subespacios compactos se conserva bajo σ-productos.

Los cubrimientos conservativos también resultan de importancia en el análisis funcional, como lo demostró Yakovlev quien probó en [Ya] que un espacio compacto con un cubrimiento conservativo por subconjuntos finitos debe ser un compacto de Eberlein. Es una práctica común en topología estudiar las propiedaddes de un espacio expresándolo como la unión de subespacios con buenas propiedades. Tkachuk demostró en [Tk3] que un gran número de propiedades topológicas no aditivas se conservan por uniones numerables en espacios de la forma $C_p(X)$. En particular, un espacio $C_p(X)$ es metrizable si se puede expresar como la unión de una familia numerable de sus subespacios primero numerables; además, el espacio X debe ser finito si $C_p(X)$ es σ-numerablemente compacto.

En el artículo [Gue] el doctorando estudió sistematicamente los espacios $C_p(X)$ representables como la unión de una familia conservativa de subespacios con buenas propiedades. En la mayoría de los casos los cubrimientos conservativos cerrados constituyen una generalización de las cubiertas cerradas numerables de modo que el estudio de los espacios $C_p(X)$ con tales cubrimientos es un buen prospecto para generalizar los resultados de Tkachuk en [Tk2] y [Tk3]. En [Gue], se demostró, entre otras cosas, que el espacio X debe ser finito si $C_p(X)$ es unión de una familia conservativa de sus subespacios compactos.

En [GT] Tkachuk y el doctorando continuaron el trabajo hecho en [Gue]. Juntos resolvieron varias preguntas abiertas planteadas en [Gue] al mostrar que un buen número de propiedades topológicas en $C_p(X)$ se conservan por las uniones de cubrimientos conservativos cerrados. En particular, para cualquier espacio X de Tychonoff, si el espacio $C_p(X)$

es la unión de una familia conservativa cerrada de sus subespacios cósmicos (o primero numerables) entonces $C_p(X)$ mismo es cósmico (o segundo numerable respectivamente). Es inmediato que si un espacio Y tiene un subespacio denso con una propiedad $\mathscr{P}$ que se conserva por uniones finitas y que todo unipuntual la tiene, entonces Y puede ser representado como la unión de una familia conservativa de subespacios con la propiedad $\mathscr{P}$. Los autores de [GT] mostraron que, en muchos casos, para los espacios $Y = C_p(X)$ el recíproco también es cierto. En particular, $C_p(X)$ admite un cubrimiento conservativo de subespacios separables (o Lindelöf) si, y sólo si, $C_p(X)$ es separable (o tiene un subespacio denso Lindelöf respectivamente).

En el caso de que un espacio $C_p(X)$ admite un cubrimiento conservativo cerrado por subespacios con una cierta propiedad $\mathscr{P}$ a menudo sucede que $C_p(X,\mathbb{I})$ tiene la propiedad $\mathscr{P}$. Tkachuk y el doctorando encontraron que un buen número de resultados clásicos acerca de ciertas propiedades $\mathscr{P}$ en $C_p(X)$ no se extienden automáticamente a espacios X tales que $C_p(X)$ tiene un cubrimiento conservativo cerrado con la propiedad $\mathscr{P}$. En particular, no es claro si en estos resultados es posible sustituir $C_p(X)$ por $C_p(X,\mathbb{I})$.

Estudiar cubrimientos conservativos de espacios de funciones ha resultado ser una tarea de grandes dimensiones y el trabajo realizado por el candidato de manera individual y en colaboración con V. Tkachuk sólo araña la superficie del tema. El candidato ha probado por ejemplo el siguiente resultado:

Corolario 3.3.10. *Si K es un espacio compacto, entonces $C_p(K)$ admite un cubrimiento conservativo por subespacios cerrados segundo numerables si, y sólo si, K es numerable.*

Aún no es claro si podemos prescindir de la hipótesis de que los elementos del cubrimiento conservativo de $C_p(K)$ en el corolario anterior son cerrados. Lo cual implica la siguiente pregunta todavía sin respuesta, que tal vez sea la más importante dentro del tema de los cubrimientos conservativos de los espacios de funciones:

> ¿Es posible cubrir al espacio $C_p([0,1])$ con una familia conservativa de subespacios segundo numerables? Ver Problema 20.

Juegos topológicos en espacios de funciones

En el contexto de las descomposiciones conservativas de los espacios topológicos Z por subespacios compactos de Z, es posible definir un juego topológico en Z de manera natural, que es una ligera variación del estudiado por Telgarsky en [Te]. En este juego el primer jugador tiene una estrategia ganadora. Por lo tanto estudiar juegos análogos en espacios de funciones abre la posibilidad de fortalecer resultados sobre cubrimientos conservativos.

En un espacio Y, consideremos una familia $\mathscr{C} \subset \exp(Y)$. Definimos el juego $\mathscr{G}(\mathscr{C},Y)$ para dos jugadores I y II quienes alternan jugadas de la siguiente manera: en la jugada número n, el Jugador I elige $C_n \in \mathscr{C}$ y el Jugador II elige un conjunto $U_n \in \tau(C_n,Y)$. El juego termina cuando se ha realizado la n-ésima jugada para cada $n \in \omega$. El Jugador I gana si $Y = \bigcup\{U_n : n \in \omega\}$; en caso contrario el ganador es el Jugador II.

Una estrategia t para el primer jugador en el juego $\mathscr{G}(\mathscr{C},X)$ sobre un espacio X se define inductivamente de la siguiente manera. En primer lugar es necesario elegir un conjunto $t(\varnothing) = F_0 \in \mathscr{C}$. Un conjunto abierto $U_0 \in \tau(X)$ se llama adecuado si $F_0 \subset U_0$. Para cada conjunto adecuado U_0 se define un conjunto $t(U_0) = F_1 \in \mathscr{C}$. Supongamos que se han definido sucesiones adecuadas finitas $(U_0,...,U_i)$ y conjuntos $t(U_0,...,U_i) \in \mathscr{C}$ para cada $i \leq n$. La sucesión finita $(U_0,...,U_{n+1})$ es adecuada siempre que la sucesión $(U_0,...,U_i)$ lo es también para cada $i \leq n$ y $F_{n+1} = t(U_0,...,U_n) \subset U_{n+1}$. Una estrategia t para el Jugador I es ganadora si asegura la victoria de este jugador en cada partido en el cual I la utiliza.

Una estrategia s para el Jugador II en el juego $\mathscr{G}(\mathscr{C},X)$ sobre un espacio X es simplemente una función que a cada sucesión finita $(F_0,\dots,F_n)$ de elementos de $\mathscr{C}$ le asigna un conjunto $U \in \tau(F_n,X)$. Una tal estrategia para el Jugador II es ganadora si le asegura la victoria en cada partida en que II la usa.

El estudio de los juegos topológicos en espacios de funciones ha probado ser fructífero. Por ejemplo, en [Gue] el autor de esta tesis probó que la σ-compacidad de los espacios de funciones se puede caracterizar por medio de la existencia en tales espacios de estrategias ganadoras para ciertos juegos topológicos estudiados previamente por R. Telgarsky en [Te] y Potozny en [Pot2].

Es un procedimiento estándar verificar que un espacio X es Lindelöf si, y sólo si, el Jugador I tiene una estrategia ganadora para el juego $\mathscr{G}(\mathscr{L},X)$ definido en el Capítulo 3 donde $\mathscr{F}$ es la familia de todos los subespacios de Lindelöf no necesariamente cerrados de X.

> ¿Es posible caracterizar otras propiedades topológicas de los espacios de funciones de manera análoga?

En [Gue] también se establece que si X es no vacío y $\mathscr{F} \subset \exp(C_p(X,\mathbb{I}))$ y además el Jugador I tiene una estrategia ganadora para el juego $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ entonces existe $F \in \mathscr{F}$ que no es denso en ninguna parte en $C_u(X,\mathbb{I})$. Un hecho análogo se establece para $C_p(X)$.

Estos hechos permiten al doctorando demostrar lo siguiente:

Corolario 3.5.18. *Supongamos que $\mathscr{P}$ es una propiedad topológica hereditaria y que $\mathscr{C}$ es una familia cerrada de subconjuntos de $C_p(X)$ o $C_p(X,\mathbb{I})$ tal que cada $C \in \mathscr{C}$ tiene $\mathscr{P}$. Si el Jugador I tiene una estrategia ganadora en el juego $\mathscr{G}(\mathscr{C},C_p(X)$ o $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ entonces $C_p(X)$ también tiene la propiedad $\mathscr{P}$.*

Observación 3.5.19. *Supongamos que κ es un cardinal infinito. Notemos que el Corolario 3.5.18 es aplicable, por ejemplo, a las siguientes propiedades: peso $\leq \kappa$, peso de red $\leq \kappa$, i-peso $\leq \kappa$, número diagonal $\leq \kappa$, carácter $\leq \kappa$, pseudocarácter $\leq \kappa$, estrechez $\leq \kappa$, amplitud $\leq \kappa$, número hereditario de Lindelöf $\leq \kappa$, densidad hereditaria $\leq \kappa$, κ-monoliticidad, metrizabilidad, la propiedad de Fréchet-Urysohn, diagonal pequeña, realcompacidad hereditaria, la propiedad de Whyburn.*

En [Tk2, Example 15] se afirma que si K es el conjunto de Cantor, entonces $C_p(K)$ tiene una familia numerable $\{F_n : n \in \omega\}$ de conjuntos cerrados tal que $\bigcup_{n\in\omega} F_n = C_p(K)$ y cada F_n tiene una π-base numerable pero $C_p(K)$ no tiene una π-base numerable. Es fácil ver que esto implica que el primer jugador tiene una estrategia ganadora en el juego $\mathscr{G}(\mathscr{F},C_p(K))$ donde $\mathscr{F}$ es la familia de todos los subconjuntos cerrados de $C_p(K)$ con π-peso numerable. Podemos concluir que si una propiedad $\mathscr{P}$ no es hereditaria y $\mathscr{F}$ es la familia de todos los subconjuntos cerrados de $C_p(X)$ que tienen la propiedad $\mathscr{P}$, si el Jugador I tiene una estrategia ganadora para el juego $\mathscr{G}(\mathscr{F},C_p(X)$ entonces $C_p(X)$ no necesariamente tiene $\mathscr{P}$.

Sin embargo, para las propiedades heredadas por subespacios cerrados podemos proceder de la misma manera que en la Sección 2 de [GT] para obtener lo siguiente. Dado un espacio no vacío X y una propiedad hereditaria por subespacios cerrados $\mathscr{P}$, llamemos $\mathscr{F}$ a la familia de todos los subespacios cerrados de $C_p(X,\mathbb{I})$ que tienen $\mathscr{P}$. Si el Jugador I tiene una estrategia ganadora para el juego $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ entonces $C_p(X,\mathbb{I})$ también tiene la propiedad $\mathscr{P}$. Podemos nombrar algunas de estas propiedades: extent $\leq \kappa$, número de Nagami $\leq \kappa$, K-analiticidad, índice de K-determinación $\leq \kappa$, índice de dominación (fuerte) por un métrico $\leq \kappa$, normalidad, secuencialidad.

Una vez más, siguiendo los argumentos presentados en la Sección 2 de [GT] podemos percatarnos de que para algunas propiedades es posible abundar. Si $\mathscr{F}$ es una familia cerrada de subconjuntos de $C_p(X,\mathbb{I})$ para la cual el Jugador I tiene una estrategia ganadora en el juego $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ y cada $F \in \mathscr{F}$ es realcompacto entonces $C_p(X)$ es realcompacto. Si cada $F \in \mathscr{F}$ es Čech-completo, entonces X es discreto. Dado un espacio X, si sucede que $\mathscr{F}$ está integrada por subespacios σ-numerablemente compactos cerrados, entonces $C_p(X,\mathbb{I})$ es numerablemente compacto. Mientras que, si los elementos de $\mathscr{F}$ son σ-compactos entonces X es discreto.

A pesar de todas las propiedades antes mencionadas que se caracterizan por medio de juegos topológicos, aún no es claro si la propiedad Lindelöf Σ puede ser caracterizada también de manera análoga. En otras palabras:

> Supongamos que el Jugador I tiene una estrategia ganadora para el juego $\mathscr{G}(\mathscr{F}, C_p(X))$ donde $\mathscr{F}$ es la familia de todos los subconjuntos cerrados de $C_p(X)$ que tienen la propiedad Lindelöf Σ. ¿Es cierto que $C_p(X)$ debe ser Lindelöf Σ? Ver Problema 21.

Problemas abiertos

A continuación enumeramos los problemas más importantes que han quedado abiertos dentro de las líneas de investigación desarrolladas en esta tesis. Hasta el momento ya se han enunciado la mayoría de tales problemas y se ha descrito el progreso realizado hacia su solución en las subsecciones previas de esta introducción.

Problema 2 [COT, Problem 3.10]. *Supongamos que X es un espacio de Tychonoff tal que $C_p(X)$ es fuertemente dominado por un espacio segundo numerable ¿Debe X ser numerable?*

Problema 2.3.1. *Supongamos que X es un espacio disperso de Eberlein-Grothendieck. ¿Debe X ser σ-discreto?*

Problema 20. *¿Es posible cubrir al espacio $C_p([0,1])$ con una familia conservativa de subespacios segundo numerables?*

Como ya se mencionó, hemos demostrado que los subespacios segundo numerables del cubrimiento conservativo en cuestión son cerrados, no es posible responder afirmativamente a la pregunta anterior.

Problema 21. *Supongamos que para un espacio X el Jugador I tiene una estrategia ganadora en el juego $\mathscr{G}(\mathscr{F}, C_p(X))$ donde $\mathscr{F}$ es una familia cerrada de subespacios Lindelöf Σ de $C_p(X)$. ¿Debe $C_p(X)$ ser un espacio Lindelöf Σ?*

En la sección correspondiente del Capítulo 2 se muestra un resultado positivo para el caso $dm(X) = \omega$ el cual incluye el caso cuando X mismo es un espacio Lindelöf Σ.

Introduction

Our notation and terminology is standard. Otherwise we refer the reader to the Notation and terminology section. Our reference books are [Ar2], [En] and [Tk6].

One of the main purposes of General Topology is to characterize and classify topological spaces. In order to achieve this goal specialists have applied several methods, theoretical knowledge and tools developed within the scope not only of the General Topology, but also of other fields of Mathematics such as Functional Analysis, Topological Algebra, Set Theory, and Game Theory to name just a few.

This is the theoretical background that gave rise to the theory of function spaces over Tychonoff spaces endowed with the topology of pointwise convergence also known as C_p-theory. These spaces ($C_p(X)$-spaces) have a very rich topological and algebraic structure which is so transcendent that it determines to a large extent the topology of Tychonoff spaces as Nagata showed by proving that two function spaces $C_p(X)$ and $C_p(Y)$ are topologically isomorphic if and only if X and Y are homeomorphic.

Furthermore, in the middle of the last century, specialists in Functional Analysis posed problems that are purely topological and arise naturally in their field. Thus, the development of the resulting techniques improved knowledge in Functional Analysis and gave birth to new tools in General Topology. From this process of mutual enrichment came the study of the classes of compact spaces like Eberlein, Talagrand, Gul'ko and Corson compacta for instance (see [Eb], [Ta], [Gu], [Co], [AMN], [Sc]). These classes of compact spaces constitute a rich source of examples in General Topology.

One of the earliest results that showed a very close relationship between Functional Analysis and General Topology was the characterization of compact subsets of Banach spaces with the weak topology (later called Eberlein compact spaces) in purely topological terms. On one hand Amir and Lindenstrauss proved that X is an Eberlein compact space if and only if there is an embedding of X into a Σ_*-product of real lines (see [AL]). Meanwhile Rosenthal characterized Eberlein compacta by means of σ-point-finite separating families of cozero sets (see [Ro]).

Corson compacta represent a larger class than Eberlein compact spaces, but they still have quite good categorical properties. For instance, if X is a Corson compact space then $C_p(X)$ is Lindelöf and $d(X) = w(X)$ (See [Ar2]). In particular, if a Corson compact space is separable then it is metrizable. It is an amazing fact that Corson compact spaces, which seemed to be a class studied exclusively in Functional Analysis, turned out to be very useful to establish characterizations of topological properties of arbitrary compact spaces. To illustrate the later we can mention a famous theorem by Shapirovsky which states that a compact space K has countable tightness if and only if there exists a continuous irreducible map from X onto a Corson compact space (see [Sh]).

A large amount of study in General Topology is devoted to the compact spaces K such that $C_p(K)$ is a Lindelöf Σ space, such a space is called a Gul'ko compactum. Research on this spaces from a purely topological point of view resulted in a profound theorem by Gul'ko which states that every compact space X such that $C_p(X)$ is Lindelöf Σ is a Corson compact space (see [Gu]). That is to say, a class that emerged within the field of General Topology happens to be an important part of the hierarchy of the classes of compact spaces studied in Functional Analysis.

Topologists obtained important results as well about Eberlein compacta. Arhangel'skii proved that if K is an Eberlein compact space then $C_p(X)$ is a $K_{\sigma\delta}$ space and in particular it is K-analytic (see [Ar2] and [Ta0]). In this line, from the point of view of the study of Banach spaces, Talagrand proved that weakly compactly generated Banach spaces are weakly K-analytic (See [Ta0]). Benyamini, Rudin and Wage proved that any continuous image of an Eberlein compact space is again an Eberlein compact space (see [BRW]).

The Lindelöf Σ property was studied in parallel both in General Topology and in Descriptive Set Theory (under the name of countable K-determined spaces). Over the last 30 years topologists and specialists in Descriptive Set Theory realized that their results were closely related. This interaction resulted in the rise of the Descriptive Set Theory in Function Spaces.

Within the theoretical background described, the candidate has pursued a few lines of research. The nature, objectives, known results and open problems of this research are described in the next subsections of this introduction. More importantly, in this chapter we also outline the most important original results the author of this text has achieved in those areas, in the course of his doctoral studies and research.

Some authors consider that a decomposition of a topological space is a family of subspaces of the space, often a partition. Such is not the case here. In this text we recover the spirit of [Tk2] and [Tk3] so that a decomposition of a topological space is understood as a process by which families of subsets of the space are obtained. This is how we understand the idea of a decomposition, which is the leading idea for all the research reported in this thesis. We do not define formally the term decomposition, certainly not as a family of subspaces of a given space. We do define formally the families we are interested in and describe the way to decompose the spaces that allows us to obtain such families.

The philosophy just described is adopted here because a strategy that has proven to be efficient in order to study and characterize topological spaces and function spaces is to decompose them to obtain certain subspaces and study the nature of such decompositions. There are several examples of how such decompositions determine to a large extent the topology of the larger spaces. For instance, splitting a space X into its compact subspaces can be done to get closure-preserving covers or M-ordered covers. In each case the existence of such decomposition has deep implications about the topological properties of the space X. Furthermore if the space that has such a compact cover, either M-ordered or closure-preserving, is of the form $C_p(Y)$ then the structure of it allows us to conclude quite strong conditions on its topology.

The results included in this thesis are presented along with some reference to where they were previously published. When no reference is offered, the corresponding result appears in this text for the first time.

This thesis has three main chapters.

In Chapter 1 the first 3 sections are devoted to settle the theoretical background within which the research was developed. Section 1.4 contains already published original results of the author of this thesis, as well as new unpublished results.

Chapter 2 starts with a section of preliminary results and the last 3 sections of it gather quite a few of the published original results on the subject obtained in collaboration with A. Avilés.

The first 2 sections of Chapter 3 include published results by the author some in collaboration with V. Tkachuk. Such content is part of the theoretical background applied to obtain the new original results yet unpublished that appear in Section 3.3. Finally in Section 3.4 published results obtained in a collaboration with A. Avilés are included.

Remaining open problems are posed in the corresponding section and some conclusions are provided as well.

We now describe in detail the contents of the thesis.

Domination in function spaces

Given a space Z the family $\mathscr{K}(Z)$ consists of all compact subsets of Z. A family $\mathscr{A}$ is called fundamental if for every $K \in \mathscr{K}(Z)$ there is $A \in \mathscr{A}$ such that $K \subset A$. If all elements of a cover $\mathscr{C}$ of X are compact then the family $\mathscr{C}$ is called compact. Whereas a family $\mathscr{B}$ is M-ordered for some space M if $\mathscr{B} = \{B_K : K \in \mathscr{K}(M)\}$ while $K \subset L$ implies $B_K \subset B_L$. A space X is dominated by a space M if it has an M-ordered compact cover. Say that X is strongly M-dominated if it has an M-ordered fundamental compact cover $\mathscr{C}$ (see [COT]).

Given a space X let M be the space obtained by giving the discrete topology to the set $\mathscr{K}(X)$ of all the compact subsets of X. It is clear that X is strongly dominated by M. The fact that every topological space is dominated by a metric space motivates the following.

Definition 1.4.1. *For a space X the metric domination index of X denoted by $dm(X)$ is the cardinal defined by*

$$dm(X) = min\{w(M) : M \text{ is a metric space that dominates } X\}.$$

Analogously, we define the index of strong metric domination.

Definition 1.4.2. *For a space X the strong metric domination index of X denoted by $sdm(X)$ is the cardinal defined by*

$$sdm(X) = min\{w(M) : M \text{ is a metric space that strongly dominates } X\}.$$

Recall that in [CMO] upper semicontinuous compact-valued maps are called uscos and in the same paper the number of K-determination of a space X is denoted by $\ell\Sigma(X)$ and defined as

$$\ell\Sigma(X) = min\{w(M) : M \text{ is a metric space and } \exists \text{ a usco onto map } \varphi : M \to \exp(X)\}.$$

The general behaviour of dm can be summarized as follows:

Proposition 1.4.3. *For any space X the following hold:*

(I) $dm(X) \leq \ell\Sigma(X)$.

(II) *If $dm(X) \leq \kappa$ then the metric domination index of any continuous image of X is also not greater than κ.*

(III) *If $dm(X) \leq \kappa$ and Y is any closed subset of X then $dm(Y) \leq \kappa$.*

(IV) *If $X = \bigcup_{i \in \omega} X_i$ and $dm(X_i) \leq \kappa$ then $dm(X) \leq \kappa$.*

(V) *$ext(X) \leq dm(X)$.*

In the case of function spaces the study of *dm* and specially *sdm* may yield conditions for metrizability as it was proved under CH in [COT, Theorem 3.10]. The authors showed that if X is compact and $C_p(X)$ is strongly dominated by a second countable space then X is countable and hence $C_p(X)$ is metrizable. The candidate has solved [COT, Problem 4.11], in other words, he has been able to prove in ZFC that for any compact space X the space $C_p(X)$ is strongly dominated by a second countable space if and only if it is metrizable and therefore X is countable. The authors of [COT] also asked if the same conclusion can be drawn if we do not assume that X is compact. The candidate has achieved a few partial answers to that and some other open problems posed in [COT].

The following result is proved in Chapter 1 and summarizes what we know so far about function spaces with countable *sdm*. Notice that fact (IV) solves [COT, Problem 4.11].

Theorem 1.4.24. *For a space X such that $C_p(X)$ is strongly dominated by a second countable space, the following hold:*

(I) *If X is separable then it is countable.*

(II) *If X is scattered then it is countable.*

(III) *Every second countable continuous image of X is countable.*

(IV) *If X is compact then it is countable.*

(V) *If X is pseudocompact then it is countable.*

(VI) *If $K \subset X$ is compact then K is scattered.*

(VII) *If X is Lindelöf-p then X is the union of countably many compact scattered subsets.*

The authors of [COT] proved that for any Tychonoff space X we have $\ell\Sigma(C_p(X)) = \omega$ if and only if $dm(C_p(X)) = \omega$. In [CMO] the authors implicitly show that if X is an angelic space then $\ell\Sigma(X) = dm(X)$, see also [CKS]. This implies in particular that for every compact or separable space X we have $\ell\Sigma(C_p(X)) = dm(C_p(X))$. Thus the following general question arises naturally:

Is $\ell\Sigma(C_p(X)) = dm(C_p(X))$ for every Tychonoff space X? See Problem 5.

Clearly the most important source of domination are upper semicontinuous compact-valued onto maps and the most important class of these maps are continuous onto maps. Thus we decided to emulate the study of the cardinal $\ell\Sigma$ performed in [CMO] and study the cosmic index defined as:

Definition 1.4.31. *For any topological space X we denote the cosmic index of X as $mi(X)$ and define it by*

$$mi(X) = min\{w(Y) : M \textit{ is a metric space that condenses onto } X\}.$$

In this context we prove that if K is a Corson compact space then $mi(K) = w(K)$, see Corollary 1.4.40.

Eberlein-Grothendieck scattered spaces

According to Arhangel'skii the Eberlein-Grothendieck spaces as those homeomorphic to some subspace of $C_p(K)$ for some compact space K. Let us notice that if X is a subset of a Banach space E with the weak topology then X embeds in $C_p(\mathbf{B}_{E^*})$ hence X is Eberlein-Grothendieck. The main purpose of Chapter 2 is to study the following problem:

Problem 2.3.1. *Are Eberlein-Grothendieck scattered spaces σ-discrete?*

A particular case of this problem was posed in [Hay] where Haydon asked if for every compact K the space $C_p(K, \{0,1\})$ is σ-discrete whenever it is scattered. This kind of question is related to the following notions introduced by J.E. Jayne, I. Namioka and C.A. Rogers in [JNR].

Definition 2.2.2. *Given a set X, a metric ρ on X and $\varepsilon > 0$, a family $\mathscr{A}$ of subsets of X is ε-small if $diam_\rho(A) < \varepsilon$ for every $A \in \mathscr{A}$. A topological space (X, τ) has the property SLD with respect to a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ the space X_n admits a τ-open cover which is ε-small. On* *the other hand a topological space (X,τ) is σ-fragmented by a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ and every $Y \subset X_n$ there exists a non-empty relative τ-open subset U of Y with $diam_\rho(U) < \varepsilon$.*

It is clear that if a topological space has SLD with respect to some metric, then it is σ-fragmented as well, but the following is an open question:

Problem 2.2.3. *Are the properties of σ-fragmentability and SLD equivalent when X is a Banach space endowed with its weak topology and with its norm metric, or when X is of the form $C_p(K)$ endowed with the uniform metric?*

This problem has its origin in the theory of renorming of Banach spaces, as these properties are conjectured as possible internal characterizations of Banach spaces admitting a norm with the Kadets-Klee property. We refer to [MOTV, Section 3.2, p.54] for information about this topic. It is easy to see that if a space with the discrete metric is SLD then it is σ-discrete and if it is σ-fragmented then it is σ-scattered. The relationship between the two problems just stated can be shown by the following question which is still open as well:

Problem 2.2.4. *If $C_p(K,\{0,1\})$ is scattered (respectively σ-discrete), does it imply that $C_p(K)$ is σ-fragmentable (respectively SLD)?*

A version of this problem states the same thing considering the weak topology of $C(K)$ instead of the pointwise convergence topology. The answer is known to be positive when K is scattered (see [Hay] and [Mtz].) Since the restriction of the uniform metric of $C_p(K)$ to $C_p(K,\{0,1\})$ is discrete, a positive answer to problems 2.3.1 and 2.2.3 combined would give a positive answer to problem 2.2.4 stated in the case of spaces of continuous functions.

We observe that it follows from known results that the Problem 2.3.1 has positive solution when X is compact: Alster proved in [Al] that an Eberlein compact scattered space K is strong Eberlein which implies that K embeds into $\{0,1\}^\Gamma$ for some Γ and $|supp(x)| < \omega$ for every $x \in K$. For each $n \in \omega$ we can define $X_n = \{x \in K : |supp(x)| = n\}$ hence we can write $K = \bigcup X_n$ where each X_n is discrete. In chapter 2 we prove some generalizations of this fact such as:

Theorem 2.2.5. *If X is an Eberlein-Grothendieck locally compact scattered space of cardinality ω_1, then X is σ-discrete.*

Also, in the same chapter we prove:

Theorem 2.2.6. *If X is an Eberlein-Grothendieck locally countable scattered space of cardinality ω_1, then X is σ-discrete.*

Remember that a transfinite sequence $\{x_\alpha : \alpha < \lambda\}$ of elements of a topological space is right-separated if for every $\mu < \lambda$ there is an open set U for which we have $U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}$. A topological space is scattered if and only if it can be written as a right separated sequence $X = \{x_\alpha : \alpha < \lambda\}$. From this point of view the last stated Theorem implies Problem 2.3.1 has a positive solution in the first non-trivial case, when $\lambda = \omega_1$.

Corollary 2.2.8. *If $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$ is a right-separated ω_1-sequence, then X is σ-discrete.*

In both of the results mentioned above, X is homeomorphic to some $X' \subset C_p(K)$ where K has weight ω_1. By [DJP, Theorem 1.2] in this case the space X is hereditarily meta-Lindelöf. This is the hypothesis that was actually assumed, so that the result on locally compact Eberlein-Grothendieck scattered spaces is proved by applying the ideas developed in [Al] to show that every open cover of a hereditarily meta-Lindelöf locally compact scattered space of height at most ω_1 has a point-finite clopen refinement, while the result on locally countable Eberlein-Grothendieck scattered spaces is proved by showing that hereditarily meta-Lindelöf locally countable scattered spaces are σ-discrete. The latter fact is stated in [HP] without proof, but the argument that they suggest does not seem to be correct. An application to function spaces has been attained by the candidate via a corollary which shows that, at least when K is scattered, the SLD property of $C_p(K)$ can be characterized as a kind of ω_1-σ-fragmentability.

The candidate has also considered another generalization of the Eberlein compact scattered spaces which is the class of Eberlein-Grothendieck Lindelöf Σ scattered spaces. In this case it is possible to show that these spaces are σ-discrete as an easy consequence of the results in [Ha] and [Ny]. But for the special subcase of the Eberlein-Grothendieck Lindelöf Čech-complete scattered spaces the candidate has actually proved their σ-compactness applying the methods of topological games developed by R. Telgarsky in [Te].

Since the results aforementioned depend very strongly on the property of hereditary metalindelöfness of the Eberlein-Grothendieck spaces considered in this section, it seems natural to ask if such property is enough for a scattered topological space to be σ-discrete. In other words:

> Is every Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space σ-discrete? See Problem 9.

The candidate has shown that this is not the case in general by constructing a hereditarily meta-Lindelöf scattered Hausdorff space X that is not σ-discrete. However, it is not clear yet whether such space X is Eberlein-Grothendieck or not.

The problem 2.3.1 has not yet been answered, and there are still several subcases to be considered. The most important of the subcases is the Lindelöf one which leads to the following natural question:

> Is every Eberlein-Grothendieck Lindelöf scattered space σ-discrete? See Problem 6.

Closure-preserving decompositions of function spaces

Recall that given a space X, a family $\mathscr{F}$ of subsets of X is called closure-preserving if for any $\mathscr{G} \subset \mathscr{F}$ we have $\overline{\bigcup G} = \bigcup\{\overline{G} : G \in \mathscr{G}\}$. It is clear that if D is a dense subspace of X then $\{D \cup \{x\} : x \in X\}$ is a closure-preserving cover of X. Also if $X = \bigcup\{F_n : n \in \omega\}$ where F_n is compact for every $n \in \omega$ then the family $\{G_n : n \in \omega\}$ where $G_n = F_0 \cup \ldots \cup F_n$ is a closure-preserving compact cover of X.

A study of spaces that can be represented as the closure-preserving union of nice subspaces has initially been done in the context of covering properties. Potozny and Junnila, and independently Kakuta, proved, in [PJ] and [Ka] respectively, that a Hausdorff space must be metacompact if it admits a closure-preserving cover by compact subspaces. Potozny constructed in [Pot] an example of a non-normal space that has such a cover. Smith and Telgarsky established in [ST] that the property of having a closure-preserving cover by compact subsets is preserved by σ-products.

It also turned out that closure-preserving covers by compact subsets are of importance in functional analysis after Yakovlev showed in [Ya] that a compact space with a closure-preserving cover by finite sets must be Eberlein compact. It is a common practice in topology to study the properties of a space by expressing it as a union of nice subspaces. Tkachuk proved in [Tk3] that many non-additive topological properties are preserved by countable unions in spaces $C_p(X)$. In particular, a space $C_p(X)$ is metrizable if it can be represented as the countable union of its first countable subspaces; besides, the space X must be finite if $C_p(X)$ is σ-countably compact.

In the paper [Gue] the candidate systematically studied the spaces $C_p(X)$ representable as the union of a closure-preserving family of subspaces with nice properties. In most cases, the closure-preserving closed covers constitute a generalization of countable closed covers so the study of spaces $C_p(X)$ with such covers gives good prospects of generalizing Tkachuk's results in [Tk2] and [Tk3]. In [Gue], it was proved, among other things, that the space X must be finite if $C_p(X)$ is the union of a closure-preserving family of its countably compact subspaces.

In [GT] Tkachuk and the candidate continued the work done in [Gue]. They solved several open questions from [Gue] by showing that quite a few topological properties in $C_p(X)$ are preserved by the unions of closure-preserving closed covers. In particular, for any Tychonoff space X, if the space $C_p(X)$ is the union of a closure-preserving closed family of cosmic (or first countable) spaces then $C_p(X)$ is itself cosmic (or second countable respectively). It is straightforward to see that if a space Y has a dense subspace with a property $\mathscr{P}$ then it can be represented as the union of a closure-preserving family of spaces with the property $\mathscr{P}$. They showed that, in many cases, for the spaces $Y = C_p(X)$ the converse is also true. In particular, $C_p(X)$ is the union of a closure-preserving family of separable (or Lindelöf) subspaces if and only if $C_p(X)$ is separable (or has a dense Lindelöf subspace respectively).

In the case when a space $C_p(X)$ admits a closure-preserving closed cover by its subspaces with a property $\mathscr{P}$ it often happens that $C_p(X,\mathbb{I})$ has $\mathscr{P}$. Tkachuk and the candidate found out that quite a few classical theorems about a property $\mathscr{P}$ in $C_p(X)$ do not extend automatically to the spaces X such that $C_p(X)$ has a closure-preserving closed cover whose elements have $\mathscr{P}$. In particular, it is not clear whether in these results we can substitute $C_p(X)$ by $C_p(X,\mathbb{I})$.

Studying closure-preserving covers of function spaces has turned out to be a huge task and the work by the candidate alone and in collaboration with V. Tkachuk has only scratched its surface. The candidate proved, for example, the following result:

Corollary 3.3.10. *If K is a compact space then $C_p(K)$ admits a closure-preserving cover by closed second countable subspaces if and only if K is countable.*

It is not yet clear if we can remove the hypothesis that the elements of the closure-preserving cover of $C_p(X)$ in Corollary 3.3.10 are closed. This implies the following open question which may be the most important concerning closure-preserving cover of function spaces:

Can $C_p([0,1])$ be represented as the union of a closure-preserving family of its second countable subspaces? See Problem 20.

Topological games in function spaces

In the context of closure-preserving decompositions of topological spaces Z by compact subspaces of Z, a topological game on Z can be defined in a natural way which is a slight variation of the one studied by Telgarsky in [Te]. In this game the first player has a winning strategy. Therefore studying analogous games in function spaces gives a possibility to strengthen some results already mentioned.

On a Tychonoff space Y, consider a family $\mathscr{C} \subset \exp(Y)$. We define the game $\mathscr{G}(\mathscr{C},Y)$ of two players *I* and *II* who take turns in the following way: at the move number n, Player *I* chooses $C_n \in \mathscr{C}$ and Player *II* chooses a set $U_n \in \tau(C_n,Y)$. The game ends after the n-th move of each player has been made for every $n \in \omega$ and Player *I* wins if the space $X = \bigcup\{U_n : n \in \omega\}$; otherwise the winner is Player *II*.

A strategy t for the first player in the game $\mathscr{G}(\mathscr{C},Y)$ on a space X is defined inductively in the following way. First the set $t(\varnothing) = F_0 \in \mathscr{C}$ is chosen. An open set $U_0 \in \tau(X)$ is legal if $F_0 \subset U_0$. For every legal set U_0 the set $t(U_0) = F_1 \in \mathscr{C}$ has to be defined. Let us assume that legal sequences $(U_0,...,U_i)$ and sets $t(U_0,...U_i)$ have been defined for each $i \leq n$. The sequence $(U_0,...U_{n+1})$ is legal if so is the sequence $(U_0,...,U_i)$ for each $i \leq n$ and $F_{n+1} = t(U_0,...,U_n) \subset U_{n+1}$. A strategy t for Player *I* is winning if it ensures victory for *I* in every play it is used.

It is customary to define a strategy s for Player *II* in the game $\mathscr{G}(\mathscr{C},Y)$ on a space X is simply a function that assigns to every finite sequence $(F_0,\ldots,F_n)$ of elements of $\mathscr{C}$ an open set $U \in \tau(F_n,X)$. Such a strategy for Player *II* is winning if it ensures victory for *II* in every play it is used.

The study of topological games in function spaces has already proven fruitful. For instance, in [Gue] the author of this thesis proved that σ-compactness of function spaces can be characterized by the existence on those spaces of winning strategies of certain topological games studied previously by R. Telgarsky in [Te] and Potozny in [Pot2].

It is standard to verify that a space X is Lindelöf if and only if Player *I* has a winning strategy for the game $\mathscr{G}(\mathscr{L},X)$ where $\mathscr{F}$ is the family of all the Lindelöf not necessarily closed subspaces of X.

> Is it possible to characterize other topological properties of function spaces in an analogous way?

In [Gue] it is also established that if X is non-empty and $\mathscr{F} \subset \exp(C_p(X,\mathbb{I}))$ and Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ then there is $F \in \mathscr{F}$ that is not nowhere dense in $C_u(X,\mathbb{I})$. An analogous fact is also established for $C_p(X)$.

This fact allows the candidate to prove the following:

Corollary 3.5.18. *Suppose that $\mathscr{P}$ is a hereditary topological property and if $\mathscr{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ such that every $C \in \mathscr{C}$ has $\mathscr{P}$. If Player I has a winning strategy in the game $\mathscr{G}(\mathscr{C},C_p(X)$ or $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ then $C_p(X)$ also has the property $\mathscr{P}$.*

Remark 3.5.19. *Suppose that κ is an infinite cardinal. Notice that Corollary 3.5.18 applies, for instance, to the following properties: weight $\leq \kappa$, network weight $\leq \kappa$, i-weight $\leq \kappa$, diagonal number $\leq \kappa$, character $\leq \kappa$, pseudocharacter $\leq \kappa$, tightness $\leq \kappa$, spread $\leq \kappa$, hereditary Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$, κ-monolithicity, metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property.*

In [Tk2, Example 15] it is stated that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that this implies that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(K))$ where $\mathscr{F}$ is the family of all the closed subspaces of $C_p(K)$ with countable π-weight. We can conclude that if a property $\mathscr{P}$ is not hereditary and $\mathscr{F}$ is the family of all the subspaces of $C_p(X)$ that have $\mathscr{P}$, if Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X)$ then $C_p(X)$ does not necessarily have $\mathscr{P}$.

Nevertheless, for properties that are inherited by closed subspaces we can proceed in a similar way as in Section 2 of [GT] to observe the following. Given a non-empty space X and a closed-hereditary property $\mathscr{P}$, call $\mathscr{F}$ the family of all the closed subspaces of $C_p(X,\mathbb{I})$ that have $\mathscr{P}$. If Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ then $C_p(X,\mathbb{I})$ also has the property $\mathscr{P}$. We can name some of these properties: extent $\leq \kappa$, Nagami number $\leq \kappa$, K-analyticity, $\ell\Sigma \leq \kappa$, $\leq \kappa$, $mi \leq \kappa$, normality, sequentiality.

Again following the arguments presented in Section 2 of [GT] we notice that for some properties we can say even more. If $\mathscr{F}$ is a closed family of subsets of $C_p(X,\mathbb{I})$ for which

Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X,\mathbb{I}))$ and every $F \in \mathscr{F}$ is realcompact then $C_p(X)$ is realcompact. If it is the case that every $F \in \mathscr{F}$ is Čech-complete subspaces, then X is discrete. Given a space X, if it happens that every $\mathscr{F}$ is integrated by σ-countably compact subspaces, then $C_p(X,\mathbb{I})$ is countably compact. Whereas if the elements of $\mathscr{F}$ are σ-compact then X is discrete.

Despite the aforementioned topological properties characterized by means of topological games, it is not yet clear if the Lindelöf Σ property can also be characterized in the same manner. In other words:

> Suppose that Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F}, C_p(X))$ where $\mathscr{F}$ is the family of all the closed Lindelöf Σ subsets of $C_p(X)$; must $C_p(X)$ be Lindelöf Σ? See Problem 21.

Open Problems

Next we summarize the most important open problems that remain to be solved within the lines of research described above and developed in this thesis. We have already mentioned most of the following problems as well as the progress towards their solution in the previous subsections of this introductory text.

Problem 2 [COT, Problem 3.10]. *Suppose that X is a Tychonoff space for which $C_p(X)$ is strongly dominated by a second countable space. Must X be countable?*

Problem 2.3.1. *Suppose that X is an Eberlein-Grothendieck scattered space. Must X be σ-discrete?*

Problem 20. *Is it possible to cover $C_p([0,1])$ with a closure-preserving family of second countable subsets?*

As mentioned previously, we have already proved that if the second countable members of the closure-preserving family are closed then it is not possible to have an affirmative answer to the previous question.

Problem 21. *Suppose that for a space X Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$. Must $C_p(X)$ be a Lindelöf Σ-space?*

In the corresponding section of Chapter 2 it is shown a positive result for the case when $dm(X) = \omega$ which includes the case when X itself is a Lindelöf Σ space.

Notation and terminology

Our notation is standard and follows [En]; our reference book on C_p-theory is [Tk6]. Unless otherwise stated, every topological space in this text is assumed to be Tychonoff. The topology of X is denoted by $\tau(X)$ and $\tau^*(X)$ is the family of non-empty open subsets of X. For $C \subset X$ the family of all open sets of X that contain C is denoted by $\tau(C,X)$; if $x \in X$ then we write $\tau(x,X)$ instead of $\tau(\{x\},X)$. Given a cardinal κ and a discrete space D of cardinality κ denote by $A(\kappa)$ the Alexandrov one-point compactification of D and by $L(\kappa)$ the one-point Lindelöfication of D. Given a space X and an element $\alpha \in X$ the map $\chi_\alpha : X \to \{0,1\}$ is defined by $\chi_\alpha(\alpha) = 1$ and $\chi_\alpha(X \setminus \{\alpha\}) \subset \{0\}$. A map $\varphi : X \to \tau^*(X)$ such that $x \in \varphi(x)$ for every $x \in X$ is called a neighbourhood assignment of X. A space X is called a D-space if for every neighbourhood assignment φ of X there is a closed discrete set D such that $X = \bigcup_{x \in D} \varphi(x)$.

For every space X we denote by υX the real-compactification of the space X. A map $f : X \to Y$ is compact covering if every compact subset of Y is the image under f of some compact subset of X.

A space X is called scattered if every non empty subspace of X contains an isolated point. For a scattered space X the set of isolated points of X is denoted by $X^{(0)}$. Given an ordinal $\alpha < |X|^+$ suppose that we have defined the set $X^{(\beta)}$ for every $\beta < \alpha$ then the $X^{(\alpha)}$ is the set of isolated points of $X \setminus \bigcup_{\beta<\alpha} X^{(\beta)}$. The height of X is equal to κ if $\kappa = min\{\alpha < |X|^+ : X^{(\alpha)} = \varnothing\}$. The space $X^{(\alpha)}$ is called the α-th scattering level of the space X.

The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X,Y)$. The space $C_p(X,\mathbb{R})$ will be abbreviated by $C_p(X)$. For every $f \in C_p(X,Y)$, define the dual map $f^* : C_p(Y) \to C_p(X)$ by $f^*(g) = g \circ f$ for every $g \in C_p(Y)$. A continuous bijection is called a condensation. If there is a condensation $\varphi : X \longrightarrow Y$ we say that X condenses onto Y. If Y is a subspace of X we denote by $\pi_Y : C_p(X) \longrightarrow C_p(Y)$ the restriction map defined by $\pi_Y(f) = f|Y$ for any $f \in C_p(X)$. Given a set A, the space $\Sigma(\mathbb{R}^A)$ defined by $\Sigma(\mathbb{R}^A) = \{f \in \mathbb{R}^A : |\{x \in A : f(x) \neq 0\}| \leq \omega\}$ is called the Σ-product of real lines of weight $|A|$, whereas $\Sigma(2^A)$ stands for the set $\{f \in \{0,1\}^A : |\{x \in A : f(x) \neq 0\}| \leq \omega\}$. Recall that every Corson compact space is a compact subspace of some Σ-product of real lines, or equivalently a compact subspace of $\Sigma(2^\kappa)$ for some κ.

Given a space X and a family $\mathscr{F} \subset \exp X$ a family $\mathscr{G} \subset \exp X$ is a refinement of $\mathscr{F}$ if $\bigcup \mathscr{F} = \bigcup \mathscr{G}$ and for every $G \in \mathscr{G}$ there is $F \in \mathscr{F}$ such that $G \subset F$. A space X is called metacompact if every open cover of X has a point-finite open refinement; whereas X is meta-Lindelöf if every open cover of X has a point-countable open refinement. Given a family $\mathscr{F}$ of subsets of a space X, for every point $x \in X$, define the order of x in $\mathscr{F}$ by $ord(x,\mathscr{F}) = |\{F \in \mathscr{F} : x \in F\}|$. A space X is weakly θ-refinable if every open cover of X has an open refinement $\mathscr{V} = \bigcup\limits_{n\in\omega} \mathscr{V}_n$ such that for each $x \in X$ there is $n \in \mathbb{N}$ such that $1 \leq ord(x,\mathscr{V}_n) < \omega$.

Given a space Z the family $\mathscr{K}(Z)$ consists of all compact subsets of Z. A family $\mathscr{A}$ is called fundamental if for every $K \in \mathscr{K}(Z)$ there is $A \in \mathscr{A}$ such that $K \subset A$. If all elements of a cover $\mathscr{C}$ of X are compact then the family $\mathscr{C}$ is called compact. Whereas a family $\mathscr{B}$ is M-ordered for some space M if $B = \{B_K : K \in \mathscr{K}(M)\}$ while $K \subset L$ implies $B_K \subset B_L$. A space X is dominated by a space M if it has an M-ordered compact cover and it is strongly dominated by M if it has an M-ordered fundamental compact cover.

Say that X is strongly M-dominated if it has an M-ordered fundamental compact cover $\mathscr{C}$. If X is a space and $\mathscr{C}$ is a cover of X then a family $\mathscr{F}$ is called a network modulo $\mathscr{C}$ if for any $C \in \mathscr{C}$ and $U \in \tau(C,X)$ there is $F \in \mathscr{F}$ with $C \subset F \subset U$. A family $\mathscr{N}$ of subsets of a space X is a network in X if it is a network modulo the cover $\{\{x\} : x \in X\}$. The network weight $nw(X)$ of a space X is the minimal cardinality of a network in X. A space X is cosmic if $nw(X) = \omega$.

A map $\varphi : Y \longrightarrow \exp(X)$ is called upper semicontinuous if for every $U \in \tau(X)$ the set $\{\varphi^{-1}(C) : C \subset U\} \in \tau(Y)$ and φ is onto if $\bigcup\{\varphi(y) : y \in Y\} = X$. If each $\varphi(y)$ is a compact subspace of X then φ is called compact-valued. An upper semicontinuous compact-valued map is called a usco map. A space X is called K-analytic if there is a usco onto map $\varphi : \omega^\omega \to X$.

A space X is Lindelöf Σ if it has a countable network modulo a compact cover of X. In Chapter 1 we will review about ten different equivalent conditions to this definition. Say that X is an $\aleph_0$-space if it has a countable network modulo $\mathscr{K}(X)$. The number of K-determination of a space X is denoted by $\ell\Sigma(X)$ and defined in [CMO] as

$$min\{w(M) : M \textit{ is a metric space and there is a usco onto map } \varphi : M \to \exp(X)\}.$$

A space X is Eberlein-Grothendieck if there is a compact space K such that X is homeomorphic to a subspace of $C_p(K)$. A σ-compact (σ-countably compact) space is the countable union of compact (countably compact) spaces.

The spread $s(X)$ of a space X is the supremum of cardinalities of the discrete subspaces of X and the cardinal invariant $\mathrm{ext}(X) = \sup\{|D| : D$ is a closed and discrete subset of $X\}$ is called the extent of the space X. The cardinal $iw(X) = min\{\kappa :$ the space X has a weaker topology of weight $\kappa\}$ is called i-weight of X, observe that it coincides with the minimum of the set $\{w(Y) :$ the space X condenses onto $Y\}$. Recall that $iw(X) \leq nw(X)$ for any space X. As usual the Suslin number (or the cellularity) of a space X is defined by the formula $c(X) = \sup\{|\mathscr{U}| : \mathscr{U} \subset \tau(X)$ and $\mathscr{U}$ is disjoint $\}$ and the density of X is $d(X) = \sup\{|D| : \overline{D} = X\}$.

1

Cardinal invariants

In this chapter we define some new cardinal invariants for topological spaces and study their relationship with known topological invariants. We will describe how our systematic study will result in complete or partial solutions to open problems on related subjects. To define our topological indexes we will proceed in the same spirit as in [Mu] and [CMO], that is we will consider a topological property, for example domination by metric spaces as defined in [COT] and then find a natural manner to quantify it. The process mentioned gave birth to the so called number of K-determination of a topological space that quantifies how far is a topological space from being Lindelöf Σ.

We begin by recalling the characteristics and techniques of two very closely related topological properties. Sections 1.1 to 1.3 are of introductory nature: we shall expose there a number of results extracted from [Tk5], [MU], [CMO] and [COT] that are relevant for our discussion. The main results are presented in Section 1.4. In Section 1.1 relevant methods and results on the Lindelöf Σ property extracted from the monograph [Tk5] are included. Whereas Section 1.2 contains extracts from the systematic study performed in [COT] of the main properties we will quantify: domination and strong domination by separable metric spaces. In that section it is also pointed out the quite clear parallel behaviour between Domination by second countable spaces and the Lindelöf Σ property. To assign a cardinal to a property may seem artificial, but as seen in [CMO] and in [MU] it can be done so smoothly that it looks almost unavoidable, and that development is reproduced in Section 1.3.

Since domination by second countable spaces is a property weaker than the Lindelöf Σ property, we decided to start the first line of research reported in this thesis by assigning a cardinal to the domination by general metric spaces property. It is in the same spirit of Section 1.3 that we proceed in Section 1.4. to generalize the topological properties of domination and strong domination. Recall that a family $\mathscr{B}$ is M-ordered for some space M if $\mathscr{B} = \{B_K : K \in \mathscr{K}(M)\}$ while $K \subset L$ implies $B_K \subset B_L$. A space X is dominated by a space M if it has an M-ordered compact cover and it is strongly dominated by M if it has an M-ordered fundamental compact cover. We begin Section 1.4 by observing that every space is strongly dominated by a metric space. Hence we can define the (strong) domination index of a given space X as the first cardinal κ for which there is a metric space of weight κ that (strongly) dominates X.

In this context we will see that the systematic study performed in [COT] outlined in Section 1.2 covers the case of spaces for which our (strong) domination index is countable. We will show that in some cases the properties reported in [COT] summarized in Section 1.2 can be generalized to larger values of the (strong) domination index. In particular we will be able to strengthen some results attained in [MU] for the number of K-determination. But our study of the (strong) domination index will also prove useful to the countable case. Domination and most importantly strong domination by second countable spaces imply complete (metrizability) as the authors of [COT] evince and we review in Section 1.2. In the special case of function spaces over compacta, strong Domination by second countable spaces is actually equivalent to metrizability. This was proved under CH in [COT] and mentioned in Section 1.2, and in Section 1.4 we prove it in ZFC. Indeed in Theorem 1.4.24 (IV) it is stated that a function space $C_p(K)$ is strongly dominated by a second countable space if and only if it is metrizable if and only if K is countable. This answers a question included in [COT].

1.1. The Lindelöf Σ property in $C_p(X)$

The shortest way to define the Lindelöf Σ property is to say that a space X is Lindelöf Σ if it is a continuous image of a space Y that maps perfectly onto a second countable space. This definition may seem technical and artificial; however, it is evident that this concept is a generalization of compactness. It takes some effort to prove that any σ-compact space and even any K-analytic space is Lindelöf Σ. One could ask why is it that this property is studied so intensely in the Descriptive Set Theory and Functional Analysis. For instance, compact spaces K for which $C_p(K)$ is Lindelöf Σ appear naturally in Functional Analysis when weakly countably K-determined Banach spaces are considered. The study of this class of Banach spaces is exhaustive due to their good categorical properties; one of the characterizations of a weakly determined Banach space is the Lindelöf Σ property of the space of functions of the unit ball of the dual space endowed with the weak* topology. Another sign of the importance of a topological notion is the amount of different equivalent definitions it has. One can easily give ten or more equivalences of compactness, but if a concept is not interesting, then it is difficult to find even two equivalent conditions to it. The Lindelöf Σ is not the exception. The next theorem illustrates this fact, it is included in the monograph [Tk5] so we include it here without proof.

Theorem 1.1.1. *[Tk5, Theorem 1] For any space X the following conditions are equivalent:*

(I) *The space X is Lindelöf Σ.*

(II) *There exists a compact space K and a space second countable M such that X is a continuous image of a closed subspace of $K \times M$.*

(III) *The space X belongs to every class that contains all compact spaces, the second countable spaces and that is invariant under closed subspaces, finite products and continuous images.*

(IV) *There exists an upper semicontinuous compact-valued onto map $\varphi : M \to \exp(X)$ for some second countable space M.*

(V) *There exists an upper semicontinuous compact-valued onto map $\varphi : P \to X$ for some subspace P of the irrational numbers.*

(VI) *There exists a compact cover $\mathscr{C}$ of the space X for which it is possible to find a family $\mathscr{N}$ that is a network mod $\mathscr{C}$ in the sense that for every $C \in \mathscr{C}$, if $U \in \tau(C,X)$ then there is $N \in \mathscr{N}$ such that $C \subset N \subset U$.*

(VII) *There exists a compact cover $\mathscr{C}$ of the space X for which it is possible to find a family $\mathscr{Q}$ of closed subsets of X, that is a network mod $\mathscr{C}$.*

(VIII) *There exists a countable family $\mathscr{F}$ of compact subsets of βX such that $\mathscr{F}$ separates X from $\beta X \setminus X$ in the sense that for each $x \in X$ and $z \in \beta X \setminus X$ exists $F \in \mathscr{F}$ for which $x \in F$ and $z \notin F$.*

(IX) *There exists a compactification bX of the space X and a countable family $\mathscr{K}$ of compact subsets of bX which separates X from $bX \setminus X$.*

(X) *There exists a space Y such that $X \subset Y$ and for some countable family $\mathscr{K}$ of compact subsets of Y for which we have $X \subset \bigcup \mathscr{K}$ and $\mathscr{K}$ separates X from $Y \setminus X$.*

The following are basic consequences of the previous theorem. They constitute classic facts about the Lindelöf Σ-property, proofs are short and can be read in [Tk5].

Corollary 1.1.2. *[See Tk5]*

(I) *Every σ-compact space, or more generally every K-analytic space has the Lindelöf Σ property.*

(II) *Every space with a countable network is Lindelöf Σ.*

(III) *Every Lindelöf Čech-complete space is Lindelöf Σ.*

(IV) *If $p : X \to exp(Y)$ is an upper semicontinuous surjective compact-valued map and X is a Lindelöf Σ space, then Y is also Lindelöf Σ. In consequence, every continuous image of a Lindelöf Σ space is a Lindelöf Σ space and every perfect pre-image of a Lindelöf Σ space is Lindelöf Σ.*

(V) *If X is a Lindelöf Σ space and $F \subset X$ is closed in X then F is also Lindelöf Σ.*

(VI) *If X_i is a Lindelöf Σ space for each $i \in \omega$ then $X = \prod_{i\in\omega} X_i$ is also Lindelöf Σ.*

(VII) *If Y is a space and $X_i \subset Y$ has the Lindelöf Σ property for every $i \in \omega$ then the set $X = \bigcap_{i\in\omega} X_i$ also has the Lindelöf Σ property.*

(VIII) *If X is a space, $X_i \subset X$ has the Lindelöf Σ property for every $i \in \omega$ and $X = \bigcup_{i\in\omega} X_i$ then X also has the Lindelöf Σ property.*

It follows from Theorem 1.1.1 (II) that the Lindelöf Σ property implies the Lindelöf property, as should be expected. Consequently for a Lindelöf Σ space X, we have X^n is Lindelöf by Corollary 1.1.2 (IV) and hence $t(C_p(X)) \leq \omega$ by [Tk6, Problem 149].

Theorem 1.1.3. *[Ar2, Theorem II.6.21] Every Lindelöf Σ space is stable.*

PROOF. Since every continuous image of a space Lindelöf Σ is Lindelöf Σ, it suffices to show that, for every Lindelöf Σ, space X, if X condenses onto a space of weight κ then $nw(X) \leq \kappa$. Fix a condensation $\phi : X \to Y$ such that Y has a base $\mathscr{B}$ $|\mathscr{B}| \leq \kappa$. If $\mathscr{F} = \{\varphi^{-1}(\overline{B}) : B \in \mathscr{B}\}$ then it is easy to see that the family $\mathscr{F}$ is T_1-separating X, i.e.,

(I) for any two distinct points $x, y \in X$ there exists $F \in \mathscr{F}$ such that $x \in F \subset X \setminus \{y\}$.

We can take a compact cover $\mathscr{C}$ of the space X for which there exists a countable network $\mathscr{N}$ with respect to $\mathscr{C}$. The family $\mathscr{Q}$ of all the finite intersections of the elements of $\mathscr{N} \cup \mathscr{F}$ has cardinality not greater than κ; we claim that $\mathscr{Q}$ is a network in X. Indeed, take a point $x \in X$ and $U \in \tau(x, X)$; there exists $C \in \mathscr{C}$ that contains x. The set $P = C \setminus U \subset X \setminus \{x\}$ is compact and for every $y \in P$ there exists $F_y \in \mathscr{F}$ such that $x \in F_y$ and $y \notin F_y$. Therefore the family $\{F_y \cap P : y \in P\}$ has empty intersection; by the compactness of P we can find a finite set $A \subset P$ such that $\bigcap\{F_y : y \in A\} \cap P = \varnothing$. The set $Q = \bigcap\{F_y : y \in A\}$ belongs to $\mathscr{Q}$ and the closed set $H = Q \setminus U$ does not meet C. The family $\mathscr{N}$ is a network with respect to $\mathscr{C}$, thus there exists $N \in \mathscr{N}$ such that $C \subset N \subset X \setminus H$. It is immediate that $E = N \cap Q$ belongs to $\mathscr{Q}$ and $x \in E \subset U$ we can conclude that $\mathscr{Q}$ is a network in X and therefore $nw(X) \leq \kappa$. □

Corollary 1.1.4. *[Ar2, Corollary II.6.34] If υX is a Lindelöf Σ space then X is ω-stable.*

PROOF. Let $f : X \to Y$ be a continuous map from X onto a space Y for which there exists a condensation $g : Y \to M$ from Y onto a second countable space M. The space Y is realcompact hence there exists a continuous map $h : \upsilon X \to Y$ with $h|X = f$. In particular, Y is a Lindelöf Σ space. Theorem 1.1.10 guarantees that Y is a stable space and therefore $nw(Y) \leq \omega$. □

In 1979 Gul'ko proved his classical result (see [Gu]) that states that if K is a compact such that $C_p(K)$ is Lindelöf Σ then K is Corson, in other words, the compact space K

embeds in a Σ-product of real lines. This theorem located the class of such compact spaces within the hierarchy of the compact spaces studied in Functional Analysis: Eberlein, Talagrand and Corson compact spaces. For this reason the compact spaces of this class are called Gul'ko compact spaces.

Baturov discovered a fundamental property of the subspaces of $C_p(X)$, for the case when X is Lindelöf Σ. Although his theorem, included next, has already been generalized in several ways, it is still one of the most cited results in C_p-theory.

Theorem 1.1.5. *[Ar2, Theorem III.6.1] If X is Lindelöf Σ then $ext(Y) = l(Y)$ for every $Y \subset C_p(X)$.*

PROOF. It is evident that $ext(Y) \leq l(Y)$ so it suffices to prove that if κ is a cardinal and $l(Y) > \kappa$ then $ext(Y) > \kappa$. Without loss of generality we can assume that X is a Lindelöf p-space because Lindelöf Σ spaces are continuous images of Lindelöf p-spaces. The space X maps perfectly onto a second countable space.

Suppose that $l(Y) > \kappa$. There exists an open cover $\mathscr{U}$ of Y that does not admit a subcover of cardinality less or equal to κ. We may suppose that the elements of $\mathscr{U}$ are of the form $[x_1,\ldots,x_n;G_1,\ldots,G_n]$ and every G_i is a rational interval for $i = 1,...,n$. Define $W_k(x,G) = [x_1,\ldots,x_k;G_1,\ldots,G_k]$ where $x = (x_1,...,x_k) \in X^k$ and $G = G_1 \times \ldots \times G_k$. The family $\mathscr{U}$ can be represented as $\mathscr{U} = \bigcup\{\mathscr{U}_n : n \in \omega\}$ where for each $n \in \mathbb{N}$ we can find $k_n \in \mathbb{N}$ and $O_n \in \tau(\mathbb{R}^{k_n})$ such that O_n is equal to a product of rational intervals and if $W_k(x,G) \in \mathscr{U}_n$ then $k = k_n$ y $G = O_n$. For each $n \in \mathbb{N}$ let $A_n = \{x \in X^{k_n} : W_{k_n}(x,G) \in \mathscr{U}_n\}$. For every $f \in C_p(X)$ and each $k \in \mathbb{N}$ call f^k the function of X^k in $\mathbb{R}^k$ that maps the point $(x_1,...,x_k)$ onto the point $(f(x_1),...,f(x_n))$. In this notation the fact that $\mathscr{U}$ covers Y can be written as follows:

$(*)$ for every $f \in Y$ we can find $n \in \mathbb{N}$ and $x \in A_n$ such that $f^{k_n}(x) \in O_n$.

The fact that no subfamily of $\mathscr{U}$ of cardinality not greater than κ covers Y can be expressed in the following way:

$(**)$ If $B_n \subset A_n$ and $|B_n| \leq \kappa$ for every $n \in \mathbb{N}$, then there exists $g \in Y$ such that $g^{k_n}(\mathscr{V}_n) \cap O_n = \varnothing$ for all $n \in \mathbb{N}$.

By transfinite recursion, we will construct a closed and discrete set

$$F = \{f_\alpha : \alpha < \kappa^+\} \subset Y.$$

Choose any $f_0 \in Y$ and suppose that given $\alpha < \kappa^+$ we have defined f_β for each $\beta < \alpha$. Fix $n \in \mathbb{N}$ and take a perfect map ϕ_{k_n} de X^{k_n} onto some second countable space M_n.

For every finite collection $\beta_1,...,\beta_r<\alpha$ consider the map $f^n_{(\beta_1,...,\beta_r)}=f^{k_n}_{\beta_1}\Delta...\Delta f^{k_n}_{\beta_r}\Delta\phi_{k_n}$. The map $f^n_{(\beta_1,...,\beta_r)}:X^{k_n}\to\mathbb{R}^{k_n\cdot r}\times M_n$ is a perfect map from X^{k_n} onto a second countable space.

The space $\mathbb{R}^{k_n\cdot r}\times M_n$ is hereditarily separable, hence we can find a countable set $S^n_{(\beta_1,...,\beta_r)}\subset A_n$ such that $f^n_{(\beta_1,...,\beta_r)}(S^n_{(\beta_1,...,\beta_r)})$ is dense in $f^n_{(\beta_1,...,\beta_r)}(A_n)$. Let

$$B^\alpha_n=\bigcup\{S^n_{(\beta_1,...,\beta_r)}:\beta_1,\dots,\beta_n<\alpha\}.$$

It is clear that $|B^\alpha_n|\le\kappa$, therefore, by the condition $(**)$ there exists $f_\alpha\in Y$ such that $f_\alpha(B^\alpha_n)\cap O_n=\varnothing$ for every $n\in\mathbb{N}$. This finishes the construction of $F=\{f_\alpha:\alpha<\kappa^+\}$.

We will show that F is closed and discrete in Y. If this is not so, then there exists $g\in Y$ such that every neighbourhood of g contains infinitely many elements of F. For some $n\in\mathbb{N}$ and $\tilde{x}\in X^{k_n}$ we have $g\in W_{k_n}(\tilde{x},O_n)$, this implies $g^{k_n}(\tilde{x})\in O_n$. The tightness of $C_p(X)$ is countable, which implies that the tightness of Y does not exceed κ. Therefore, there exists $\alpha'<\kappa^+$ such that $g\in\overline{\{f_\alpha:\alpha<\alpha'\}}$. Denote by α_0 the minimum of such $\alpha'<\kappa^+$ and define $P=\{f_\alpha\in W_{k_n}(\tilde{x},O_n):\alpha<\alpha_0\}$. The set

$$(g^{k_n})^{-1}(O_n)\cap(\bigcap\{(f^{k_n})^{-1}(f^{k_n}(\tilde{x})):f\in P\}$$

contains $\tilde{x}$ so it is not empty. Let $T=\bigcap\{(f^{k_n})^{-1}(f^{k_n}(\tilde{x})):f\in P\}\setminus(g^{k_n})^{-1}(O_n)$.

There are two possible cases:

Case 1: The set T is empty. Let $\phi_{k_n}(\tilde{x})=\tilde{m}$. Recall that our map $\phi_{k_n}:X^{k_n}\to M_n$ is perfect, hence $\phi^{-1}_{k_n}(\tilde{m})$ is compact. Since the set $\phi^{-1}_{k_n}(O_n)$ is open, there exists a finite set $\{f_{\beta_1},\dots,f_{\beta_r}\}\subset P\}$ such that

$$\Phi=[(\bigcap_1^r(f^{k_n}_{\beta_i})^{-1}(f^{k_n}_{\beta_i}(\tilde{x})))\cap\phi^{-1}_{k_n}(\tilde{m})]\subset(g^{k_n})^{-1}(O_n).$$

The set Φ is the pre-image of the point $(f^{k_n}_{\beta_1}(\tilde{x}),\dots,f^{k_n}_{\beta_r}(\tilde{x}),\tilde{m})$ under the map $f^n_{(\beta_1,...,\beta_r)}$. Observe that the map $f^n_{(\beta_1,...,\beta_r)}$ is perfect, in particular it is closed and that $(g^{k_n})^{-1}(O_n)$ is a neighbourhood of the set Φ. Since $\tilde{x}\in A_n\cap\Phi$ the whole pre-image of a point of the set $f^n_{(\beta_1,...,\beta_r)}(S^n_{(\beta_1,...,\beta_r)})$ is contained in $(g^{k_n})^{-1}(O_n)$ hence it is possible to find a point $x'\in S^n_{(\beta_1,...,\beta_r)}\cap(g^{k_n})^{-1}(O_n)$ which shows that $g\in W_{k_n}(x',O_n)$. However, by the construction, if $\alpha>\alpha^*=max\{\beta_1,\dots,\beta_r\}$ then $f_\alpha\notin W_{k_n}(x',O_n)$. It is clear that $\alpha^*<\alpha_0$ and that $g\in\overline{\{f_\alpha:\alpha<\alpha_0\}}$ implies that $g\in\overline{\{f_\alpha:\alpha<\alpha^*\}}$ which contradicts the choice of α_0.

Case 2: The set T is not empty. Take a point $x''\in T$. It is immediate that the point $g^{k_n}(x'')\ne g^{k_n}(\tilde{x})$ because $g^{k_n}(\tilde{x})\in O_n$. Besides $f^{k_n}(x'')=f^{k_n}(\tilde{x})$ for every $f\in P$. This implies that $g\notin\overline{P}$ which is a contradiction. Hence the set F is closed and discrete in Y. □

Theorem 1.1.6. *[Ar2, Theorem IV.9.5] Given a space X, if the space $C_p(X)$ is Lindelöf Σ then υX is Lindelöf Σ.*

Theorem 1.1.7. *[Ar2, Theorem IV.9.8] Given a space X, if the space $C_p(X)$ is Lindelöf Σ then $C_p(X)$ is ω-monolithic.*

Theorem 1.1.8. *[See Tk5] If ω_1 is a caliber of a space X, then the space $C_p(X)$ is Lindelöf Σ if and only if X is cosmic.*

1.2. Domination by second countable spaces

In this section we present some of the results by Cascales, Orihuela and Tkachuk on spaces dominated by second countable ones.

Theorem 1.2.1. *[COT, Theorem 2.1, see also COr]*

(a) Every Lindelöf Σ*-space is dominated by a second countable space;*

(b) if X *is dominated by a second countable space then any continuous image of* X *is also dominated by a second countable space;*

(c) if X *is dominated by a second countable space then any closed subspace of* X *is also dominated by a second countable space;*

(d) if $X = \bigcup_{i\in\omega} X_i$ *and* X_i *is dominated by a second countable space for all* $i \in \omega$ *then* X *is dominated by a second countable space;*

(e) if X_i *is dominated by a second countable space for all* $i \in \omega$ *then the space* $X = \prod_{i\in\omega} X_i$ *is dominated by a second countable space;*

(f) if X *is a space and* $Y_i \subset X$ *is dominated by a second countable space for all* $i \in \omega$ *then* $Y = \bigcap_{i\in\omega} Y_i$ *is also dominated by a second countable space;*

(g) a space X *is Lindelöf* Σ *if and only if it is Dieudonné complete (i.e., homeomorphic to a closed subspace of a product of metrizable spaces) and dominated by a second countable space;*

(h) if X *is dominated by a second countable space then* $ext(X) = \omega$.

PROOF. Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \longrightarrow X$ for some second countable space M. If we let $F_K = \bigcup\{\varphi(x) : x \in K\}$ for any compact set $K \subset M$ then it happens that the family $\mathscr{F} = \{F_K : K \in \mathscr{K}(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies that also $F_K \subset F_L$. We have shown that $\mathscr{F}$ is an M-ordered compact cover of X.

To prove (b) suppose that X is a space dominated by some second countable space M. There is an M-ordered compact cover $\{F_K : K \in \mathscr{K}(M)\}$ of the space X. If Z is a closed subspace of X then $\{F_K \cap Z : K \in \mathscr{K}(M)\}$ is evidently an M-ordered compact cover of Z.

To settle (c) consider a space X that has a compact cover $\{F_K : K \in \mathscr{K}(M)\}$ which is M-ordered and a space Y that is a continuous image of X under a map φ. It is immediate that $\{\phi(F_K) : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of Y.

To see that (d) is true suppose that X_i has an M_i-ordered cover by compact subsets $\mathscr{F}_i = \{P(K,i) : K \in \mathscr{K}(M_i)\}$ for some second countable space M_i for every $i \in \omega$. The space $M = \bigoplus_{i\in\omega} M_i$ is second countable; we identify every M_i with the corresponding clopen subset of M. Given any $K \in \mathscr{K}(M)$ the set $N_K = \{i \in \omega : K \cap M_i \neq \varnothing\}$ is finite so it is clear that the set $F_K = \bigcup\{P(K\cap M_i, i) : i \in N_K\}$ is compact. It is immediate that the family $\{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of X.

Let us prove (e). For each $i \in \omega$ fix a second countable space M_i and an M_i-ordered compact cover $\mathscr{F}_i = \{Q(K,i) : K \in \mathscr{K}(X_i)\}$ of the space X_i. For the space $M = \prod_{i\in\omega} M_i$ let $p_i : M \longrightarrow M_i$ be the natural projection for every $i \in \omega$. Given any $K \in \mathscr{K}(M)$, the set $F_K = \prod\{Q(p_i(K), i) : i \in \omega\}$ belongs to $\mathscr{K}(X)$. It is an easy exercise that the family $\{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of X.

It is standard to deduce (f) from (c) and (e); the statement of (g) was proved in [COr]. If X is dominated by a second countable space and D is a closed discrete subspace of X then D is also dominated by a second countable space by (c). Since D is also Dieudonné complete, it must be Lindelöf and hence countable by (g). This shows that $ext(X) = \omega$, i.e., (h) is proved. $\square$

Proposition 1.2.2. *[COT, Proposition 2.5] Suppose that X is dominated by a second countable space M and a collection $\{F_K : K \in \mathscr{K}(M)\}$ witnesses this. Take a countable base $\mathscr{B}$ of M such that the union and the intersection of any finite subfamily of $\mathscr{B}$ belongs to $\mathscr{B}$. For each $K \in \mathscr{K}(M)$ take a countable outer base $\mathscr{B}_K = \{U_n : n \in \omega\}$ such that for each $n \in \omega$ we have $U_{n+1} \subset U_n$; then $F_K \subset C_K = \bigcup\{G(U) : U \in \mathscr{B}_K\}$. If $S = \{y_n : n \in \omega\} \subset X$ is a sequence such that $y_n \in G(U_n)$ for all $n \in \omega$ then:*

(a) The set $\overline{S}$ is compact and hence the set D of cluster points of S is non-empty.

(b) There exists a compact set Q_K such that $D \subset Q_K \subset C_K$.

Proposition 1.2.3. *[COT, Proposition 2.6] Suppose that X is dominated by a second countable space M and a collection $\{F_K : K \in \mathscr{K}(M)\}$ witnesses this. Take a countable base $\mathscr{B}$ of M such that the union and the intersection of any finite subfamily of $\mathscr{B}$*

belongs to $\mathscr{B}$. For each $U \in \mathscr{B}$ let $G(U) = \bigcup\{F_K : K \in \mathscr{K}(M)$ and $K \in U\}$. Then there exists a family $\mathscr{C}$ in the space X with the following properties:

(a) If $C \in \mathscr{C}$ and $A \subset C$ is a countable set then the set $\overline{A}$ is compact and $\overline{A} \subset C$; in particular, each $C \in \mathscr{C}$ is countably compact;

(b) For every $K \in \mathscr{K}(M)$ there exists a set $C_K \in C$ such that $F_K \subset C_K$ and hence $\mathscr{C}$ is a cover of X;

(c) The family $\mathscr{N} = \{G(U) : U \in \mathscr{B}\}$ is a countable network with respect to $\mathscr{C}$.

Corollary 1.2.4. *[COT, Corollary 2.7] Suppose that, in a space X, every relatively countably compact set has compact closure. Then X is dominated by a second countable space if and only if it has the Lindelöf Σ property. In particular, an angelic X is dominated by a second countable space if and only if X is Lindelöf Σ.*

Theorem 1.2.5. *[COT, Theorem 2.8] Suppose that Z is a compact space of countable tightness. Then a set $X \subset Z$ is dominated by a second countable space if and only if X has the Lindelöf Σ property.*

Theorem 1.2.6. *[COT, Theorem 2.9] If K is a compact space with $t(K) \leq \omega$ and $K^2 \setminus \Delta$ is dominated by a second countable space then $w(K) \leq \omega$.*

The following theorem was proved assuming $MA(\omega_1)$ by Cascales, Orihuela and Tkachuk, but the same proof can be done by assuming only $\omega_1 < \mathfrak{d}$.

Theorem 1.2.7. *[COT, Theorem 2.12] Assume $\omega_1 < \mathfrak{d}$ and suppose that X is a compact space such that $X^2 \setminus \Delta$ is $\mathbb{P}$-dominated. Then X has a small diagonal and hence $t(X) = \omega$.*

PROOF. Suppose that $A = \{z_\alpha : \alpha < \omega_1\} \subset X^2 \setminus \Delta$ and $\alpha \neq \beta$ implies $z_\alpha \neq z_\beta$. Fix a $\mathbb{P}$-directed cover $\{K_p : p \in \mathbb{P}\}$ of compact subsets of $X^2 \setminus \Delta$. Take $p_\alpha \in \mathbb{P}$ such that $z_\alpha \in K_{p_\alpha}$ for any $\alpha \in \omega_1$. It follows from $\omega_1 < \mathfrak{d}$ that there exists $p \in \mathbb{P}$ such that $p_\alpha \leq^* p$ for any $\alpha \in \omega_1$. The set $P = \{K_q : q \in \mathbb{P}$ and $q =^* p\}$ is σ-compact and $A \subset P$. Consequently, there is $q \in \mathbb{P}$ for which $K_q \cap A$ is uncountable; therefore the set $K_q \cap A$ witnesses the small diagonal property of X. Since no space with a small diagonal can have a convergent ω_1-sequence, it follows from [JuS, Theorem 1.2] that X has no free sequences of length ω_1, i.e., $t(X) < \omega$. $\square$

Corollary 1.2.8. *Under $\omega_1 < \mathfrak{d}$, if X is a compact space such that $X^2 \setminus \Delta$ is dominated by a Polish space then X is metrizable.*

PROOF. Apply [COT, Proposition 2.2] to see that the space $X^2 \setminus \Delta$ is dominated by $\mathbb{P}$ so $t(X) \leq \omega$ by Theorem 1.2.7 and hence X is metrizable by Theorem 1.2.6. $\square$

Hodel established in [Ho, Corollary 4.13] that any hereditarily Lindelöf Σ-space is cosmic.

The following fact is an immediate consequence of [Tk4, Proposition 2.7].

Proposition 1.2.9. *[COT, Proposition 2.14] If X is a space which has a countable network modulo a cover of X by countably compact sets then $C_p(X)$ is Lindelöf Σ-framed, i.e., there is a Lindelöf Σ-space L such that $C_p(X) \subset L \subset \mathbb{R}^X$.*

Theorem 1.2.10. *[COT, Theorem 2.15] A space $C_p(X)$ is dominated by a second countable space if and only if it is Lindelöf Σ.*

PROOF. We must only prove necessity. Suppose that $C_p(X)$ is dominated by a second countable space M and fix a family $\{F_K : K \in \mathscr{K}(M)\}$ which witnesses this. It follows from Proposition 1.2.3 and Proposition 1.2.7 that $C_p(C_p(X))$ is Lindelöf Σ-framed. We can now apply [Ok, Theorem 3.5] to conclude that $\upsilon(C_p(X))$ is a Lindelöf Σ-space and hence υX is a Lindelöf Σ-space by [Ok, Corollary 3.6]. Let $\pi : C_p(\upsilon X) \longrightarrow C_p(X)$ be the restriction map. If $G_K = \pi^{-1}(F_K)$ then G_K is compact for any $K \in \mathscr{K}(M)$ (see [Tk4, Theorem 2.6]). It is clear that $\mathscr{G} = \{G_K : K \in \mathscr{K}(M)\}$ is a cover of $C_p(\upsilon X)$ which shows that $C_p(\upsilon X)$ is dominated by M. By Proposition 1.2.3 we can find a countable network $\mathscr{N}$ modulo a cover $\mathscr{C}$ of the space $C_p(\upsilon X)$ such that every $C \in \mathscr{C}$ is countably compact. Every countably compact subset of $C_p(\upsilon X)$ is compact by [Ar2, Proposition IV.9.10] so $\mathscr{C}$ consists of compact subsets of $C_p(\upsilon X)$ and hence $C_p(\upsilon X)$ is a Lindelöf Σ-space. Therefore $C_p(X)$ is also Lindelöf Σ-space being a continuous image of $C_p(\upsilon X)$. $\square$

Proposition 1.2.11. *[COT, Proposition 3.3]*

(a) If X is strongly dominated by a second countable space and Y is a compact-covering image of X then Y is strongly dominated by a second countable space;

(b) every $\aleph_0$-space is strongly dominated by a second countable space;

(c) if X is strongly dominated by a second countable space then every closed subspace of X is also strongly dominated by a second countable space;

(d) if X_i is strongly dominated by a second countable space for every $i \in \omega$ then $\prod_{i\in\omega} X_i$ is strongly dominated by a second countable space;

(e) if X is a space and $Y_i \subset X$ is strongly dominated by a second countable space for each $i \in \omega$ then $Y = \bigcap_{i\in\omega} Y_i$ is also strongly dominated by a second countable space.

PROOF. Suppose that X is strongly dominated by a second countable space M and $f : X \longrightarrow Y$ is a compact-covering map. Let $\{F_K : K \in \mathscr{K}(M)\}$ be the family which witnesses that X is strongly dominated by M and consider the family $\mathscr{F} = \{f(F_K) : K \in \mathscr{K}(M)\}$. It is clear that $\mathscr{F}$ consists of compact subsets of Y and $K \subset L$ implies $f(F_K) \subset f(F_L)$. If P is a compact subset of Y then there exists a compact subset $Q \subset X$ such that $f(Q) = P$. Pick a set $K \in \mathscr{K}(M)$ such that $Q \subset F_K$ and observe that $P = f(Q) \subset f(F_K)$. Therefore the family F witnesses that Y is strongly dominated by M, i.e., we proved (a).

The item (b) follows from (a) and the fact that every $\aleph_0$-space is a compact-covering image of a second countable space [Mi, Theorem 11.4].

The proof of (c) is straightforward and can be left to the reader. Next assume that X_i is strongly dominated by a second countable space M_i and hence we can fix the respective family $\mathscr{F}_i = \{F_i(K) : K \in \mathscr{K}(Mi)\}$ for any $i \in \omega$. The space $M = \prod_{i\in\omega} M_i$ is second countable; we now let $\pi : M \longrightarrow M_i$ be the natural projection for each $i \in \omega$. If $K \in \mathscr{K}(M)$ then $F_K = \prod_{i\in\omega} F_i(\pi(K))$ is easily seen to be a compact subset of $X = \prod_{i\in\omega} X_i$. Let $\pi : X \longrightarrow X_i$ be the natural projection for every $i \in \omega$. The family $\mathscr{F} = \{F_K : K \in \mathscr{K}(M)\}$ witnesses that X is strongly dominated by M. Indeed, if Q is a compact subset of X then we can choose $K_i \in \mathscr{K}(Mi)$ such that $\pi(Q) \subset F_i(K_i)$ for each $i \in \omega$; for the set $K = \prod_{i\in\omega} K_i$ we have $Q \subset F_K$. It is immediate that $K \subset L$ implies $F_K \subset F_L$ so we settled (d).

As to (e), observe that Y is homeomorphic to a closed subspace of $\prod_{i\in\omega} Y_i$ so we can apply (c) and (d) to finish the proof. □

Theorem 1.2.12. *[COT, Theorem 3.6] The following conditions are equivalent for any space X:*

(a) X is an $\aleph_0$-space;

(b) X is strongly dominated by a second countable space and $iw(X) \leq \omega$;

(c) X is submetrizable and strongly dominated by a second countable space.

PROOF. Every $\aleph_0$-space X is cosmic and hence $iw(X) \leq \omega$; this, together with Proposition 1.2.10 (b), shows that $(a) \Rightarrow (b)$. The implication $(b) \Rightarrow (c)$ being trivial assume that X is submetrizable and strongly dominated by a second countable space. It follows from [COr, Theorem 4] that X is a Lindelöf Σ-space so its weaker metrizable topology must be second countable, i.e., $iw(X) \leq \omega$. Fix an M-ordered family $\{F_K : K \in \mathscr{K}(M)\}$ of compact subsets of X such that every $L \in \mathscr{K}(X)$ is contained in some F_K. Apply Proposition 1.2.3 to find a family $\mathscr{C}$ of countably compact (and hence compact) subsets of X such that some countable family $\mathscr{N}$ is a network modulo $\mathscr{C}$ and, for every $K \in \mathscr{K}(M)$ there exists $C_K \subset C$ such that $F_K \subset C_K$. In particular, the family $\mathscr{C}$ swallows all compact subsets of X. Taking the closures of the elements of $\mathscr{N}$ we will still have a network modulo $\mathscr{C}$ so we can assume, without loss of generality, that $\mathscr{N}$ consists of closed subsets of X. Fix a second countable topology μ on the set X such that $\mu \subset \tau(X)$. The space (X,μ) has a countable closed network $\mathscr{P}$ modulo all compact subsets of (X,μ). Observe that the identity map $id : X \longrightarrow (X,\mu)$ is continuous and hence any compact subset of X is also compact in (X,μ). Consider the family $\mathscr{Q}$ of all finite unions and finite intersections of the elements of the family $\mathscr{P} \cup \mathscr{N}$; we claim that $\mathscr{Q}$ is a network for all compact subsets of X.

Indeed, take any $L \subset \mathscr{K}(X)$ and $U \in \tau(L,X)$. There exists $C \in \mathscr{C}$ such that $L \subset C$. The set $C \setminus U$ does not meet L so there exists $P \in \mathscr{P}$ such that $L \subset P$ and $P \cap (C \setminus U) = \varnothing$. The set $P' = P \setminus U$ does not meet C so we can find a set $N \in \mathscr{N}$ such that $C \subset N \subset X \setminus P'$. The set $Q = N \cap P$ belongs to Q and $L \subset Q \subset U$ so the family Q witnesses that X is an $\aleph_0$-space. $\square$

Given an infinite cardinal κ say that a space X is κ-hemicompact if there exists a family $\mathscr{F}$ of compact subsets of X such that $|\mathscr{F}| \leq \kappa$ and $\mathscr{F}$ swallows all compact subsets of X, i.e., for any $K \in \mathscr{K}(X)$ there exists $F \in \mathscr{F}$ such that $K \subset F$. Observe that a space is hemicompact if and only if it is ω-hemicompact. Denote by $\mathbb{D}$ the set $\{0,1\}$.

Theorem 1.2.13. *[COT, Theorem 3.9] The σ-product $S_k = \{x \in \mathbb{D}^\kappa : |x^{-1}(1)| < \omega\}$ of the space $\mathbb{D}^\kappa$ is not κ-hemicompact for any infinite cardinal κ.*

PROOF. Denote by u the point of $\mathbb{D}^\kappa$ which is identically zero on κ and hence $u^{-1}(1) = \varnothing$. Take any family $\mathscr{F} = \{F_\alpha : \alpha < \kappa\}$ of compact subsets of S_κ. The set S_κ is not compact so we can pick a point $x_0 \in S_\kappa \setminus F_0$. Proceeding inductively assume that $\alpha < \kappa$ and we have chosen a set $\{x_\beta : \beta < \alpha\}$ with the following properties:

(I) $x_\beta \in S_\kappa \setminus F_\beta$ for any $\beta < \alpha$;

(II) The family $\{x_\beta^{-1}(1) : \beta < \alpha\}$ is disjoint.

Observe that the set $A = \bigcup\{x_\beta^{-1}(1) : \beta < \alpha\}$ has cardinality strictly less than κ. Therefore the subspace $Y = \{x \in S_\kappa : x(A) = 0\}$ is not compact so we can choose a point $x_a \in Y \setminus F_a$; it is immediate that the conditions (I) and (II) are still satisfied for the set $\{x_\beta : \beta \leq \alpha\}$. Thus we can construct a set $\{x_a : \alpha < \kappa\}$ for which the properties (I) and (II) hold for any $\alpha < \kappa$. It follows from (II) that the set $K = \{x_\beta : \beta < \kappa\} \cup \{u\}$ is compact; the property (I) shows that $x_\beta \in K \setminus F_\beta$ for any $\beta < \kappa$ and therefore no element of the family $\mathscr{F}$ swallows the set K. □

Theorem 1.2.14. *[COT, Theorem 3.10] Under the Continuum Hypothesis (CH) if a space X is compact and $C_p(X)$ is strongly dominated by a second countable space then X is countable and hence $C_p(X)$ is second countable.*

PROOF. Apply Theorem 1.2.10 to see that $C_p(X)$ is a Lindelöf Σ-space and hence X is Gul'ko compact. If the space X is not scattered then we can find a countable dense-in-itself set $A \subset X$. The space $K = A$ is compact, second countable and metrizable [Ar3, Theorem 7.21] so $C_p(K)$ embeds in $C_p(X)$ as a closed subspace [Ar3, Theorem 4.1]. This implies, by Proposition1.2.11 (c), that $C_p(K)$ is strongly dominated by a second countable space. Since $iw(C_p(K)) \leq nw(C_p(K)) = \omega$, we can apply Theorem 1.2.12 to convince ourselves that $C_p(K)$ is an $\aleph_0$-space so K is countable by [Mi, Proposition 10.7]. However, K has no isolated points; this contradiction shows that X has to be scattered. The set D of isolated points of the space X is dense in X; if D is countable then X is second countable so we can apply Theorem 1.2.12 again to see that $C_p(X)$ is an $\aleph_0$-space and hence X is countable by [Mi, Proposition 10.7]. Therefore we can assume that $\kappa = |D| \geq \omega_1$; consider the space Y which is obtained from X by contracting the set $F = X \setminus D$ to a point. It is evident that Y is a compact space with a unique non-isolated point, i.e., Y is homeomorphic to the one-point compactification $A(\kappa)$ of a discrete space of cardinality κ. The space Y is a continuous closed image of X so $C_p(Y)$ is homeomorphic to a closed subspace of $C_p(X)$. Thus $C_p(Y) \simeq C_p(A(\kappa))$ is strongly dominated by a second countable space.

It is an easy exercise that the space $C_p(A(\kappa))$ is homeomorphic to the Σ_*-product $\Omega = \{x \in \mathbb{R}^\kappa$: the set $\{\alpha < \kappa : |x(\alpha)| \geq \varepsilon\}$ is finite for any $\varepsilon > 0\}$ of the space $\mathbb{R}^\kappa$. Furthermore, $\Omega \cap \mathbb{D}^\kappa = S_\kappa = \{x \in \mathbb{D}^\kappa : x^{-1}(1)$ is finite$\}$ so S_κ is a closed subset of Ω; in particular, S_κ is strongly dominated by a second countable space M. We can take a family $\mathscr{F} = \{F_K : K \in \mathscr{K}(M)\}$ of compact subsets of S_κ which witnesses this. However, $|\mathscr{K}(M)| \leq \mathfrak{c} = \omega_1$ so $|\mathscr{F}| \leq \omega_1$ and hence S_κ is ω_1-hemicompact; since $\kappa \geq \omega_1$, we have obtained a contradiction with Theorem 1.2.13. □

1.3. The number of K-determination of a topological space

In this section we recall some properties of the cardinal invariant $\ell\Sigma$ introduced in [CMO] as a measure of how far is a space from having the Lindelöf Σ property.

Definition 1.3.1. *[CMO, Definition 2] Let Y be a topological space. The number $\ell\Sigma(Y)$ of K-determination of Y is defined as the smallest cardinal number $\mathfrak{m}$ for which it is possible to find a metric space (M,d) of weight $\mathfrak{m}$ and a usco map $\phi : M \longrightarrow Y$ such that $Y = \bigcup\{\phi(x) : x \in M\}$.*

Let us note that the class of spaces Y's with $\ell\Sigma(Y) = \omega$ is the class that authors in functional analysis refer to as countably K-determined spaces [JR, Sec. 5.1]; whereas in topology, this class of spaces is referred to as Lindelöf Σ-spaces. Here is a brief description of the contents of this section. We start by giving a characterization of the existence of usco maps $\phi : X \longrightarrow Y$ whose range covers Y in terms of families of closed sets in βY that determine Y. We study the behaviour of $\ell\Sigma(\cdot)$ on subspaces, unions, products, usco images, when taking generated subspaces and closures in Banach spaces, etc. In particular, we prove that for a compact space K we always have that $\ell\Sigma(C_p(K)) \leq w(C_p(K))$.

Theorem 1.3.2. *[CMO, Theorem 5] Let Y be a topological space and $\mathfrak{n} \leq \mathfrak{m}$ two cardinal numbers. The following statements are equivalent:*

(a) It is possible to find a topological space X with $w(X) \leq \mathfrak{m}$ and $\chi(X) \leq \mathfrak{n}$ and a usco onto map $\phi : X \longrightarrow \exp(Y)$.

(b) It is possible to find a topological subspace $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ as well as a usco onto map $\Phi : \Sigma \longrightarrow \exp(Y)$.

(c) There is a family of closed subsets $\mathscr{A} = \{A_i : i \in \mathfrak{m}\}$ in βY, with the property that for every $y \in Y$ there is a subset $L \subset \mathfrak{m}$ with $|L| \leq \mathfrak{n}$ such that

$$y \in \bigcap_{l \in L} A_l \subset Y.$$

The proposition that follows summarizes some properties of the cardinal invariant $\ell\Sigma$.

Proposition 1.3.3. *[CMO, Proposition 7] For topological spaces $Y, (Y_j)_{j \in J}$ and Z the following properties hold:*

(a) If Y is a metric space then $w(Y) = \ell\Sigma(Y)$;

(b) if J is a finite or countable set then

$$\ell\Sigma(\prod_{j \in J} Y_j) \leq \sup_{j \in J} \ell\Sigma(Y_j);$$

(c) let $Z \subset Y$ be a closed subspace, then $\ell\Sigma(Z) \leq \ell\Sigma(Y)$;

(d) if $\phi : Y \longrightarrow \exp(Z)$ is a usco onto map then $\ell\Sigma(Z) \leq \ell\Sigma(Y)$; in particular, if Z is a continuous image of Y then $\ell\Sigma(Z) \leq \ell\Sigma(Y)$;

PROOF. For (a) it suffices to show $\ell(Y) \leq \ell\Sigma(Y)$. Let $\phi : M \longrightarrow \exp(Y)$ be a usco map from some metric space onto Y. Let $\mathscr{O} = \{O_i : i \in I\}$ be an open cover of Y. Since $\phi(x)$ is compact for each $x \in X$, there exists a finite set of indexes $i_1^x; i_2^x, \ldots, i_{n_x}^x \in I$ such that $\phi(x) \subset O_{i_1^x} \cup O_{i_2^x} \cup \ldots \cup O_{i_{n_x}^x}$. We define now the open set $O(x) = \bigcup_{j=1}^{n_x} O_{i_j^x}$. Since ϕ is a usco map, there exists an open neighbourhood $U(x)$ of x such that $\phi(U(x)) \subset O(x)$. Since, $\ell(M) = w(M)$, see [Tk6, Problem 2.14], and $U = \{U(x) : x \in M\}$ is an open cover of M, there exists a subset $S \subset M$ with $|S| \leq w(M)$ such that $M = \bigcup x \in \bigcup U(x)$. Hence, we have

$$Y = \bigcup_{x \in S} \phi(U(x)) \subset \bigcup_{x \in S} O(x) = \bigcup_{x \in S} = \bigcup_{j=1}^{n_x} O_{i_j^x}.$$

Thus we have $\ell(Y) \leq w(M)$ and consequently, $\ell(Y) \leq \ell\Sigma(Y)$.

To verify (b) note that for each $j \in J$ there exists a metric space M_j for which we have $w(M_j) = \ell\Sigma(Y_j)$ and a usco onto map $\phi_j : M_j \longrightarrow \exp(Y_j)$. The metric space $M = \prod_{j\in J} M_j$ with the product topology satisfies that $w(M) \leq \sup_{j\in J} w(M_j) = \sup_{j\in J} \ell\Sigma(Y_j)$ and also, we can see that the map $\phi : M \longrightarrow \exp(\prod_{j\in J} Y_j$ defined by $\phi(x) := \prod_{j\in J} \phi_j(x_j), x = (x_j)_{j\in J}$, is a usco onto map, see [En, Theorem 3.2.10]. Hence $\ell\Sigma(\prod_{j\in J} Y_j) \leq \sup_{j\in J} \ell\Sigma(Y_j)$.

The statements (c) and (d) straightforwardly follow from the definitions. $\square$

In [COT, Theorem 3.11] it is proved that a compact space K is metrizable if and only if $K^2 \setminus \Delta$ is strongly dominated by a second countable space. The following generalization of that theorem is introduced in [CMO].

Theorem 1.3.4. *[CMO, Theorem 21] Let K be a compact space and* $\mathfrak{m}$ *a cardinal number. The following statements are equivalent:*

(a) $w(K) \leq \mathfrak{m}$;

(b) There exists a metric space M with $w(M) \leq \mathfrak{m}$ and a family $\mathscr{O} = \{O_L : L \in \mathscr{K}(M)\}$ of open sets in $K \times K$ that is basis of the neighbourhoods of Δ such that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathscr{K}(M)$;

(c) $(K \times K) \setminus \Delta$ *is strongly dominated by a metric M with $w(M) \leq \mathfrak{m}$.*

PROOF. The implication $(a) \Longrightarrow (b)$ goes as follows. Assuming that (a) holds, we have that $d(C_u(K)) = w(K) \leq \mathfrak{m}$, see [Ar2, Theorem I.1.5]. Hence we can suppose that the set $\{f_i : i \in A\} \subset C_u(K)$ where $|A| = \mathfrak{m}$ is dense in $C_u(K)$. Define $M := A$ endowed with the discrete topology and for every compact (hence finite) set $L \subset M$ we consider

$$O_L := \bigcap_{i\in L} \{(x;y) \in K \times K : |f_i(x) - f_i(y)| < \frac{1}{|L|}\}.$$

Each O_L is open in $K \times K$ and it is easily proved that O_{L_1} subset O_{L_2} whenever L_2 subset L_1 in $\mathscr{K}(M)$. On the other hand, since

$$O_L \subset \bigcap_{i\in L} \{(x;y) \in K \times K : |f_i(x) - f_i(y)| \leq \frac{1}{|L|}\}.$$

the density of $\{f_i : i \in A\}$ in $C_u(K)$ implies that $\Delta = \bigcap\{\overline{O_L} : L_2 \in \mathscr{K}(M)\}$; this last equality and a standard compactness argument allow us to conclude that $\{O_L : L_2 \in \mathscr{K}\}$ is a basis for the open neighbourhoods of Δ in $K \times K$, and therefore (b) is satisfied.

The equivalence of (b) and (c) is easily established by taking complements and defining $F_L := (K \times K) \setminus O_L$ when the $O_L^{\prime}s$ are given and $O_L := (K \times K) \setminus F_L$ when the F_L's are given. To finish we prove that (b) implies (a). Let us assume that (b) holds and given $m \in \mathbb{N}$ and a sequence $(L_1, L_2, \dots)$ in $\mathscr{K}(M)$ we define

$$\varphi(m, L_1, L_2, \dots) := \bigcap_{n \in \mathbb{N}} \{f \in mB_{C(K)} : |f_i(x) - f_i(y)| \leq \frac{1}{n} \text{ for all } (x;y) \in O_{L_n}\}$$

Note that each $\varphi(m, L_1, L_2, \dots)$ is a bounded, closed and equicontinuous family of $C_u(K)$. Therefore, Ascoli's theorem, see [Ke, p. 234], implies that $\varphi(m, L_1, L_2, \dots)$ is compact in $C_u(K)$. If $(\mathscr{K}(M), h)$ is the lattice of compact subsets of M with the Hausdorff distance, then $w(\mathscr{K}(M), h) = w(M)$, see [Sr, Proposition 2.4.14]. Therefore the product $M' := \mathbb{N} \times_{n \in \mathbb{N}} (\mathscr{K}(M), h)$ of countably many copies of $(\mathscr{K}(M), h)$ and $\mathbb{N}$ is still a metric space with $w(M') = w(M)$. Note that the definition of $\varphi(m, L_1, L_2, \dots)$ multivalued map $\varphi : M' \longrightarrow \mathscr{K}(C_u(K))$.

Being $\mathscr{O}$ a basis of neighbourhoods of Δ implies that $C(K) = \bigcap\{\varphi(x) : x \in M'\}$. On the other hand φ has the following property:

$[P]$ If a sequence $(x_k)_k$ converges in M', then $\bigcup\{\varphi(x_k) : k \in \mathbb{N}\}$ is relatively compact in $C_u(K)$.

Indeed, if $x_k := (m_k; L_1^k; L_2^k, \dots)$ converges when $k \longrightarrow \infty$ to $x = (m; L_1; L_2, \dots)$ in M' then there is $l \in \mathbb{N}$ with $m_k \leq l$ for every $k \in \mathbb{N}$ and $S_n := \bigcup_k L_n^k$ is a compact subspace of M for every $n \in \mathbb{N}$ after [Mil, Lemma 1.11.2]. The decreasing order in $\mathscr{O}$ easily implies that

$$\{\varphi(x_k) : k \in \mathbb{N}\} \subset \varphi(l, S_1, S_2 \dots)$$

and therefore property $[P]$ is proved. An appeal to [COr, Corollary 3.1] provides us with an upper semicontinuous map $\psi : M' \longrightarrow \mathscr{K}(C_u(K))$ such that $\varphi(x) \subset \psi(x)$ for every $x \in M'$. Thus $C(K) = \bigcup\{\psi(x) : x \in M'\}$. Summarizing, we have proved the inequality $\ell\Sigma(C_u(K)) \leq w(M') = \mathfrak{m}$. Since $C_u(K)$ is metric we have

$$\mathfrak{m} \geq \ell\Sigma(C_u(K)) = \ell(C_u(K)) = d(C_u(K)) = w(K)$$

and the proof is over. $\square$

1.4. Domination in $C_p(X)$

The concepts of domination and strong domination by second countable spaces have been systematically studied in the contexts of Functional Analysis and General Topology. For instance, Cascales and Orihuela proved, with some other terminology, in [CO] that a compact space K is metrizable if and only if $K^2 \setminus \Delta$ is strongly dominated by the irrationals. Meanwhile, M. Talagrand proved in [Ta] that for a compact space K we have $C_p(K)$ is dominated by the irrationals if and only if $C_p(K)$ is K-analytic. Later in [Tk1], V. Tkachuk generalized Talagrand's result for any space $C_p(X)$ with X a Tychonoff space. In [COT] it is shown that strong domination by second countable spaces sometimes yields conditions for the metrizability of some spaces. For the case of function spaces it was proved under CH in [COT, Theorem.3.10] that if K is compact and $C_p(K)$ is strongly dominated by a second countable space then K is countable and hence $C_p(K)$ is metrizable.

In [KM] the authors obtain (in ZFC) a result somewhat analogous to [COT, Theorem.3.10] when the weak topology of $C(K)$ is considered for a compact space K. The authors consider $C_w(K)$ spaces that admit a fundamental compact cover whose elements are the images under an upper semi-continuous map onto $C_w(K)$ of the points of some second countable space. They prove (with a different terminology) in [KM, Proposition 4.1] that such spaces are in fact strongly dominated by second countable spaces in the sense of [COT].

We will apply some of the ideas contained in [KM] to solve [COT, Problem 4.11], that is, we will prove in ZFC that for any compact space K the space $C_p(K)$ is strongly dominated by a second countable space if and only if it is metrizable and therefore K is countable. The authors of [COT] also ask in [COT, Problem 4.10] if the same conclusion can be drawn without assuming that K is compact. In the next subsection we will provide a few partial answers to that and some other open problems posed in [COT].

1.4.1. Domination index, definition and applications

In this section we introduce for any space X the cardinal invariant $dm(X)$ which extends the concept of domination by a second countable space.

Given a space X let M be the space obtained by giving $\mathscr{K}(X)$ the discrete topology. It is clear that X is strongly dominated by M. The fact that every topological space is dominated by a metric space motivates the following.

Definition 1.4.1. *[Gue1, Definition 3.1] For a space X the metric domination index of X denoted by $dm(X)$ is the cardinal defined by*

$$dm(X) = min\{w(M) : M \text{ is a metric space that dominates } X\}.$$

Analogously, we define the index of strong metric domination.

Definition 1.4.2. *[Gue1, Definition 3.2] For a space X the strong metric domination index of X denoted by $sdm(X)$ is the cardinal defined by*

$$sdm(X) = min\{w(M) : M \text{ is a metric space that strongly dominates } X\}.$$

We will establish its relationship with other known cardinal functions such as $\ell\Sigma(X)$ the number of K-determination of a topological space X. In this case it is easy to see that $dm(X) \leq \ell\Sigma(X)$ but both invariants coincide in spaces where countably compact subsets are compact, for example angelic spaces.

These observations turn out to be useful when considering domination and specially strong domination of function spaces. In particular, we can prove that for a compact space K the space $C_p(K)$ is strongly dominated by a second countable space if and only if it is metrizable if and only if K is countable. We will also consider spaces $C_p(X)$ strongly dominated by second countable ones when X is a Tychonoff, not necessarily compact space. We will show that the study of this general case can be reduced to the the study of the case when X is a Lindelöf Σ space.

The properties of a space X for which $dm(X) \leq \omega$ have been systematically studied in [COT, Theorem 2.1] and most of them generalize to larger values of $dm(X)$. We can also point out that the relation between the invariants $dm(X)$ and $\ell\Sigma(X)$ is almost the same as the one between the concepts of domination by second countable spaces and the Lindelöf Σ property. To describe this behaviour it suffices to generalize the corresponding arguments in [COT]. Next we summarize such generalizations.

Proposition 1.4.3. *[Gue1, Proposition 3.3] For any space X the following hold:*

(I) $dm(X) \leq \ell\Sigma(X)$.

(II) *If $dm(X) \leq \kappa$ then the metric domination index of any continuous image of X is also not greater than κ.*

(III) *If $dm(X) \leq \kappa$ and Y is any closed subset of X then $dm(Y) \leq \kappa$.*

(IV) *If $X = \bigcup_{i \in \omega} X_i$ and $dm(X_i) \leq \kappa$ then $dm(X) \leq \kappa$.*

(V) *If $X = \prod_{i \in \omega} X_i$ and $dm(X_i) \leq \kappa$ then $dm(X) \leq \kappa$.*

(VI) $\mathrm{ext}(X) \leq dm(X)$.

PROOF. Suppose that $\ell\Sigma(X) = \kappa$. We can find a metric space M and a compact valued usco onto map $\varphi : M \longrightarrow X$. If we let $F_K = \bigcup\{\varphi(x) : x \in K\}$ for any compact set $K \subset M$ then the family $\mathscr{F} = \{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of X. Then (I) is proved.

To settle (II) and (III) just observe that if $\{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of some space X and Y is a continuous image of X under a map φ, then the family $\{\phi(F_K) : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of Y. Also, if Z is a closed subset of X then $\{F_K \cap Z : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of Z.

To verify (IV) suppose that for every $i \in \omega$ there is a family $\mathscr{F}_i = \{P(K,i) : K \in \mathscr{K}(M_i)\}$ which is an M_i-ordered compact cover of X_i for some metric space M_i with $w(M_i) \leq \kappa$. The space $M = \bigoplus_{i \in \omega} M_i$ has weight not greater than κ; we identify every M_i with the corresponding clopen subset of M. Given any $K \in \mathscr{K}(M)$ the set $N_K = \{i \in \omega : K \cap M_i \neq \varnothing\}$ is finite so the set $F_K = \bigcup_{i \in \omega} P(K \cap M_i, i) : i \in N_K$ is compact. It is immediate that the family $\{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of X.

Now for each $i \in \omega$ fix a metric space M_i with $w(M_i) \leq \kappa$ and an M_i-ordered compact cover $\mathscr{F}_i = \{Q(K,i) : K \in \mathscr{K}(X_i)\}$ of the space X_i. The space $M = \prod_{i \in \omega} M_i$ is metrizable and $w(M) \leq \kappa$. Let $p_i : M \longrightarrow M_i$ be the natural projection for every $i \in \omega$. Given any $K \in \mathscr{K}(M)$, the set $F_K = \prod\{Q(p_i(K), i) : i \in \omega\}$ belongs to $\mathscr{K}(X)$. It follows that the family $\{F_K : K \in \mathscr{K}(M)\}$ is an M-ordered compact cover of X and (V) is settled.

The fact (VI) can be deduced as follows. Take a discrete subspace $D \in X$ it follows from (III) and (I) that $dm(D) \leq dm(X) \leq \ell\Sigma(X)$. Apply [CMO, Proposition 7 (i), (iii)] to obtain $dm(D) \leq \ell\Sigma(D) \leq \ell(D) = |D|$. $\square$

Suppose $\ell\Sigma(X) \leq \kappa$ implies certain property $\mathscr{P}$ for some space X. Observe that from Proposition 1.4.3 (I) the condition $dm(X) \leq \kappa$ is weaker than $\ell\Sigma(X) \leq \kappa$ so it is worth to ask ourselves if $dm(X) \leq \kappa$ also implies $\mathscr{P}$. An affirmative answer for a property $\mathscr{P}$ constitutes a generalization for the corresponding result concerning the cardinal $\ell\Sigma$. We will find some such generalizations.

The following definition is due to Arhangel'skii.

Definition 1.4.4. *For any space X the class $\mathscr{E}(X)$ consists of all continuous images of $X \times K$ where K is a compact space. If κ is a cardinal, the so called κ-hull of X is the space $O_\kappa(X) = (X \times D(\kappa))^\kappa$ where $D(\kappa)$ is the discrete space of cardinality κ. A closed hereditary class $\mathscr{P}$ is called κ-$perfect$ if for every space $X \in \mathscr{P}$ it happens that $O_\kappa \in \mathscr{P}$ and $\mathscr{E}(X) \subset \mathscr{P}$.*

As a consequence of Proposition 1.4.3 (II, II, V) we have the following.

Remark 1.4.5. *For any cardinal κ the class of all the topological spaces X for which $dm(X) \leq \kappa$ is ω-$perfect$.*

We can apply now [Ar Theorem IV.2.14] to generalize [Mu, Teorema 2.4.10].

Corollary 1.4.6. *For any compact space K the following properties are equivalent:*

(I) *$dm(C_p(K)) \leq \kappa$.*

(II) *There exists $Y \subset C_p(K)$ such that Y is dense in $C_p(K)$ and $dm(Y) \leq \kappa$.*

(III) *There exists $Y \subset C_p(K)$ such that Y separates the points of K and $dm(Y) \leq \kappa$.*

(IV) *The space K embeds into a space Y with $dm(Y) \leq \kappa$.*

Propositions 2.4.11 and 2.4.12 of [Mu] can also be generalized applying Proposition 1.4.3 (II) and (III).

Proposition 1.4.7. *Let K be a compact space, if L is a closed subspace of K or a continuous image of K then $dm(C_p(L)) \leq dm(C_p(K))$.*

PROOF. Suppose L is a closed subset of K, then by [Tk6, Problem 152] the restriction map $\pi_L : C_p(K) \longrightarrow C_p(L)$ defined by $\pi_L(f) = f|L$ for any $f \in C_p(K)$ maps continuously $C_p(K)$ onto $C_p(L)$. By Proposition 1.4.3 (II) we have $dm(C_p(L)) \leq dm(C_p(K))$. Now if L is a continuous image of K then by compactness of K and [Tk6, Problem 163] the space $C_p(L)$ is homeomorphic to a closed subspace of $C_p(K)$ and Proposition 1.4.3 (III) finishes the proof. □

Analogously we can generalize the proposition 2.4.13 of [Mu].

Proposition 1.4.8. *Let K_n be a compact space such that $dm(C_p(K_n)) \leq \kappa$ for every $n \in \omega$ then $dm(C_p(\prod_{n\in\omega} K_n)) \leq \kappa$.*

PROOF. Let $\pi_m : \prod_{n\in\omega} K_n \longrightarrow K_m$ be the projection determined by m. Define the following map $p_m : C_p(K_m) \longrightarrow C_p(\prod_{n\in\omega} K_n)$ by $p_m(f) = f \circ \pi_m$. Since p_m is continuous, we can apply Proposition 1.4.3 (II) and (IV) to see that

$$dm(p_m(C_p(K_m)) \leq \kappa$$

and

$$dm(\bigcup_{n\in\omega} p_m(C_p(K_m))) \leq \kappa.$$

Finally $\bigcup_{n\in\omega} p_m(C_p(K_m))$ separates the points of $\prod_{n\in\omega} K_n$ so by Corollary 1.4.6 we can conclude $dm(C_p(\prod_{n\in\omega} K_n)) \leq \kappa$. □

In [Tk1] Tkachuk constructed a space X which is not Lindelöf but $dm(X) \leq \omega$. Also, in [CMO] the authors attribute to J. Pelant the example introduced in their paper of a space for which $\omega = dm(X) < \ell\Sigma(X)$. However, in [COT] it is proved that for any space X we have $dm(C_p(X)) \leq \omega$ if and only if $\ell\Sigma(C_p(X)) \leq \omega$, and the authors also show that the equivalence $dm(Y) \leq \omega$ if and only if $\ell\Sigma(Y) \leq \omega$ is true for any space in which every relatively countably compact subset has compact closure, for instance in angelic spaces.

Actually both cardinal invariants dm and $\ell\Sigma$ coincide in the class of angelic spaces. This fact was implicitly proved in [CMO]. Here we give an argument based on the ideas in [COT]. First we use the ideas contained in [COT] to prove the following lemma which is a characterization of the spaces X for which $\ell\Sigma(X) \leq \kappa$.

Lemma 1.4.9. *If κ is a cardinal and a space X has a compact cover $\mathscr{C}$ for which there is a network $\mathscr{N}$ of cardinality κ such that for each $K \in \mathscr{C}$ there is a countable family $\mathscr{N}(K) \subset \mathscr{N}$ that is a network for K in the sense that for every $U \in \tau(K,X)$ there is $N \in \mathscr{N}(K)$ with $K \subset \mathbb{N} \subset U$; then $\ell\Sigma(X) \leq \kappa$.*

PROOF. We will first show that for every $C \in \mathscr{C}$ we have

$$\bigcap\{cl_{\beta X}(N) : N \in \mathscr{N}_C\} = cl_{\beta X}(\bigcap\{N : N \in \mathscr{N}_C\} = C.$$

It is clear that $C \subset \bigcap\{cl_{\beta X}(N) : N \in \mathscr{N}_C\}$. So we need to prove only that the set $\bigcap\{cl_{\beta X}(N) : N \in \mathscr{N}_C\} \subset C$. If this is not so, take a point $x \in (\bigcap\{cl_{\beta X}(N) : N \in \mathscr{N}_C\}) \setminus C$. There is $U \in \tau(x, \beta X)$ such that $cl_{\beta X}(U) \cap C = \varnothing$; let $V = (\beta X \setminus cl_{\beta X}(U)) \cap X$. It is clear that $V \in \tau(C,X)$, thus there exist $N \in \mathscr{N}_C$ such that $C \subset N \subset V \subset (\beta X \setminus cl_{\beta X}(U))$ and hence $cl_{\beta X}(N) \subset cl_{\beta X}((\beta X \setminus cl_{\beta X}(U)))$ which implies that $cl_{\beta X}(N) \cap U = \varnothing$, a contradiction.

Now the family $\mathscr{F} = \{cl_{\beta X}(N) : N \in \mathscr{N}\}$ has cardinality κ and consists of compact subsets of βX. For each $C \in \mathscr{C}$ the family $\mathscr{F}_C = \{cl_{\beta X}(N) : N \in \mathscr{N}_C\}$ is countable and $C = \bigcap \mathscr{F}_C$. Finally, for every $y \in X$ there is $C \in \mathscr{C}$ such that $y \in C = \bigcap \mathscr{F}_C$ hence we can apply Theorem 1.3.2 to conclude that $\ell\Sigma(X) \leq \kappa$. □

Remark 1.4.10. *[Gue1, Remark 3.4] If X is a space in which the relatively compact spaces have compact closure then $dm(X) = \ell\Sigma(X)$. In particular for any angelic space the invariants dm and $\ell\Sigma$ coincide.*

PROOF. First we generalize [COT, Proposition 2.5] and [COT, Proposition 2.6].

Claim 1. *Suppose that X is dominated by a metric space M and a collection $\{F_K : K \in \mathscr{K}(M)\}$ witnesses this. Take a σ-discrete base $\mathscr{B}$ of M. For each $K \in \mathscr{K}(M)$ take a countable outer base $\mathscr{B}_K = \{U_n : n \in \omega\}$ such that for each $n \in \omega$ we have $U_{n+1} \subset U_n$ and U_n is the union of a finite family of elements of $\mathscr{B}$. Let $\mathscr{B}' = \bigcup\{\mathscr{B}_K : K \in \mathscr{K}(M)\}$ and for each $U \in \mathscr{B}'$ let $G(U) = \bigcup\{F_K : K \in \mathscr{K}(M)$ and $K \in U\}$. Fix $K \in \mathscr{K}(M)$; then $F_K \subset C_K = \bigcup\{G(U) : U \in \mathscr{B}_K\}$. If $S = \{y_n : n \in \omega\} \subset X$ is a sequence such that $y_n \in G(U_n)$ for all $n \in \omega$ then:*

(a) The set $\overline{S}$ is compact and hence the set D of cluster points of S is non-empty.

(b) There exists a compact set Q_K such that $D \subset Q_K \subset C_K$.

PROOF OF THE CLAIM. Take a set $K_n \in \mathscr{K}(\mathscr{M})$ such that $K_n \subset U_n$ and $y_n \in F_{K_n}$ for any $n \in \omega$. It is straightforward that the set $L_m = K \cup (\bigcup\{K_i : i \geq m\})$ is compact for any $m \in \omega$. The sequence $\{y_n\}$ is eventually in the compact set F_{L_m} which shows that the set $\overline{S}$ is compact, $D \neq \varnothing$ and $D \subset F_{L_m}$ for any $m \in \omega$. Therefore D is contained in the compact set $Q_K = \bigcap\{F_{L_m} : m \in \omega\} \subset C_K$ as promised.

Claim 2. *Suppose that X is dominated by a metric space M and a collection $\{F_K : K \in \mathscr{K}(M)\}$ witnesses this. Take a σ-discrete base $\mathscr{B}$ of M. For each $K \in \mathscr{K}(M)$ take a countable outer base $\mathscr{B}_K = \{U_n : n \in \omega\}$ such that for each $n \in \omega$ we have $U_{n+1} \subset U_n$ and U_n is the union of a finite family of elements of $\mathscr{B}$. Let $\mathscr{B}' = \bigcup\{\mathscr{B}_K : K \in \mathscr{K}(M)\}$ and for each $U \in \mathscr{B}'$ let $G(U) = \bigcup\{F_K : K \in \mathscr{K}(M)$ and $K \in U\}$. Then there exists a family $\mathscr{C}$ in the space X with the following properties:*

(a) If $C \in \mathscr{C}$ and $A \subset C$ is a countable set then the set $\overline{A}$ is compact and $\overline{A} \subset C$; in particular, each $C \in \mathscr{C}$ is countably compact;

(b) For every $K \in \mathscr{K}(M)$ there exists a set $C_K \in C$ such that $F_K \subset C_K$ and hence $\mathscr{C}$ is a cover of X;

(c) The family $\mathscr{N} = \{G(U) : U \in \mathscr{B}'\}$ is a network with respect to $\mathscr{C}$, moreover for each $C \in \mathscr{C}$ there is a countable subfamily $\mathscr{N}_C \subset \mathscr{N}$ such that for every $V \in \tau(C,X)$ there is $N \in \mathscr{N}_C$ such that $C \subset N \subset V$.

PROOF OF THE CLAIM. If $K \in \mathscr{K}(M)$; then $F_K \subset C_K = \bigcup\{G(U) : U \in \mathscr{B}_K\}$. Let the family $\mathscr{C} = \{C_K : K \in \mathscr{K}(M)\}$; it is clear that (b) holds for C_K.

Take $K \in \mathscr{K}(M)$ and $\mathscr{N}_{C_K} = \{G(U_n) : n \in \omega\}$ is not a network for C_K then we can choose a point $y_n \in G(U_n) \setminus W$ for some $W \in \tau(C_K, X)$. The sequence y_n must have a cluster point in C_K by Claim 1 which contradicts the fact that $\{y_n\} \subset X \setminus W$ while $C_k \subset W$. Therefore the family $\mathscr{C}$ has the property (c).

Furthermore, if $A \subset C_K$ is countable then $A = \{y_n : n \in \omega\}$. It is clear $y_n \in G(U_n)$ for all $n \in \omega$ and hence we can apply Claim 2 to see that $\overline{A} = \overline{\{y_n : n \in \omega\}}$ is compact. If $x \in \overline{A} \setminus C_K$ then $x \in \overline{A} \setminus A$ and hence x is a cluster point of the sequence $S = \{y_n\}$. However, all cluster points of S belong to C_K by Claim 1. This contradiction shows that $\overline{A} \subset C_K$ so (a) is proved.

Now returning to our solution, we can apply the previous lemma to the compact cover $\mathscr{C}$ to conclude the proof. $\square$

Notice that for function spaces we have the following. If X is a compact or a separable space then $dm(C_p(X)) = \ell\Sigma(C_p(X))$ because $C_p(X)$ is angelic. On the other hand, for every Tychonoff space X it happens that $\omega = dm(C_p(X))$ if and only if $C_p(X)$ is a Lindelöf Σ space. This observation motivates the question:

> Is it true that $dm(C_p(X)) = \ell\Sigma(C_p(X))$ for every Tychonoff space X? (see Problem 5)

From Remark 1.4.10 we can obtain a partial affirmative answer to the previous question. First let us observe that if $iw(X)$ is countable for some space X then $dm(X) = \ell\Sigma(X)$.

An application of Remark 1.4.10 is the following generalization of [Mu, Teorema 2.5.4].

Theorem 1.4.11. *Let K be a compact subspace of $C_p(Y)$ for some space Y such that $dm(Y) \leq \kappa$ then K is strongly κ-monolithic.*

PROOF. Take $A \subset K$ with $|A| \leq \kappa$. Let $L = \overline{A}$. By [Tk6, Problem 166] the evaluation map $E^L : Y \longrightarrow C_p(L)$ defined by $E^L(y) = e^L_y$ where $e^L_y : L \longrightarrow \mathbb{R}$ is defined by $e^L_y(f) = f(y)$ for any $f \in L$; is continuous. Besides $E^L(Y) \subset C_p(L)$ separates the points of L therefore $dm(E^L(Y)) \leq \kappa$ implies $\ell\Sigma(C_p(L)) = dm(C_p(L)) \leq \kappa$.

We now apply inequality 2.17 in the proof of [Mu, Teorema 2.5.4] to observe that $nw(C(L), \tau_p(A)) \leq |A| \leq \kappa$. Finally by [Mu, Proposición 2.3.14] we have the inequality $d(C_p(L)) \leq max\{nw(C(L), \tau_p(A)), \ell\Sigma\} = \kappa$. It follows $w(L) = iw(L) = d(C_p(L)) \leq \kappa$. $\square$

As a consequence we can extend [Mu, Corolario 2.5.5].

Corollary 1.4.12. *If $dm(X) \leq \omega$ and $K \subset C_p(X)$ is compact and separable then K is metrizable.*

The following generalizes Theorem 1.2.5 of Section 1.2.

Theorem 1.4.13. *Suppose that Z is a space of countable tightness. Then for any set $X \subset Z$ we have $dm(X) = \ell\Sigma(X)$.*

PROOF. Fix any set $X \subset Z$ and assume that $dm(X) \leq \kappa$. For any set $A \subset X$ we denote by $cl_X(A)(or\ cl_Z(A))$ the closure of the set A in the space X (or in Z respectively). By Claim 2 , there exists a cover $\mathscr{C}$ of the space X such that if $C \in \mathscr{C}$ and $A \subset C$ is a countable set then the set $\overline{A}$ is compact and $\overline{A} \subset C$; in particular, each $C \in \mathscr{C}$ is countably compact; and we can find a family $\mathscr{N}$ which is a network with respect to $\mathscr{C}$, and for each $C \in \mathscr{C}$ there is a countable subfamily $\mathscr{N}_C \subset \mathscr{N}$ such that for every $V \in \tau(C,X)$ there is $N \in \mathscr{N}_C$ such that $C \subset N \subset V$. If $C \in \mathscr{C}$ and C is not closed in Z then we can find a point $x \in cl_Z(C) \setminus C$. By countable tightness of Z, there exists a countable $A \subset C$ such that $x \in cl_Z(A)$. The set $F = cl_X(A) \subset C$ is compact and hence closed in Z; as a consequence, $x \in cl_Z(A) \subset F \subset C$. This contradiction shows that every $C \in \mathscr{C}$ is compact being closed in X. Thus the family $\mathscr{N}$ is a network with respect to the compact cover $\mathscr{C}$ and for each $C \in \mathscr{C}$ there is a countable subfamily $\mathscr{N}_C \subset \mathscr{N}$ such that for every $V \in \tau(C,X)$ there is $N \in \mathscr{N}_C$ such that $C \subset N \subset V$ which says that $\ell\Sigma(X) \leq \kappa$. □

The following is a generalization of Theorem 1.2.6.

Theorem 1.4.14. *If K is a compact space with $t(K) \leq \omega$ and $dm(K^2 \setminus \Delta) \leq \kappa$ then $w(K) \leq \kappa$.*

PROOF. The space K^2 also has countable tightness [Ar2, Theorem 2.3.3] so we can apply Theorem1.4.13 to the set $K^2 \setminus \Delta \subset K \times K$ to conclude that $\ell\Sigma(K^2 \setminus \Delta) \leq \kappa$; implies that $w(K) \leq \kappa$. □

The following definition is due to Kąkol, López-Pellicer and Okunev, see [KLO].

Definition 1.4.15. *A space X is $\Sigma(\kappa)$-quasi-Suslin if it is possible to define a set-valued map $T : \Sigma \longrightarrow \exp(X)$ from a non-empty subset $\Sigma \subset D(\kappa)^\omega$ covering X such that if the sequence $\alpha_n \longrightarrow \alpha$ in Σ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$. If $\Sigma = \omega^\omega$, then X is called a quasi-Suslin space.*

Clearly $\Sigma(\kappa)$-quasi-Suslin spaces are a generalization of quasi-Suslin spaces introduced by Valdivia in [Va].

To show the relationship between domination and $\Sigma(\kappa)$-quasi-Suslin spaces we will proceed as in the proof of Claim 1 of Remark 1.4.10 and use some of the ideas in Section 1.6 of [Mu].

Theorem 1.4.16. *Given an infinite cardinal κ, if $dm(X) \le \kappa$ for some space X, then X is $\Sigma(\kappa)$-quasi-Suslin.*

PROOF. Take a metric space M of weight κ that dominates X and a compact collection $\{F_K : K \in \mathscr{K}(M)\}$ witnesses this. Take a σ-discrete base $\mathscr{B}$ of M. For each $K \in \mathscr{K}(M)$ take a countable outer base $\mathscr{B}_K = \{U_n : n \in \omega\}$ such that for each $n \in \omega$ we have $U_{n+1} \subset U_n$ and U_n is the union of a finite family of elements of $\mathscr{B}$. Let $\mathscr{B}' = \bigcup\{\mathscr{B}_K : K \in \mathscr{K}(M)\}$ and for each $U \in \mathscr{B}'$ let $G(U) = \bigcup\{F_K : K \in \mathscr{K}(M)$ and $K \in U\}$. Fix $K \in \mathscr{K}(M)$; then $F_K \subset C_K = \bigcup\{G(U) : U \in \mathscr{B}_K\}$.

Let d be the metric of M. The Vietoris Topology in $M' = (\mathscr{K}(M), d_H)$ is a metric topology generated by the Hausdorff metric, defined by the formula

$$d_H(K,K') = \text{ínf}\{r > 0 : K' \subset B_d(K,r) \textit{ and } K \subset B_d(K',r)\}$$

where $B_d(K,r) = \{x \in M : d(x,K) = \text{ínf}\{d(x,x') : x' \in K\} = r\}$. Define the set valued map $T : M' \longrightarrow \exp(X)$ by $T(K) = C_K$. Suppose that $K_n \longrightarrow K$ in M' and that $y_n \in C_{K_n}$. For every $n \in \omega$ there is $m_n \in \omega$ such that $K_i \subset U_n \in \mathscr{B}_K$ for every $i > m_n$. By construction $y_{m_n} \in G(U_n)$ for all $n \in \omega$. Put $S = \{y_{m_n} : n \in \omega\} \subset X$ and let D be the set of cluster points of S. Apply Claim 1, Remark 1.4.10 to find a compact space Q_K such that $D \subset Q_K \subset C_K$.

It follows from [Mu, Corolario 1.6.10] that $(\mathscr{K}(M), d_H)$ is a metric space of weight κ therefore it is a continuous image of a subspace $\Sigma \subset D(\kappa)^\omega$. We can conclude that X is $\Sigma(\kappa)$-quasi-Suslin. $\square$

In the rest of this section we will deal mostly with strong domination by metric spaces, so we list the properties of spaces strongly dominated by some metric space M.

Proposition 1.4.17. *For any space X the following hold:*

(I) *If $sdm(X) \le \kappa$ and the space Y is a compact-covering continuous image of X then $sdm(Y) \le \kappa$;*

(II) *if $sdm(X) \le \kappa$ and Y is a closed subspace of X then $sdm(Y) \le \kappa$;*

(III) *if $sdm(X_i) \le \kappa$ for every $i \in \omega$ then $sdm(\prod_{i\in\omega} X_i) \le \kappa$;*

(IV) *if X is a space and $Y_i \subset X$ with $sdm(Y_i) \le \kappa$ for each $i \in \omega$ then $sdm(\bigcap_{i\in\omega} Y_i) \le \kappa$.*

PROOF. Suppose $\varphi : X \longrightarrow Y$ is compact covering and that X is strongly dominated by M. Take an M-ordered fundamental compact cover $\{F_K : K \in \mathscr{K}(M)\}$ of X. Since φ is compact covering it follows that $\{\varphi(F_K) : K \in \mathscr{K}(M)\}$ is an M-ordered fundamental compact cover of Y.

If $\{F_K : K \in \mathscr{K}(M)\}$ is a fundamental M-ordered compact cover of some space X and Y is a closed subset of X then $\{F_K \cap Y : K \in \mathscr{K}(M)\}$ is a fundamental M-ordered compact cover of Y.

Assume that X_i is strongly dominated by a metric space M_i with $w(M_i) \leq \kappa$ and fix a respective family $F_i = \{F_i(K) : K \in \mathscr{K}(M_i)\}$ for any $i \in \omega$. The space $M = \prod_{i\in\omega} M_i$ is metrizable and $w(M) \leq \kappa$; let $\pi : M \longrightarrow M_i$ be the natural projection for each $i \in \omega$. If $K \in \mathscr{K}(M)$ then $F_K = \prod_{i\in\omega} F_i(\pi(K))$ is easily seen to be a compact subset of $X = \prod_{i\in\omega} X_i$. Let $\pi : X \longrightarrow X_i$ be the natural projection for every $i \in \omega$. The family $\mathscr{F} = \{F_K : K \in \mathscr{K}(M)\}$ witnesses that X is strongly dominated by M. Indeed, if Q is a compact subset of X then we can choose $K_i \in \mathscr{K}(Mi)$ such that $\pi(Q) \subset F_i(K_i)$ for each $i \in \omega$; for the set $K = \prod_{i\in\omega} K_i$ we have $Q \subset F_K$. It is immediate that $K \subset L$ implies $F_K \subset F_L$ so we settled (III).

Now observe that $Y = \bigcap_{i\in\omega} Y_i$ is homeomorphic to a closed subspace of $\prod_{i\in\omega} Y_i$ so we can apply (II) and (III) to finish the proof. □

We will now apply what we know about metric (strong) domination to the study of function spaces.

Lemma 1.4.18. *[Gue1, Lemma 3.6] Given an infinite cardinal κ, if X is a space of cardinality κ with a unique non-isolated point and $C_p(X)$ is strongly dominated by some space M then $\mathbb{R}^\kappa$ is dominated by M.*

PROOF. Let $\{x_\alpha : \alpha \in \kappa\}$ be an enumeration of the set of isolated points of X. For each $K \in \mathscr{K}(C_p(X))$ and every $\alpha < \kappa$ let $m_\alpha(K) = max\{f(\alpha) : f \in K\}$ and

$$n_\alpha(K) = min\{f(\alpha) : f \in K\}.$$

Define the map $\varphi : \mathscr{K}(C_p(X)) \to \mathscr{K}(\mathbb{R}^\kappa)$ by $\varphi(K) = \prod_{\alpha<\kappa} [n_\alpha(K), m_\alpha(K)]$. Clearly $\varphi(K) \in \mathscr{K}(\mathbb{R}^\kappa)$. Observe that for every K and L in $\mathscr{K}(C_p(X))$ if $K \subset L$ then we have $\varphi(K) \subset \varphi(L)$. Moreover if $C \in \mathscr{K}(\mathbb{R}^\kappa)$ then it is possible to find a family of real closed intervals $\{[a_\alpha, b_\alpha] : \alpha :\in \kappa\}$ such that $C \subset \prod_{\alpha<\kappa} [a_\alpha, b_\alpha]$. If we call $u \in C_p(X)$ the function such that $u(x) = 0$ for every $x \in X$ then the set

$$K = \{a_\alpha \chi_{x_\alpha} : \alpha < \kappa\} \cup \{b_\alpha \chi_{x_\alpha} : \alpha < \kappa\} \cup \{u\}$$

is a compact subset of $C_p(X)$ and we have $C \subset \varphi(K)$.

Since $C_p(X)$ is strongly dominated by a metric space M there is a fundamental M-ordered compact cover $\mathscr{F} = \{K_L : L \in \mathscr{K}(M)\} \subset \mathscr{K}(C_p(X))$. Therefore it is possible to find $L \in \mathscr{K}(M)$ such that $K \subset K_L$ and hence $C \subset \varphi(K_L)$. It follows that the family $\{\varphi(K_L) : L \in \mathscr{K}(M)\}$ witnesses that M strongly dominates $\mathbb{R}^\kappa$. □

Definition 1.4.19. *Define the cardinal*

$$\mathfrak{l} = min\{\gamma : \mathbb{R}^\gamma \textit{ does not contain a closed discrete set of cardinality } \gamma\},$$

observe that $\omega_1 < \mathfrak{l}$.

Corollary 1.4.20. *[Gue1, Corollary 3.7] Suppose* $\kappa < \mathfrak{l}$. *If* X *is a space of cardinality* κ *with a unique non-isolated point and* $C_p(X)$ *is strongly dominated by some metric space* M *then* $\kappa \leq w(M)$.

PROOF. Observe that $\kappa \leq ext(\mathbb{R}^\kappa)$ and apply Lemma 1.4.18 and Proposition 1.4.3. □

Remark 1.4.21. *[Gue1, Remark 3.8] Notice that Lemma 1.4.18 and Corollary 1.4.20 imply that if* $sdm(C_p(X)) \leq \kappa$ *for some cardinal* $\kappa < \mathfrak{l}$ *and a space with a unique non-isolated point* X, *then* $|X| \leq \kappa$.

PROOF. Suppose $\kappa < |X| = \lambda$. Apply Lemma 1.4.20 to deduce that $\mathbb{R}^\lambda$ is dominated by a metric space M with weight equal to κ. Since $\kappa \leq \mathfrak{l}$ there is a closed discrete space $D \subset \mathbb{R}^\lambda$ of cardinality λ. By Proposition 1.4.3 (III) the space D is dominated by M and by Proposition 1.4.3 (IV) we have $|D| \leq w(M) = \kappa < \lambda$ a contradiction. □

Given an arbitrary cardinal κ it is not yet quite clear what properties of a space X can be deduced from the fact that $sdm(C_p(X)) \leq \kappa$. Nevertheless we can summarize what we know so far.

Theorem 1.4.22. *[Gue1, Theorem 3.9] For a space X such that $sdm(C_p(X)) \leq \kappa$, the following hold:*

(I) *The space X has at most κ many isolated points.*

(II) *If X is scattered then $d(X) \leq \kappa$.*

(III) *For every Eberlein-Grothendieck Čech-complete scattered space X we have the equality $c(X) = w(X) \leq \kappa$.*

PROOF. To settle (I), let $D \subset X$ be the set of isolated points of X. Identify $X \setminus D$ with a single point obtaining a space Y which is a quotient image of X and has a unique non isolated point. The space $C_p(Y)$ embeds in $C_p(X)$ as a closed subspace [Tk6, Problem 163(iii)] and therefore $sdm(C_p(Y)) \leq \kappa$. Apply Remark 1.4.21 to conclude it is countable and so is D.

The statement (II) is an immediate consequence of (I).

Recall that $c(X) = w(X) \leq \kappa$ for every Eberlein-Grothendieck Čech-complete scattered space and apply (I) to prove (III). □

Focussing on the countable case, we can say much more about a space X such that $sdm(C_p(X)) \leq \omega$. First let us prove that the study of function spaces for which we have $sdm(C_p(X)) \leq \omega$ can be reduced to the case when X is a Lindelöf Σ space.

Theorem 1.4.23. *[Gue1, Theorem 3.10] For any X the space $C_p(X)$ is strongly dominated by a second countable space if and only if $C_p(\upsilon X)$ is strongly dominated by a second countable space.*

PROOF. Suppose that $C_p(X)$ is strongly dominated by a second countable space M and fix a family $\{F_K : K \in \mathscr{K}(M)\}$ which witnesses this. It follows from [COT, Theorem 2.15] that $C_p(X)$ and therefore υX is Lindelöf Σ by [Ar2, Theorem IV.9.5]. Consider the restriction map $\pi : C_p(\upsilon X) \to C_p(X)$. If $G_K = \pi^{-1}(F_K)$ then G_K is compact for any $K \in \mathscr{K}(M)$ (see [Tk4, Theorem 2.6]). It is clear that $\mathscr{G} = \{G_K : K \in \mathscr{K}(M)\}$ is a cover of $C_p(\upsilon X)$. Also if $C \subset C_p(\upsilon X)$ is compact then there is $K \in \mathscr{K}(M)$ such that $\pi(C) \subset F_K$ and therefore $C \subset \pi^{-1}(F_k) \in \mathscr{G}$. This shows that $C_p(\upsilon X)$ is strongly dominated by M.

Now assume that $C_p(\upsilon X)$ is strongly dominated by a second countable space. The restriction map $\pi : C_p(\upsilon X) \to C_p(X)$ is compact-covering as follows from [Tk4, Theorem 2.6]. By Proposition 1.4.17 (I) the space $C_p(X)$ is strongly dominated by a second countable space. $\square$

Theorem 1.4.24. *[Gue1, Theorem 3.11] For a space X such that $C_p(X)$ is strongly dominated by a second countable space, the following hold:*

(I) *If X is separable then it is countable.*

(II) *If X is scattered then it is countable.*

(III) *Every second countable continuous image of X is countable.*

(IV) *If X is compact then it is countable.*

(V) *If X is pseudocompact then it is countable.*

(VI) *If $K \subset X$ is compact then K is scattered.*

(VII) *If X is Lindelöf-p then X is the union of countably many compact scattered subsets.*

PROOF. The statement (I) is a direct consequence of the equality $\omega = d(X) = iw(C_p(X))$ ([Tk6, Problem 173]) which by [COT, Theorem 3.6] implies $C_p(X)$ is an $\aleph_0$-space and hence X is countable by [Mi, Proposition 10.7].

The fact (II) follows immediately from Theorem 1.4.22 (I) and (I).

To establish (III) let $f : X \to Y$ be continuous onto and Y second countable. It is possible to find a space Z and continuous maps $h : X \to Z$ and $g : Z \to Y$ such that h is $\mathbb{R}$-quotient, g is a condensation and $f = g \circ h$. Since $dm(C_p(Z)) \leq sdm(C_p(Z)) \leq \omega$, it follows from [COT, Theorem 2.15] that $C_p(Z)$ is Lindelöf Σ and ω-monolithic by [Ar2, Theorem IV.9.8]. Since the space Z condenses onto a second countable space, we have the following equalities $\omega = iw(Z) = d(C_p(Z)) = nw(C_p(Z)) = nw(Z) = d(Z)$. Apply (I) to conclude that Z is countable as well as Y.

To deduce (IV) notice that the compact space X has to be scattered by (III) and apply (II).

To prove (V) observe that $\upsilon X = \beta X$, so $sdm(C_p(\beta X)) \leq \omega$ by Theorem 1.4.23, now apply (IV) to finish the proof.

To obtain (VI) note that by the fact (III) every continuous real image of K is countable.

If X is Lindelöf-p then there is a second countable space M that is a perfect image of X. By (III) the space M is countable which means that X is the union of countably many compact subsets and each one of them is scattered by (VI). Thus (VII) is proved. □

Notice that Theorem 1.4.24 (IV) provides a positive answer to Problem 4.11 of [COT]. Also the rest of the statements in Theorem 3.11 give some partial answers to Problems 4.9 and 4.10 of [COT]. Still we cannot fully answer Problem 4.10 but Theorem 1.4.23 shows that the general situation reduces to the case when X is Lindelöf Σ.

It is clear that if X is a discrete space and $sdm(X) \leq \kappa$ then $|X| = \kappa$. This simple fact has the following consequences.

Theorem 1.4.25. *[Gue1, Theorem 3.12] For a space X in which every subspace Y has $sdm(Y) \leq \omega$, the following hold:*

(I) *The spread of X is countable.*

(II) *If X is scattered of countable height then X is countable.*

(III) *If X is scattered meta-Lindelöf then X is countable.*

(IV) *If X is a scattered D-space of height at most ω_1 then X is countable.*

(V) *If X is a D-space then X is Lindelöf.*

PROOF. The statement (I) is an immediate consequence of [COT, Theorem 3.6] and the fact that every $\aleph_0$-space is cosmic.

To obtain (II) simply observe that by (I) every scattering level of X is countable.

To prove (III) let ω_1 be the scattering height of X. For each $\alpha < \omega_1$ denote by $X^{(\alpha)}$ the α-th scattering level of the space X. Recall that for every $\alpha < \kappa$ and each $x \in X^{(\alpha)}$ there is $U_x \in \tau(x,X)$ such that $U_x \cap X^{(\alpha)} = \{x\}$ and $U_x \cap X^{(\beta)} = \varnothing$ for every $\beta > \alpha$. Let $\mathscr{U}$ be a point-countable refinement of the cover $\{U_x : x \in X\}$. By (I) the set $X^{(0)}$ is countable. It is easy to see that for every $x \in X^0$ there is $\beta_x < \omega_1$ such that $x \in U \in \mathscr{U}$ implies $U \cap X^{(\beta_x)} = \varnothing$. It follows that the height of X is at most equal to the supremum of $\{\beta_x : x \in X^{(0)}\}$ which is countable. This contradiction shows that the height of X is countable and by (II) so is X.

To establish (IV), let κ be the scattering height of X. As in the proof of (III), for each $\alpha < \omega_1$ denote by $X^{(\alpha)}$ the α-th scattering level of the space X and for every $\alpha < \kappa$ and each $x \in X^{(\alpha)}$ assign $U_x \in \tau(x,X)$ such that $U_x \cap X^{(\alpha)} = \{x\}$ and $U_x \cap X^{(\beta)} = \varnothing$ for every $\beta > \alpha$. Since X is a D-space there is a closed and discrete set $A \subset X$ such that $X \subset \bigcup_{x\in A} U_x$. Therefore the height of X cannot exceed $\beta = \sup\{\text{height of } U_x : x \in A\}$. Each U_x has countable height and A is countable by (I) thus β is countable and so is X by (II).

If X is a D-space and we take any open cover $\mathscr{U}$ of X then to each $x \in X$ we can assign $U_x \in \mathscr{U} \cap \tau(x,X)$. Hence there is a closed and discrete space A such that $X \subset \bigcup_{x\in A} U_x$. By (I) the set A is countable and the family $\{U_x : x \in A\}$ is a countable subcover of $\mathscr{U}$ of the space X. $\square$

Notice that Theorem 1.4.25 provides partial answers to Problems 4.15, 4.16, 4.17, 4.18, 4.19 and 4.20 of [COT]. It has come to the author's knowledge that P. Gartside and J. Morgan have completely solved Problems 4.14-4.20 of [COT].

1.4.2. Complete domination

If M is a metric space of weight κ for some cardinal κ we can consider that M is a subset of $D(\kappa)^\omega$. In this subsection we study the case when $M = D(\kappa)^\omega$ and X is a space dominated by M.

In [KLO] the spaces (strongly) dominated by $M = D(\kappa)^\omega$ are referred to as spaces having a $\mathbb{P}(\kappa)$-compact resolution (swallowing compact sets).

The authors of [COT] finish their paper proving that if for some compact space K we have $sdm(K^2 \setminus \Delta) \le \omega$ then such K is metrizable. They point out that the most interesting problem remaining in their study is to determine whether the hypothesis that the domination of $K^2 \setminus \Delta$ is strong can be removed to obtain the same conclusion. We still do not know if this is true, but next we give some observations that might be useful.

Recall that given an infinite cardinal λ, a space X is λ-hemicompact if there exists a fundamental family $\mathscr{F}$ of compact subsets of X such that $|\mathscr{F}| = \lambda$. Observe that a space is hemicompact if and only if it is ω-hemicompact.

Proposition 1.4.26. *If K is a compact space such that $K^2 \setminus \Delta$ is dominated by a space M which is λ-hemicompact, then $w(K) \le \lambda$.*

PROOF. Suppose that M dominates $X = K^2 \setminus \Delta$ and take a fundamental compact cover $\mathscr{F}$ of M such that $|\mathscr{F}| = \lambda$. There is family $\{K_L : L \in \mathscr{K}(M)\}$ which is an M-ordered compact cover of X, it is clear that $\Delta = \bigcap\{X_L : L \in \mathscr{F}\}$. It follows that $w(K) = \chi(\Delta, X) = \psi(\Delta, X) \leq \lambda$. □

Observe that every metric space of weight κ is κ^ω-hemicompact.

Remark 1.4.27. *In particular, under CH, if there is a non-metrizable compact space K such that* $dm(K^2 \setminus \Delta) \leq \omega$, *then* $w(K) = \omega_1$.

Remark 1.4.28. *By Corollary 1.2.8, if there is a non-metrizable compact space K such that* $K^2 \setminus \Delta$ *is dominated by* ω^ω *then* $\omega_1 = \mathfrak{d}$. *It follows that* $w(K) = \omega_1$.

We can conclude that we should start looking for an example of a non-metrizable compact space K such that $dm(K^2 \setminus \Delta) \leq \omega$ among the compact spaces of weight ω_1 that are not Corson compact spaces. The first evident candidate is $\omega_1 + 1$ which is not a suitable example.

Proposition 1.4.29. *The following inequality holds* $\omega < dm((\omega_1 + 1)^2 \setminus \Delta)$.

PROOF. Notice that $\{(\alpha, \alpha + 1) : \alpha < \omega_1\}$ is an uncountable closed and discrete subset of $(\omega_1 + 1)^2 \setminus \Delta$ that hence witnesses that $\omega < ext((\omega_1 + 1)^2 \setminus \Delta) \leq dm(\omega_1 + 1)^2 \setminus \Delta$. □

Recall that Christensen proved in [Chr, Theorem 3.3] that a second countable space is strongly dominated by ω^ω if and only if it is completely metrizable.

We decided to investigate if Christensen's result can be generalized in the following sense:

> Suppose that a non separable metric space M of weight κ has a $\mathbb{P}(\kappa)$-compact resolution swallowing compact sets, does it imply that M is completely metrizable?

We can show that this is not so by presenting the following theorem that appeared as an external contribution of the author of this thesis to the paper [KLO].

Theorem 1.4.30. *[KLO, Theorem 17] If M is a metric space of weight* κ *then M is strongly dominated by* $D(\kappa^\omega)^\omega$.

PROOF. It suffices to find a fundamental family of compact sets

$$\{K_\alpha : \alpha \in ([\kappa^\omega]^{<\omega})^\omega\}.$$

Since M is a metric space then it contains at most κ^ω compact subsets.

Therefore we can write $\{C_\beta : \beta \in \xi\}$ the family of all the non void compact subsets of M for some $\xi \leq \kappa^\omega$. Let $L_\beta = C_\beta$ if $\beta < \xi$ and $L_\beta = \varnothing$ if $\xi \leq \beta < \kappa^\omega$. Let $\alpha \in ([\kappa^\omega]^{<\omega})^\omega$ so $\alpha = (s_0, s_1, ...)$ with s_n a finite subset of κ^ω. Define $K_\alpha = \bigcup\{L_\beta : \beta \in s_0\}$. Clearly the family $\{K_\alpha : \alpha \in ([\kappa^\omega]^{<\omega})^\omega\}$ is as required. □

As a consequence, if $\kappa = \kappa^\omega$ (for example $\kappa = \mathfrak{c}$) then any metric space (completely metrizable or not) is strongly dominated by $(D(\kappa))^\omega$.

1.4.3. Compact continuous images of metric spaces

As we have seen a source of domination is the presence of usco maps defined on metric spaces. Since continuous maps are a special case of usco maps we will study the case when a space X is a continuous image of a metric space M. First observe that for any topological space (X, τ), if X_d is the set X with the discrete topology then the identity map on X is a condensation and thus for any topological space X there is a metric space M that condenses onto X. Therefore we can introduce the following cardinal invariant for topological spaces:

Definition 1.4.31. *For any topological space X we denote the cosmic index of X as $mi(X)$ and define it by*

$$mi(X) = min\{w(Y) : M \text{ is a metric space that condenses onto } X\}.$$

Proposition 1.4.32. *For any space X the following hold:*

(I) $nw(X) \leq mi(X)$.

(II) *If $mi(X) \leq \kappa$ and $Y \subset X$ then $mi(Y) \leq \kappa$.*

(III) *If $mi(X) \leq \kappa$ then the cosmic index of any continuous image of X is also not greater than κ.*

(IV) *If* $X = \bigcup_{n\in\omega} X_n$ *and* $mi(X_n) \le \kappa$ *then* $mi(X) \le \kappa$.

(V) *If* $X = \prod_{n\in\omega} X_n$ *and* $mi(X_n) \le \kappa$ *then* $mi(X) \le \kappa$.

PROOF. Suppose that M is a metric space that condenses onto X by a condensation f and $\mathscr{B}$ is a base of M. For every $U \in \tau(X)$ the set $f^{-1}(U)$ is open in M therefore there is $\mathscr{B}_U \subset \mathscr{B}$ such that $f^{-1}(U) = \bigcup \mathscr{B}_U$. Since $\mathscr{B}$ is a bijection we have $U = \{f(B) : B \in \mathscr{B}_U\}$. We can conclude that the family $\{f(B) : B \in \mathscr{B}\}$ is a network in X and (I) is settled.

If $f : M \longrightarrow X$ is a condensation and M is a metric space, then for any $Y \subset X$ we have that $M' = f^{-1}(Y)$ is a metric space with $w(M') \le w(M)$ and $f|_{M'} : M' \longrightarrow Y$ is a condensation. Hence we have proved (II).

Suppose that $g : X \longrightarrow Y$ is continuous and M is a metric space that condenses onto X by a condensation f. For every $y \in Y$ take a single $m_y \in (f \circ g)^{-1}(y)$ and put $M' = \{m_y : y \in Y\}$. The space M' is metric and $w(M') \le w(M)$. Clearly $(f|_{M'} \circ g) : M' \longrightarrow Y$ condenses M' onto Y.

Assume $X = \bigcup_{n\in\omega} X_n$ and $mi(X_n) \le \kappa$ for every $i \in \omega$. Define $Y_0 = X_0$ and for $n > 0$.

$$Y_n = X_n \setminus \dot{\bigcup_{m<n}} X_m.$$

It is clear that $X = \bigcup_{n\in\omega} Y_n$ and by (III) we have $mi(Y_n) \le \kappa$. Therefore, for each $n \in \omega$ we can find a metric space M_n with $w(M_n) \le \kappa$ and a condensation $f_n : M_n \longrightarrow Y_n$. It follows that $M = \bigoplus_{n\in\omega} M_n$ is a metric space with $w(M) \le \kappa$ and $f : M \longrightarrow Y$ defined by $f(m) = f_n(m)$ if $m \in M_n$ is a condensation.

Analogously suppose that $X = \prod_{n\in\omega} X_n$ and $mi(X_n) \le \kappa$ for every $i \in \omega$. It is possible to find a family $\{M_n : n \in \omega\}$ of ω pairwise disjoint metric spaces such that $w(M_n) \le \kappa$ and $f_n : M_n \longrightarrow X_n$ is a condensation for each $n \in \omega$. It follows that $M = {}_{n\in\omega}\prod M_n$ is a metric space with $w(M) \le \kappa$ and $f : M \longrightarrow X$ defined by $f(m)(n) = f_n(m(n))$ condenses M onto X. $\square$

It follows from the definition that for any space X we have $\ell\Sigma(X) \le mi(X)$. So we decided to investigate for which spaces do these cardinal invariants coincide. First let us trivially observe that they do not in general, for if K is any non metrizable compact space then $\ell\Sigma(K) = \omega < w(K) \le mi(K)$ by Proposition 1.4.32 (I).

Nevertheless, we have a positive result for the case of submetrizable spaces.

Theorem 1.4.33. *If X is a submetrizable space then $\ell\Sigma(X) = mi(X)$.*

PROOF. Let μ be the metrizable coarser topology on X. Suppose that $\ell\Sigma(X) = \kappa$ and take a metric space M for which there is a compact valued usco onto map $\varphi : M \longrightarrow \exp(X)$. It is clear that the map $\varphi : M \longrightarrow \exp((X,\mu))$ is still usco compact valued and onto, therefore we have $w((X,\mu)) = \ell((X,\mu)) \leq \ell\Sigma((X,\mu)) \leq \kappa$. The space

$$T = \{(m,x) \in M \times (X,\mu) : x \in \varphi(m)\}$$

is a metric space of weight not greater than κ. Define the map $p : T \longrightarrow X$ by $p(m,x) = x$ for each $(m,x) \in T$ We will show that p is continuous at every $(m,x) \in T$. Indeed, let $(m,x) \in T$ and $U \in \tau(x,X)$, since $\varphi(m) \setminus U$ is compact there is $B \in \mu(x,X)$ such that $\overline{B} \cap \varphi(m) \setminus U = \varnothing$ hence $V = \overline{B} \cap X \setminus U \in \tau(\varphi(m),X)$. Thus the set $O = \{m \in M : \varphi(m) \in W\}$ is an open neighbourhood of M in T and finally the set $W = O \times B \cap T$ is an open neighbourhood of $(m,x) \in T$ and $p(W) \subset U$. We can conclude that p is continuous. □

By Proposition 1.4.32 (I) we have $nw(X) \leq mi(X)$ for any X, also $nw(X) \leq w(X)$ so we can ask ourselves for which spaces do we have $mi(X) = w(X)$. Again first we observe that these cardinal invariants differ in general. Indeed consider the space $X = C_p(\mathbb{I})$ we have $w(X) = |I| = \mathfrak{c}$ but $iw(X) = d(I) = \omega$ which implies that X is submetrizable and, by Theorem 1.4.33, we have $mi(X) = nw(X) = \omega$.

Nevertheless we have some positive results for certain classes of compact spaces.

Let $D(\kappa)$ denote the discrete space of cardinality κ.

Theorem 1.4.34. *For every cardinal κ, the space $D(\kappa)^\omega$ maps continuously onto $C_p(A_\kappa)$. Hence, every Eberlein compact space is the continuous image of a metric space of the same weight.*

PROOF. Let

$$Z = \left\{x = (x_n, x^n)_{n<\omega} \in D(\kappa)^\omega \times C_p(A_\kappa) : n < m \Rightarrow x_n \neq x_m, |x^n| \geq |x^m|\right\}$$

which is a closed subspace of $D(\kappa)^\omega$, hence a retract of it. Let $f : Z \longrightarrow K$ be defined so that $f(z)$ has the value $x^n \in \mathbb{R}$ on the coordinate $x_n \in \kappa$, and is zero on all other coordinates. This is a continuous onto function. □

The previous method of proof is not applicable to prove that Σ-products are continuous images of metric spaces of the same weight. This is because in such case we cannot prove that the corresponding function f is continuous.

Still, for the Corson compacta of small weight we will obtain a positive result.

Definition 1.4.35. *By $cf[\kappa]^\omega$ we denote the least cardinal of a cofinal family of countable subsets of κ.*

It is easy to verify inductively that $cf[\aleph_n]^\omega = \aleph_n$ whenever $n \in \mathbb{N}$. However, for $cf[\aleph_\omega]^\omega$ the best we know is Shelah's bound $cf[\aleph_\omega]^\omega \leq \aleph_{\aleph_4}$.

Theorem 1.4.36. *The space $\Sigma(\mathbb{R}^\kappa)$ is a continuous image of $D(cf[\kappa]^\omega)^\omega$. Hence, every Corson compact space of weight κ is a continuous image of a metric space of weight $cf[\kappa]^\omega$.*

PROOF. Let $\mathscr{F}$ be a cofinal family of countable subsets of κ, and for each $A \in \mathscr{F}$, let $\phi_A :\longrightarrow \omega$ be a bijection. Consider the function

$$f : D(\mathscr{F}) \times \mathbb{R}^\omega \longrightarrow \Sigma(\mathbb{R}^\kappa)$$

so that $f(A, x_n)$ has value $x_{\phi(a)}$ on coordinate $a \in A$, and is zero elsewhere. Then f is a continuous onto map, and its domain is a complete metric space of weight $cf[\kappa]^\omega$, hence a continuous image of $D(cf[\kappa]^\omega)^\omega$. $\square$

Corollary 1.4.37. *Every Corson compact space of weight $\aleph_n$, $n \in \mathbb{N}$, is a continuous image of a metric space of the same weight.*

For the case of an arbitrary Corson compact space, we first make some useful observations.

Lemma 1.4.38. *Given a cardinal κ, there exists a metric space M of weight κ and a continuous map $f : M \to \Sigma(2^\kappa)$ if and only if there exists a family $\mathscr{F} \subset \exp\kappa$ of cardinality κ such that for each countable $A \subset \kappa$ it is possible to find a countable $F(A) \subset \mathscr{F}$ such that $A = \bigcap F(A)$.*

PROOF. Suppose that $f : Z \longrightarrow S$ is a continuous onto map, where $Z \subset \kappa^{\omega}$ and $S = \Sigma(2^{\kappa})$. Then to every $s \in \kappa^{<\omega}$ we can associate two sets $A_s, B_s \subset \kappa$,

$$A_s = \{i < \kappa : f(x)(i) = 1 \text{ for all } x \succ s\}$$

$$B_s = \{i < \kappa : f(x)(i) = 0 \text{ for all } x \succ s\}$$

Then $A_s \cap B_s = \emptyset$, and $A_s \subset A_t$, $B_s \subset B_t$ whenever $s \prec t$. Moreover, if $A = \bigcup_{n\in\omega} A_{x|n}$ and $B = \bigcup_{n\in\omega} B_{x|n}$ for every $x \in Z$, then by continuity

$$\chi_A = f(x) \in S$$

$$\chi_B = 1 - f(x)$$

Indeed, take $x \in Z$ and suppose that $f(x)(i) = 1$ for some $i \in \kappa$. The set $U = \{y \in K : y(i) = 1\} \in \tau(f(x), S)$ therefore, by continuity of f at x, it is possible to find $n \in \omega$ such that the set $V = \{z \in Z : z|n = x|n\}$ which is a basic open neighbourhood of x has the property that $f(V) \subset U$. It follows that $i \in A_{x|n} \subset A$ hence $\chi_A(i) = 1$. Conversely if $\chi_A(i) = 1$ then there is $n \in \omega$ such that $i \in A_{x|n}$ since $x \succ x|n$ by the definition of $A_{x|n}$ we have $f(x)(i) = 1$.

Now suppose that $i \notin A$, we will show $i \in B$. For every $n \in \omega$ we have $i \notin A_{x|n}$ and therefore we can find $z_n \in Z$ such that $z_n \succ x|n$ and $f(z_n)(i) = 0$. The sequence $(z_n)_{n\in\omega}$ converges to x therefore the constant sequence $(f(z_n)(i))_{n\in\omega}$ converges to $f(x)(i)$ and therefore $f(x)(i) = 0$. As before the set $U = \{y \in S : y(i) = 0\} \in \tau(f(x), S)$ and it is possible to find $n \in \omega$ such that the set $V = \{z \in Z : z|n = x|n\}$ which is a basic open neighbourhood of x has the property that $f(V) \subset U$ implying $i \in B_{x|n} \subset B$. Again if $\chi_B(i) = 1$ then there is $n \in \omega$ such that $i \in B_{x|n}$ since $x \succ x|n$ by the definition of $B_{x|n}$ we have $f(x)(i) = 0$.

It follows that the family $\mathscr{F} = \{S \setminus \kappa^{<\omega}\}$ is as promised.

Now suppose that every countable subset of κ is a countable intersection of a family $\mathscr{F} \subset \exp(\kappa)$ of κ many subsets of κ.

Consider the set $T = \{(s, \kappa \setminus F) : s \in [\kappa]^{<\omega}, F \in \mathscr{F} \cup \emptyset, s \subset F\}$. Now define and order in T by $(s, G) < (s', G')$ if $s \subset s'$ and $G \subset G'$. The set $(T, <)$ is now partially ordered so that it is a tree of countable height. Denote by $[T] = \{r \subset T : r \text{ is a maximal linearly ordered subset of } T\}$ the family of the branches of $(T, <)$. We can give $[T]$ the structure of a metric space of weight κ by defining for every $r = (r_0, r_1, \ldots), r' = (r'_0, r'_1, \ldots) \in [T]$ the following distance $d(r, r') = 2^{-m(r,r')}$ where $m(r, r') = min\{n \in \omega : r_n \neq r'_n\}$. It is clear that $([T], d)$ is a metric space of weight κ. Now let $M = \{r \in [T] : (s,G) \in r \text{ implies } \kappa \setminus G \in \mathscr{F}\}$ and call $F_r = \bigcap\{\kappa \setminus G : (s,G) \in r \text{ for some } s \in [\kappa]^{<\omega}\}$. Finally define $f : M \to S$ by $f(r) = \chi_{F_r}$, clearly this map is continuous and the lemma is proved. □

The existence of the families in Lemma 1.4.38 is the combinatorial translation of a Σ-product being a continuous image of a metric space of the same weight.

For the general case we have the following.

Theorem 1.4.39. *For every cardinal κ there is a metric space of weight at most κ that condenses onto $\Sigma(2^\kappa)$.*

PROOF. Clearly the result holds for $\kappa = \omega$. Now suppose that for every cardinal $\lambda < \kappa$ there is a metric space M_κ of weight at most κ that condenses onto $\Sigma(2^\lambda)$. We distinguish two different cases, first assume $\omega < cof(\kappa)$. The space $D = \{\chi_F : F \in [\kappa]^{<\omega}\}$ is dense in $\Sigma(2^\kappa)$ and has cardinality κ, moreover if we let $D_\lambda = \{\chi_F : F \in [\lambda]^{<\omega}\}$ for each cardinal $\lambda < \kappa$ then $\overline{D_\lambda}$ is homeomorphic to $\Sigma(2^\lambda)$. We have $\Sigma(2^\kappa) = \bigcup\limits_{\lambda<\kappa} \overline{D_\lambda}$ and therefore $\bigoplus\limits_{\lambda<\kappa} \overline{D_\lambda}$ maps continuously onto $\Sigma(2^\lambda)$. The discrete union $X = \bigoplus\limits_{\lambda<\kappa} M_\lambda$ is a metric space of weight κ that condenses onto $\bigoplus\limits_{\lambda<\kappa} \overline{D_\lambda}$. It follows that there is a metric space $M \subset X$ of weight at most κ that condenses onto $\Sigma(2^\kappa)$.

For the case $\omega = cof[\kappa]$ we will apply an argument communicated to the author by M. Kojman.

We will now suppose that $\omega = cof[\kappa]$. Define $\alpha_{-1} = 0$ and take any increasing sequence $\{\alpha_n : n \in \omega\}$ cofinal in κ. By the induction hypothesis and Lemma 1.4.46, for every interval $[\alpha_{n-1}, \alpha_n)$ there is a family $\mathscr{F}_n \subset \exp([\alpha_{n-1}, \alpha_n))$ of cardinality $|\alpha_n|$ with the property that every countable subset of the interval is a countable intersection of such family.

Define $\mathscr{F} = \{Z \subset \kappa : Z = F_0 \cup \ldots \cup F_k \cup [\alpha_n, \kappa) : F_i \in \mathscr{F}_i, n \in \omega, k < n\}$. We will show that $\mathscr{F}$ has the property that for each countable $A \subset \aleph_\omega$ it is possible to find a countable $F(A) \subset \mathscr{F}$ such that $A = \bigcap F(A)$. The elements of $\mathscr{F}$ are subsets of κ, also $|\mathscr{F}| = \sum\limits_{m<\omega} \prod\limits_{n<m} |\alpha_n| = \sum\limits_{m<\omega} |\alpha_n| = \kappa$.

Now, given a countable $X \subseteq \kappa$ we have $X = \bigcup_{n\in\omega} X_n$ where $X_0 = X \cap \alpha_0$ and also $X_n = X \cap [\alpha_{n-1}, \alpha_n)$. For each $n \in \omega$ there exists a countable family

$$\mathscr{F}_n(X_n) = \{Y^n_m : m \in \omega\} \subset \mathscr{F}_n$$

such that $X_n = \bigcap \mathscr{F}_n(X_n)$. For each $m \in \omega$ let $Z_m = [\bigcup_{n \leq m} Y^n_m] \cup [\alpha_m, \kappa) \subset \kappa$. It follows that $Z_m \in \mathscr{F}$ for every $m \in \omega$ and $\bigcap_{m \in \omega} Z_m = X$. Apply Lemma 1.4.38 to conclude that there is a metric space of weight at most κ that condenses onto $\Sigma(2^\kappa)$. $\square$

Corollary 1.4.40. *Every Corson compact space is the continuous image of a metric space of the same weight.*

2

Eberlein-Grothendieck spaces

As pointed out in the Introduction chapter, in [Ar2] Arhangel'skii defined the class of Eberlein-Grothendieck topological spaces as the one that consists of all the spaces X for which there exists a compact space K such that X is homeomorphic to a subspace of $C_p(K)$. In this chapter we will study decompositions of Eberlein-Grothendieck spaces by discrete subsets. More accurately we will investigate whether Eberlein-Grothendieck scattered spaces are σ-discrete or not. It happens to be so in the case of scattered Eberlein compact spaces.

We will obtain some partial original results. To achieve them we shall recall in Section 2.1 a few important theoretical preliminary facts. The subsequent sections contain the original results attained that are collected in the paper [AG]. In Section 2.3 we deal with a generalization of the Eberlein compact scattered spaces which is the class of Eberlein-Grothendieck Lindelöf Σ scattered spaces. In this case we show that these spaces are σ-discrete as an easy consequence of the results in [Ha] and [Ny].

Since the results in Section 2.3 depend very strongly on the property of hereditary metalindelöfness of the Eberlein-Grothendieck spaces considered in that section, we decided to ask if such property is enough for a scattered topological space to be σ-discrete. In other words, is every hereditarily meta-Lindelöf scattered space σ-discrete? We show in Section 2.4 that this is not the case in general by constructing a hereditarily meta-lindelöf scattered space that is not σ-discrete.

2.1. Properties of Eberlein-Grothendieck spaces

Recall that a compact space K is called strong Eberlein compact space if K is homeomorphic to a subspace of the product $\Sigma_*(2^\kappa) = \{x \in 2^\kappa : |x^{-1}(1)| < \omega\}$. In this section we will review some classical results that will be useful and even crucial in Chapter 2. We start with a classical result by K. Alster which states that Corson compact scattered spaces are strong Eberlein compact. We will not reproduce Alster's proof, but instead we will proceed as G. Gruenhage does in [Gr0] which is basically an explanation of Alster's argument.

Gruenhage extracts a fundamental set-theoretic fact from Alster's proof. We state and prove such main lemma. The notation $A_\alpha \nearrow A$ means that for some ordinal κ, $(A_\alpha)_{\alpha<\kappa}$ is an increasing sequence of sets whose union is A.

Lemma 2.1.1. *[Gr0, Lemma 2.1] Let M be a set, and let $\phi : \exp(M) \longrightarrow \exp(M)$ satisfy*

(I) $|\phi(A)| \leq max(|A|, \omega)$;

(II) $A_\alpha \nearrow A \Longrightarrow \phi(A_\alpha) \nearrow \phi(A)$.

Call $A \subset M$ ϕ-closed if $\phi(A) \subset A$. Suppose $\mathscr{P}$ is a property of subsets of M. Then M has property $\mathscr{P}$ whenever $\mathscr{P}$ satisfies the following properties:

(I) $\mathscr{P}$ holds for all countable ϕ-closed sets;

(II) $\mathscr{P}$ holds for all sets which are increasing unions of ϕ-closed sets satisfying $\mathscr{P}$.

PROOF. The proof is by induction on $|M|$. If $|M| \leq \omega$, then M satisfies $\mathscr{P}$ by property (I).

Suppose $|M| = \kappa > \omega$, and the lemma holds for all sets of cardinality less than κ. Let $M = \{m_\alpha : \alpha < \kappa\}$. We inductively construct $(M_\alpha)_{\alpha<\kappa}$ such that, for each α,

(a) $M_\alpha \supset \{m_\beta : \beta < \alpha\}$;

(b) $|M_\alpha| < max\{|\alpha|, \omega\}$;

(c) M_α is ϕ-closed.

Let $M_0 = \varnothing$. Suppose $\alpha < \kappa$, and M_β has been defined for all $\beta < \alpha$. If α is a limit ordinal, let $M_\alpha = \bigcup\limits_{\beta<\alpha} M_\beta$. If $\alpha = \alpha' + 1$, let $M_{\alpha,0} = M_{\alpha'} \cup \{m_\alpha\}$, and let

$$M_{\alpha,n+l} = M_{\alpha n} \cup \phi(M_{\alpha,n}).$$

Then let $M_\alpha = \bigcup_{n\in\omega} M_{\alpha,n}$. It is easily checked that (a)-(c) hold. (Property (c) follows from property (2) in the statement of the lemma.) Now each M_α and $\phi|\exp(M_\alpha)$ satisfies the conditions of the lemma. Hence each M_α satisfies $\mathscr{P}$, and so by property (ii) of $\mathscr{P}$, we conclude that M satisfies $\mathscr{P}$. □

We will use the previous lemma to prove the following result that appears in Alster's paper [Al] as a proposition, but given its importance we will state it as a theorem as Gruenhage does in [Gr0].

Before we proceed, observe that if K is a compact scattered space of height γ, then the "top level" $X^{(\gamma-1)}$ of X is finite (being closed discrete in K). In the sequel, we will let $T(X)$ denote this top level of the compact scattered space X.

Theorem 2.1.2. *[Gr, Theorem 4.1 (Alster)] If $\mathscr{U}$ is a point-countable collection of compact scattered open subsets of a space X, then $\mathscr{U}$ has a point-finite clopen refinement.*

PROOF. We apply Lemma 2.1.1, with $M = \mathscr{U}$. For $\mathscr{V} \subset \mathscr{U}$ let

$$T(\mathscr{V}) = \bigcup\{T(Z) : Z = \cap\mathscr{F}, \mathscr{F} \text{ a finite subset of } \mathscr{V}\}$$

and

$$\phi(\mathscr{V}) = \{U \in \mathscr{U} : U \cap T(\mathscr{V}) \neq \varnothing\}.$$

It is routine to check that ϕ satisfies (I) and (II) of Lemma 2.1.1. Let $\mathscr{P}(\mathscr{V})$ be the property that $\mathscr{V}$ has a point-finite clopen refinement. Then $\mathscr{P}(\mathscr{V})$ holds for countable $\mathscr{V}$ since $\cap\mathscr{V}$ is σ-compact. Suppose $\mathscr{V}_\alpha \nearrow \mathscr{V}, \alpha < \kappa$, where $\mathscr{V}$ is ϕ-closed and satisfies $\mathscr{P}$. To complete the proof, we need to show that $\mathscr{V}$ has a point-finite clopen refinement. Now each $\mathscr{V}_\alpha$ has a point-finite clopen refinement $\mathscr{W}_\alpha$. Let

$$\mathscr{W}_\alpha^{'} = \{W \in \mathscr{W}_\alpha \nsubseteq \bigcup_{\beta<\alpha} (\bigcup \mathscr{V}_\beta)\}$$

and let $\mathscr{W}' = \bigcup_{\alpha<\kappa} \mathscr{W}_\alpha'$. Then $\mathscr{W}'$ is a refinement of $\mathscr{V}$. It remains to prove that $\mathscr{W}'$ is point-finite.

If $\mathscr{W}'$ is not point-finite, then there exists $x \in X$ and $W_{\alpha(n)} \in \mathscr{W}_{\alpha(n)}$ with $x \in \bigcap_{n\in\omega} W_{\alpha(n)}$. Since each $\mathscr{W}_\alpha$ is point-finite, we can assume that $(\alpha(n))_{n\in\omega}$ is a strictly increasing sequence of distinct ordinals. Let $W_{\alpha(n)} \subset V_n \in \mathscr{V}_{\alpha(n)}$. If γ_k is the height of the scattered space

$\bigcap_{i\le k} V_i$ then $(\gamma_k)_{k\in\omega}$ is a nonincreasing sequence of ordinals, so $\gamma_m = \gamma_{m+1} = \ldots$ for some $m \in \omega$. Then $V_{m+l} \cap T(\bigcap_{i\le m} V_i) \neq \varnothing$. But then if $\widehat{\mathscr{V}} = \bigcup_{\beta<\alpha(m+1)} \mathscr{V}_\beta$ we have $V_{m+1} \in \widehat{\mathscr{V}}$ since $\widehat{\mathscr{V}}$ is ϕ-closed and contains each $V_i, i \le m$. This contradicts $W_{\alpha(m+1)} \subset V_{m+1}$ and $W_{\alpha(m+1)} \not\subseteq \bigcup_{\beta<\alpha(m+1)} (\bigcup \mathscr{V}_\beta)$. □

Theorem 2.1.3. *[Al, Theorem] If K is a scattered, compact space, which has a point-countable separating family of open F_σ-sets (it is the same as to say that K is a compact subset of a Σ-product of intervals), then it is a strong Eberlein compact space.*

PROOF. We shall prove the theorem by transfinite induction with respect to the ordinal number α defined by

$$T(K) = K^\alpha.$$

If $\alpha = 0$ then K is finite, the theorem holds. Let us suppose that the theorem holds for every $\beta < \alpha$ and that $T(K) = K^\alpha$.

It is easy to see that we can assume without loss of generality that $|T(K)| = 1$. Put $T(K) = \{a\}$.

Let $\mathscr{H}$ be a point-countable separating family of open F_σ-sets in K. Since K is zero-dimensional we can assume that the elements of $\mathscr{H}$ are clopen.

If $\{H \in \mathscr{H} : a \in H\} = \{H_n : n = 1,2,...\}$ then

$$\mathscr{U} = \{H \in \mathscr{H} : a \notin H\} \cup \{K \setminus H_n : n = 1,2,...\}$$

is an open, point-countable cover of $K \setminus \{a\}$ consisting of compact sets.

By Theorem 2.1.2 there is a point-finite, open cover $\mathscr{V}$ of $K \setminus \{a\}$ consisting of compact sets.

By the inductive assumption, every $V \in \mathscr{V}$ has a point-finite family $\mathscr{U}(V)$ of clopen sets, separating points in V.

Put $\mathscr{L} = \mathscr{V} \cup (\bigcup\{\mathscr{U}(V) : V \in \mathscr{V}\})$. It is easy to see that $\mathscr{L}$ is a point-finite family of clopen sets separating points so the proof of the theorem is finished. □

It is a known fact that every Eberlein compact scattered space has a point-countable separating family of clopen sets. Thus we obtain the following corollary.

Corollary 2.1.4. *[Al, Corollary 3] An Eberlein compact is a strong Eberlein compact if and only if it is scattered.*

So far we have seen that point-countable separating open families of subsets of the Eberlein-Grothendieck spaces imply strong properties of those spaces. Establishing when such a space has point-countable open cover can be achieved as shown by Dow, Junnila and Pelant who proved that if K is a compact space with weight ω_1 then $C_p(K)$ is hereditarily meta-Lindelöf.

First we need the following result.

Lemma 2.1.5. *[DJP, Lemma 1.1] Let τ and μ be two topologies on X such that $\tau \subset \mu$ and the topology μ is metrizable. Assume that there exists a cover $\{F_\alpha : \alpha \in \omega_1\}$ of X such that the following conditions hold for every $\alpha \in \omega_1$:*

(I) F_α is τ-closed;

(II) F_α is μ-separable;

(III) $\bigcup_{\beta<\alpha} F_{\beta+1}$ is a μ-dense subset of F_α.

Then the topology τ is hereditarily meta-Lindelöf.

PROOF. Note that the conditions of the lemma are hereditary with respect to τ-open sets, and that to prove the lemma it therefore suffices to show that τ is meta-Lindelöf. Let d be a metric on X which induces the topology μ; we may assume that d is bounded. For all $x \in X$ and $r < 0$, denote by $B(x,r)$ the d-ball of radius r with center x. Let $\mathscr{O}$ be an open cover of (X,τ). For every $x \in X$, let

$$r_x = \sup\{r : \text{there exists } O \in \mathscr{O} \text{ such that } B(x,r) \subset O\},$$

and note that $r_x < 0$, because $\tau \subset \mu$; further, let $O_x \in \mathscr{O}$ be such that $B(x, \frac{3}{4}r_x) \subset O_x$.

For every a $\alpha \in \omega_1$ let C_a be a countable μ-dense subset of $F_{\alpha+1} \setminus F_\alpha$. We show that the family

$$\mathscr{U} = \{O_z \setminus F_\alpha : \alpha \in \omega_1 \text{ and } z \in C_\alpha\}$$

is a point-countable open refinement of $\mathscr{O}$ in (X,τ). Clearly, $\mathscr{U}$ is a point-countable family, and for every $U \in \mathscr{U}$, the set U is open and it is contained in some set of the family $\mathscr{O}$; hence we need only show that $\mathscr{U}$ covers X. Let $x \in X$ and denote by η_x the least $\eta \in \omega_1$

such that $x \in F_\eta$. We show that there exists $\alpha < \eta_x$ such that $F_{\alpha+1} \setminus F_\alpha \cap B(x, \frac{3}{4}r_x) \neq \varnothing$. If η_x is a successor, let α be the predecessor of η_x, and note that then $x \in B(x, \frac{3}{4}r_x)$. If η_x is a limit ordinal, then it is a consequence of condition (iii) that there exists a minimal ordinal $\alpha < \eta_x$ such that $F_{\alpha+1} \cap B(x, \frac{3}{4}r_x) \neq \varnothing$; it now follows from minimality of α that we have $F_{\alpha+1} \setminus F_\alpha \cap B(x, \frac{3}{4}r_x) \neq \varnothing$. Since C_α is dense in $F_{\alpha+1} \setminus F_\alpha$, it follows from the foregoing that there exists $z \in C_\alpha \cap B(x, \frac{3}{4}r_x)$. We show that $x \in O_z$. Since $z \in B(x, \frac{3}{4}r_x)$ and $B(x, \frac{3}{4}r_x) \subset O_x$, we have that $B(z, \frac{3}{4}r_x) \subset O_z$ and hence that $r_z \geq \frac{1}{2}r_x$; further, since $B(z, \frac{3}{4}r_z) \subset O_z$, we have that $B(z, \frac{3}{4}r_x) \subset O_z$ and hence that $x \in O_z$. Since $\alpha < \eta_x$, we have that $x \in F_\alpha$. It follows that $x \in O_z \setminus F_\alpha \in \mathscr{U}$. □

Remark 2.1.6. *[DJP, Remark] Note that if the topologies τ and μ and the family $\{F_\alpha : \alpha \in \omega_1\}$ satisfy the conditions of the Lemma 2.1.5, then for every topology ρ lying between τ and μ, the topologies ρ and μ and the family $\{F_\alpha : \alpha \in \omega_1\}$ also satisfy the conditions for the topologies; as a consequence, every such topology ρ is hereditarily meta-Lindelöf.*

Theorem 2.1.7. *[DJP, Theorem 1.2(B)] If K is a compact space of weight ω_1 then $C_p(K)$ is hereditarily meta-Lindelöf.*

PROOF. Denote by μ the norm topology of $C(K)$ and by τ the pointwise topology. We have that $d((C_u(K)) = w(K) \leq \omega_1$. Let $\{f_\alpha : \alpha < \omega_1\}$ be a dense subset of $C_u(K)$ such that f_0 is a non-zero constant function . For each $\alpha < \omega_1$, let F_α be the μ-closed subalgebra of $C_u(K)$ generated by the set $\{f_\beta : \beta < \alpha\}$. It is a straightforward consequence of the Stone-Weierstrass Theorem that each F_α is τ-closed, and it is easy to check that τ, μ and $\{F_\alpha : \alpha < \omega_1\}$ also satisfy the other conditions of Lemma 2.1.5; therefore the topology τ is hereditarily meta-Lindelöf. Since the weak topology of $C(K)$ lies between the topologies τ and μ, it follows by the remark made after Lemma 2.1.5 that the space $C_w(K)$ is also hereditarily meta-Lindelöf . □

It follows that every Eberlein-Grothendieck space of cardinality ω_1 is hereditarily meta-Lindelöf. Indeed, suppose that $X \subset C_p(K)$ where K is a compact space such that $w(K) > \omega_1 = |X|$. Let $\varphi(p)(f) = f(p)$ for any $p \in K$ and $f \in X$. In [Tk6, Problem 166] it is proved that $\varphi(p) \in C_p(X) \subset \mathbb{R}^{\omega_1}$, for any $p \in K$ and that the map $\varphi : K \to C_p(X)$ is continuous. The map φ thus defined is called the diagonal product of the elements of X. Let $L = \varphi(K) \subset \mathbb{R}^{\omega_1}$, it is clear that $w(L) = \omega_1$. Besides, the dual map defined by the function $\varphi^* : C_p(L) \to C_p(K)$ defined by $\varphi^*(f) = f \circ \varphi$ embeds $C_p(L)$ into $C_p(K)$ as a closed subspace (see [Tk6, Problem 163]). For every $g \in X$, if $\pi_g : L \to \mathbb{R}$ is the projection of L onto the factor determined by g, then $g = \varphi^*(\pi_g)$. It follows that the space X embeds into $C_p(L)$ which is hereditarily meta-Lindelöf. We can conclude that every Eberlein-Grothendiek space of cardinality ω_1 is hereditarily meta-Lindelöf.

2.2. Eberlein-Grothendieck scattered spaces

As mentioned earlier in the introductory chapter and at the beginning of this one, the rest of Chapter 2 contains original results on the topic at hand obtained by the author of this text in collaboration with A. Avilés. Most of the results in the following sections in this chapter are collected in the paper [AG].

Notice that if X is a subset of a Banach space E with the weak topology and $\mathbf{B}_{E^*}$ is the unit ball of the dual space of E endowed with the weak* topology, then X embeds in $C_p(\mathbf{B}_{E^*})$ and hence X is Eberlein-Grothendieck. The main purpose of this section is to present the following problem:

Problem 2.2.1. *Are Eberlein-Grothendieck scattered spaces σ-discrete?*

A very similar problem was posed in [Hay] where Haydon asked if for every compact K the space $C_p(K, \{0,1\})$ is σ-discrete whenever it is σ-scattered. As mentioned in the Introduction Chapter, questions of this sort are related to the following notions introduced by J.E. Jayne, I. Namioka and C.A. Rogers in [JNR].

Definition 2.2.2. *Given a set X, a metric ρ on X and $\varepsilon > 0$, a family $\mathscr{A}$ of subsets of X is ε-small if $diam_\rho(A) < \varepsilon$ for every $A \in \mathscr{A}$. A topological space (X, τ) has the property SLD with respect to a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ the space X_n admits a τ-open cover which is ε-small. Whereas a topological space (X, τ) is σ-fragmented by a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ and every $Y \subset X_n$ there exists a non-empty relative τ-open subset U of Y with $diam_\rho(U) < \varepsilon$.*

Clearly, if a topological space has SLD with respect to some metric, then it is σ-fragmented as well. However, the following is still an open question:

Problem 2.2.3. *Are the properties of σ-fragmentability and SLD equivalent when X is a Banach space endowed with its weak topology and with its norm metric, or when X is of the form $C_p(K)$ endowed with the uniform metric?*

Thus Problem 2.3.1 can be viewed as the discrete version of Problem 2.2.3. Actually the next question remains open as well:

Problem 2.2.4. *If $C_p(K,\{0,1\})$ is σ-scattered (respectively σ-discrete), does it imply that $C_p(K)$ is σ-fragmentable (respectively SLD)?*

When considering the weak topology of $C(K)$ instead of the pointwise convergence topology, the answer to the corresponding problem is known to be positive when K is scattered (see [Hay] and [Mtz]). Since the restriction of the uniform metric of $C_p(K)$ to $C_p(K,\{0,1\})$ is discrete, a positive answer to Problems 2.3.1 and 2.2.4 combined would give a positive answer to Problem 2.2.3 for $C_p(K)$ spaces.

An easy observation is the fact that Problem 2.3.1 has positive solution when X is separable, because by monolithicity X is cosmic so it is countable being scattered.

The first non-trivial positive partial solution to Problem 2.3.1 follows from known facts when X is compact: as shown in the previous sections, Alster proved that an Eberlein compact scattered space K is strong Eberlein which implies that K embeds into $\{0,1\}^\Gamma$ for some Γ and $|supp(x)| < \omega$ for every $x \in K$. For each $n \in \omega$ we can define $X_n = \{x \in K : |supp(x)| = n\}$ so we can write $K = \bigcup X_n$ where each X_n is discrete. In this section we will prove some generalizations of this fact.

In this chapter, we give some partial positive answers to Problem 2.3.1. The first one is the following:

Theorem 2.2.5. *[AG, Theorem 1.4] If X is an Eberlein-Grothendieck locally compact scattered space of cardinality ω_1 then X is σ-discrete.*

In the same section we will prove also our second result which states:

Theorem 2.2.6. *[AG, Theorem 1.5] If X is an Eberlein-Grothendieck locally countable scattered space of cardinality ω_1, then X is σ-discrete.*

Definition 2.2.7. *A transfinite sequence $\{x_\alpha : \alpha < \lambda\}$ of elements of a topological space is right-separated if for every $\mu < \lambda$ there is an open set U such that*

$$U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}.$$

A topological space is scattered if and only if it can be written as a right-separated sequence $X = \{x_\alpha : \alpha < \lambda\}$. From this point of view, Theorem 2.2.6 implies that Problem 2.3.1 has a positive solution in the first non-trivial case, when $\lambda = \omega_1$.

Corollary 2.2.8. *[AG, Corollary 1.6] Suppose that K is a compact space and take a right-separated ω_1-sequence $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$, then X is σ-discrete.*

In both of our results mentioned above, X is homeomorphic to some $X' \subset C_p(K)$ where K has weight ω_1. By [DJP, Theorem 1.2] in this case the space X is hereditarily meta-Lindelöf. This is the hypothesis that we shall actually assume, so that Theorem 2.2.5 is proved by applying the ideas developed in [Al] to show that every open cover of a hereditarily meta-Lindelöf locally compact scattered space has a point-finite clopen refinement, while Theorem 2.2.6 is proved by showing that hereditarily meta-Lindelöf locally countable scattered spaces are σ-discrete. The latter fact is stated in [HP] without proof, but the argument that they suggest does not seem to be correct. Section 3 ends with a corollary that, at least when K is scattered, the SLD property of $C_p(K)$ can be characterized as a kind of ω_1-σ-fragmentability.

2.3. Eberlein-Grothendieck small spaces

Let us remind ourselves of the main problem of this Chapter:

Problem 2.3.1. *Are Eberlein-Grothendieck scattered spaces σ-discrete?*

Also we recall the following concept that will be very relevant:

Definition 2.3.2. *A space X is called weakly θ-refinable if every open cover of X has an open refinement $\mathscr{U} = \bigcup\{\mathscr{U}_n : n \in \omega\}$ such that for every $x \in X$ there is $n \in \omega$ such that $(x, \mathscr{U}_n) = |\{U \in \mathscr{U}_n : x \in U\}|$ is finite.*

The first mandatory step is to pose the problem as one of covering properties; we can do it by recalling that Nyikos established in [Ny, Theorem 3.4] that a scattered space is σ-discrete if and only if it is hereditarily weakly θ-refinable. It is clear that this result implies that every metrizable scattered space is σ-discrete as it was proved by Telgarsky in [Te, Theorem 12.1] by other methods. Yakovlev showed in [Ya] that every Eberlein compact space is hereditarily σ-metacompact hence hereditarily weakly θ-refinable, so it suffices to invoke [Ar, Corollary III.4.2] to conclude that every Eberlein-Grothendieck pseudocompact scattered space is σ-discrete.

In [DJP, Theorem 1.2] it was proved by Dow, Junnila and Pelant that if K is a compact space of weight ω_1 then $C_p(K)$ is hereditarily meta-Lindelöf. It follows that every Eberlein-Grothendieck space of cardinality ω_1 is hereditarily meta-Lindelöf. Indeed, suppose that $X \subset C_p(K)$ where K is a compact space such that $w(K) > \omega_1 = |X|$. Define $\varphi(p)(f) = f(p)$ for any $p \in K$ and $f \in X$. In [Tk2, Problem 166] it is proved that $\varphi(p) \in C_p(X) \subset \mathbb{R}^{\omega_1}$, for any $p \in K$ and that the map $\varphi : K \to C_p(X)$ is continuous. The map φ thus defined is called the diagonal product of the elements of X. Let $L = \varphi(K) \subset \mathbb{R}^{\omega_1}$; it is clear that $w(L) = \omega_1$. Besides, the function $\varphi^* : C_p(L) \to C_p(K)$ defined by $\varphi^*(f) = f \circ \varphi$ embeds $C_p(L)$ into $C_p(K)$ as a closed subspace (see [Tk2, Problem 163]). For every $g \in X$, if $\pi_g : L \to \mathbb{R}$ is the projection of L onto the factor determined by g, then $g = \varphi^*(\pi_g)$. It follows that the space X embeds into $C_p(L)$ which is hereditarily meta-Lindelöf. We can conclude that every Eberlein-Grothendiek space of cardinality ω_1 is hereditarily meta-Lindelöf.

In the rest of this section we will strongly exploit hereditary metalindelöfness in Eberlein-Grothendieck spaces combined with a fundamental fact proved by Alster in [Al] which establishes that every locally countable family of compact scattered open subspaces of a space X has a point-finite clopen refinement. In order to apply these results to the study of locally compact scattered spaces we will apply some of the ideas introduced in [Sp] to prove the following result.

Theorem 2.3.3. *[AG, Theorem 3.1] Every open cover of a locally compact scattered hereditarily meta-Lindelöf space X has a point-finite clopen refinement.*

PROOF. Let $\mathscr{O}$ be an open cover of X. For each $\alpha < \kappa$ denote by $X^{(\alpha)}$ the α-th scattering level of X. Recalling that a scattered locally compact space is zero dimensional, it is easy to see that the open cover $\mathscr{O}$ of X has a refinement $\mathscr{U}'$ consisting of compact clopen subsets of X such that for every $\alpha < \kappa$ and each $x \in X^{(\alpha)}$ there is $U'_x \in \mathscr{U}'$ such that $U'_x \cap X^{(\alpha)} = \{x\}$ and $U'_x \cap X^{(\beta)} = \varnothing$ for every $\beta > \alpha$. There is a point-countable open refinement $\mathscr{U}$ of the cover $\mathscr{U}'$. We will show that $\mathscr{U}$ has a point-finite clopen refinement.

Let κ be the height of X. Note that for every $U \in \mathscr{U}$ and every $x \in U^{(\alpha)} = U \cap X^{(\alpha)}$ there is a compact clopen $V_x^U \subset U$ such that $V_x^U \cap U^{(\alpha)} = \{x\}$ and $V_x^U \cap U^{(\beta)} = \varnothing$ for every $\beta > \alpha$. Let $\mathscr{V}_\alpha^U = \{V_x^U : x \in U^{(\alpha)}\}$.

We will proceed by induction on κ.

Let us assume that the height of X is a successor ordinal $\kappa = \beta + 1$ and that the result holds for every locally compact scattered hereditarily meta-Lindelöf space of height not larger than β.

For each $x \in X^{(\beta)}$ take $U_x \in \mathscr{U}$ with $x \in U_x$. Observe that by the definition of $\mathscr{U}'$ the level β of every element of $\mathscr{U}'$ is a singleton, this implies that $U_x \cap X^{(\beta)} = \{x\}$ because $U_x \in \mathscr{U}$ and $\mathscr{U}$ is a refinement of $\mathscr{U}'$. Also the family $\mathscr{V}_\beta^{U_x}$ has only one member which we will call V_x for every $x \in X^{(\beta)}$. Since the family $\mathscr{W} = \{U_x : x \in X^{(\beta)}\} \subset \mathscr{U}$ is point-countable so is the family $\mathscr{V} = \{V_x : x \in X^{(\beta)}\}$ which consists of scattered compact subspaces of X. Apply [Al, Proposition] to find a point-finite clopen refinement $\mathscr{C}_\mathscr{W}$ of the family $\mathscr{V}$.

Let $\mathscr{W}' = \{U_x \setminus V_x : x \in X^{(\beta)}\} \cup \{U \in \mathscr{U} :$ the height of U is lower than $\kappa\}$. It is clear that $Y = \bigcup \mathscr{W}'$ is a locally compact scattered hereditarily meta-Lindelöf open subspace of X of height not larger than β. Refining the open cover $\mathscr{W}'$ of Y with compact clopen sets and applying the induction hypothesis we can find a point-finite refinement $\mathscr{C}_{\mathscr{W}'}$ consisting on compact clopen sets. It follows that $\mathscr{C}_\mathscr{W} \cup \mathscr{C}_{\mathscr{W}'}$ is a point-finite clopen refinement of $\mathscr{U}$.

Now, suppose that κ is a limit ordinal and that the result holds for every locally compact scattered hereditarily meta-Lindelöf space of height lower than κ.

In this case, every $U \in \mathscr{U}$ is a hereditarily meta-Lindelöf locally compact scattered space of height γ lower than κ, therefore we can find a point-finite clopen refinement $\mathscr{C}_U$ of the family $\mathscr{V}^U = \bigcup\limits_{\alpha \in \gamma} \mathscr{V}_\alpha^U$. The family $\mathscr{U}$ is point-countable, it follows that the family $\mathscr{C} = \bigcup\limits_{U \in \mathscr{U}} \mathscr{C}_U$ is a point-countable clopen refinement of $\mathscr{U}$ and its elements are compact scattered subspaces of X. Again by [Al, Proposition] the cover $\mathscr{C}$, and hence $\mathscr{U}$ has a point-finite clopen refinement. $\square$

In the proof of [Ny, Theorem 3.4] Nyikos proved implicitly the following fact.

Fact 1. *Let κ be a limit ordinal and suppose that X is a scattered space of height κ such that every $U \in \tau(X)$ of height less than κ is σ-discrete. The space X is σ-discrete if it is weakly θ-refinable.*

Metacompactness implies weak θ-refinability, therefore we obtain the following consequence.

Theorem 2.3.4. *[AG, Theorem 3.2] Every hereditarily meta-Lindelöf locally compact scattered space X is σ-discrete.*

If instead of local compactness we assume local countability we still obtain the desired σ-discreteness.

Theorem 2.3.5. *[AG, Theorem 3.3] Every scattered hereditarily meta-Lindelöf locally countable space is σ-discrete.*

PROOF. Let $\mathscr{C}$ be a point-countable cover of the space X such that every $C \in \mathscr{C}$ is countable. For each $x \in X$, define $U_0^x = \bigcup\{C \in \mathscr{C} : x \in C\}$. Suppose that for every $x \in X$ we have defined the countable open set $U_n^x \in \tau(x,X)$. Let $U_{n+1}^x = \bigcup\{U_0^y : y \in U_n^x\}$ and $V_x = \bigcup\{U_n^x : n \in \omega\}$. Notice that by this construction, for every $x \in X$ and every $C \in \mathscr{C}$ if $C \cap V_x \neq \varnothing$ then $C \subset V_x$.

Now if $x \in V_y$ then there is $n \in \omega$ such that $x \in U_n^y$ which implies that $U_0^x \subset U_{n+1}^y$. It follows that $V_x \subset V_y$. On the other hand, by the construction, $x \in U_n^y$ also implies that $y \in U_n^x$ and thus $V_y \subset V_x$. We can conclude that if $V_y \cap V_x \neq \varnothing$ then $V_y = V_x$.

We have shown that the cover $\{V_x : x \in X\}$ induces a partition $\mathscr{P} = \{P_\alpha : \alpha \in I\}$ of the space X by countable open subsets of X. We can write $P_\alpha = \{x_n^\alpha : n \in \omega\}$ and define $D_n = \{x_n^\alpha : \alpha \in I\}$ for every $n \in \omega$. It is clear that $X = \bigcup\{D_n : n \in \omega\}$ and that each D_n is discrete because P_α isolates x_n^α in D_n. □

We have already observed that Eberlein-Grothendieck spaces of cardinality ω_1 are hereditarily meta-Lindelöf. This observation allows us to obtain Theorem 2.2.5, Theorem 2.2.6 and Corollary 2.2.8 as consequences of Theorems 2.3.4 and 2.3.5.

As we already mentioned in the Introduction, when K is a compact scattered space, the σ-fragmentability of $C_p(K)$ is equivalent to $C_p(K,\{0,1\})$ being σ-scattered, and analogously the SLD property of $C_p(K)$ is equivalent to $C_p(K,\{0,1\})$ being σ-discrete, cf. [Mtz]. The following definition is a slight variation of [Fa, Definition 5.1.1].

Definition 2.3.6. *[AG, Definition 3.4] Let X be a space with a metric ρ. Given $\varepsilon > 0$ and an ordinal α, an ε-open partitioning of length α of the space X is an increasing family $\{U_\beta : \beta < \alpha\} \subset \tau^*(X)$ that covers X and has the property that if $\gamma < \alpha$ then $diam_\rho(U_\gamma \setminus \bigcup_{\beta<\gamma} U_\beta) < \varepsilon$.*

Corollary 2.3.7. *[AG, Corollary 3.5] If K is a scattered compact space of weight ω_1 then $C_p(K)$ is SLD if and only if for every $\varepsilon > 0$ there exists a family $\{X_n : n \in \omega\}$ that covers $C_p(K,\{0,1\})$ and for each $n \in \omega$ the set X_n considered as a subspace of the metric space $C_u(K)$ has an ε-open partitioning of length ω_1.*

PROOF. We can consider that $\varepsilon < 1$; there exists a family $\{X_n : n \in \omega\}$ that covers $C_p(K)$ and for each $n \in \omega$ the set X_n has a ε-open partitioning $\{U^n_\alpha : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ we have $|U^n_\alpha \cap C_p(K,\{0,1\})| \leq 1$ which implies that for each $n \in \omega$, the space $X_n \cap C_p(K,\{0,1\})$ is an ω_1-right-separated sequence and hence σ-discrete by Corollary 2.2.8 so $C_p(K,\{0,1\})$ is σ-discrete as well. Apply [Mtz, Theorem 6] to conclude that $C_p(K)$ is SLD.

Assume that $C_p(K)$ is SLD and take $\varepsilon > 0$ and a countable cover $\{X_n : n \in \omega\}$ such that for each $n \in \omega$ the space X_n has an ε-small open cover. For every $n \in \omega$ we have $l(X_n) \leq w(C_u(K)) = w(K) = \omega_1$, therefore X_n has an ε-small open cover $\mathscr{V}_n$ of cardinality ω_1. Let $\mathscr{V}_n = \{V^n_\beta : \beta < \omega_1\}$ and define $U^n_\beta = \bigcup_{\alpha \leq \beta} V^n_\alpha$. The family $\{U^n_\beta : \beta < \omega_1\}$ is an ε-open partitioning of length ω_1. $\square$

2.4. Example

In Section 2.3 we applied the hereditary metalindelöfness of certain scattered spaces to prove they are σ-discrete. It is not very clear whether this property implies σ-discreteness of scattered Tychonoff spaces. This is not true for general spaces as we can deduce from the following example.

Example 2.4.1. *[AG, Example 5.1] There exists a Hausdorff scattered space which is hereditarily meta-Lindelöf that is not σ-discrete.*

PROOF. We will define a hereditarily meta-Lindelöf T_1-topology on the ordinal $\omega_1 \cdot \omega_1$, and in the last part of the construction we will modify this topology to make it Hausdorff. First we will define some auxiliary sets. Fix an ordinal $\gamma < \omega_1$ and define $V_{(\alpha,\gamma)}$ as follows:

- $V_{(\alpha,\gamma)} = \varnothing$ if $\alpha < \gamma$.

- $V_{(\gamma,\gamma)} = [\omega_1 \cdot \gamma, \omega_1 \cdot (\gamma+1))$.
- $V_{(\gamma+\beta,\gamma)} = [\omega_1 \cdot \gamma+\beta, \omega_1 \cdot (\gamma+1))$ for $\beta < \omega_1$.

Define $V_\alpha = \bigcup_{\gamma<\omega_1} V_{(\alpha,\gamma)}$.

Observe that the family $\{V_\alpha : \alpha < \omega_1\}$ thus defined is point-countable. Indeed, let $\xi \in \omega_1 \cdot \omega_1$. There are countable ordinals ς, ρ such that $\xi = \omega_1 \cdot \varsigma + \rho$. It is clear that $\xi \notin V_\alpha$ for $\alpha > (\varsigma+\rho)+1$.

For every $\lambda < \omega_1 \cdot \omega_1$ let $U_\lambda = [0,\lambda)$ and $W_\alpha^\lambda = V_\alpha \cap U_\gamma$. It is easy to see that the family $\mathscr{W} = \{W_\alpha^\lambda : \alpha < \omega_1, \lambda < \omega_1 \cdot \omega_1\}$ is a base for a topology τ on $\omega_1 \cdot \omega_1$. Note that by this construction $U_\lambda \in \tau$ for each $\lambda \in \omega_1 \cdot \omega_1$; hence the space $(\omega_1 \cdot \omega_1, \tau)$ with the order of $\omega_1 \cdot \omega_1$ is right-separated, and therefore this space is scattered.

To verify that the space $(\omega_1 \cdot \omega_1, \tau)$ is hereditarily meta-Lindelöf it suffices to show that for any family $\mathscr{C} \subset \mathscr{W}$ there is a point-countable family $\mathscr{O} \subset \tau$ such that $\bigcup \mathscr{C} = \bigcup \mathscr{O}$.

Take a family of basic open sets $\mathscr{C} = \{W_\alpha^\lambda : (\alpha,\lambda) \in I\} \subset \mathscr{W}$. For each $\alpha < \omega_1$ define $C_\alpha = \bigcup \mathscr{C} \cap V_\alpha$ and $J_\alpha = \{\lambda \in \omega_1 \cdot \omega_1 : W_\alpha^\lambda \in \mathscr{C}\}$. Notice that the family $\{C_\alpha : \alpha < \omega_1\}$ is point-countable, therefore it will be enough to show a point-countable family $\mathscr{O}_\alpha$ such that $C_\alpha = \bigcup \mathscr{O}_\alpha$ for each $\alpha < \omega_1$.

There are three possible mutually exclusive cases:

(I) If there is $\xi \in J_\alpha$ such that $W_\alpha^\lambda \subset W_\alpha^\xi$ for every $\lambda \in J_\alpha$ then define $\mathscr{O}_\alpha = \{W_\alpha^\xi\}$;

(II) If J_α is bounded in V_α and $cf(J_\alpha) = \omega$, then let J'_α be a countable cofinal subset of J_α and $\mathscr{O}_\alpha = \{W_\alpha^\lambda : \lambda \in J'_\alpha\}$.

(III) If the cofinality of the set J_α is ω_1 then it is possible to find an increasing ω_1-sequence of ordinals $\{\lambda_\eta : \eta < \omega_1\} \subset (\omega_1 \cdot \beta, \omega_1 \cdot (\beta+1))$, with $\beta+1 \leq \alpha+1$, that is cofinal in J_α. For each $\eta < \omega_1$ there is $\delta(\eta) < \omega_1$ such that $\lambda_\eta = \omega_1 \cdot \beta + \delta(\eta)$. Let $\mathscr{O}_\eta = \{W_\varsigma^{\lambda(\eta+1)} : \beta + \delta(\eta) \leq \varsigma < \beta + \delta(\eta+1)\}$. Define $\mathscr{O}_\alpha = \bigcup_{\eta<\omega_1} \mathscr{O}_\eta$. To see that in this case the family $\mathscr{O}_\alpha$ is point-countable, take $\rho \in \bigcup \mathscr{O}_\alpha$. Since each $\mathscr{O}_\eta$ is countable, it suffices to show that the set $\{\eta < \omega_1 : \rho \in \bigcup \mathscr{O}_\eta\}$ is countable. We can find countable ordinals ζ and ξ such that $\rho = \omega_1 \cdot \zeta + \xi$. Observe that by the definition of each $\delta(\eta)$ the family $\{\delta(\eta) : \eta < \omega_1\}$ is increasing and cofinal in ω_1. Thus there is $\eta < \omega_1$ such that $\zeta + \xi + 1 < \delta(\eta)$. It is clear that $\rho \notin \mathscr{O}_{\eta'}$ for every $\eta' > \eta$.

To see that the constructed space is not σ-discrete, suppose that $(\omega_1 \cdot \omega_1, \tau) = \bigcup_{n\in\omega} D_n$. For each $\alpha < \omega_1$ define $D_n^\alpha = D_n \cap [\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha+1))$. For every $\alpha \in \omega_1$ we can find $\Phi(\alpha) \in \omega$ such that $|D_{\Phi(\alpha)}^\alpha| = \omega_1$. Thus, there is $m \in \omega$ such that $|\{\alpha < \omega_1 : |D_m^\alpha| = \omega_1\}| = |\Phi^{-1}(m)| = \omega_1$. It is possible to find ordinals $\beta \in \omega_1$ y $\gamma \in D_m$ such that $|\{\alpha \in \beta : |D_m^\alpha| = \omega_1\}| = \omega$ and $\omega_1 \cdot \beta < \gamma \in D_m$.

If $\gamma \in W_\delta^\xi \in \tau$ then for every $\alpha < \beta$ such that $|D_m^\alpha| = \omega_1$ there exist $\lambda < \omega_1$ such that $W_\delta^\xi \cap [\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha+1)) = [\omega_1 \cdot \alpha + \lambda, \omega_1 \cdot (\alpha+1))$. Since $|D_m^\alpha| = \omega_1$, it follows that $D_m \cap [\omega_1 \cdot \alpha + \lambda, \omega_1 \cdot (\alpha+1)) \neq \varnothing$. This shows that the set D_m is not a discrete subspace of $(\omega_1 \cdot \omega_1, \tau)$.

The topology just defined on $\omega_1 \cdot \omega_1$ is not Hausdorff. Indeed, any two uncountable ordinals cannot be separated by disjoint open subsets. So take an injection φ of $\omega_1 \cdot \omega_1$ into some second countable space M. Let $\mathscr{B}$ be a countable base for M. The family $\{\varphi(B) \cap W : B \in \mathscr{B}, W \in \mathscr{W}\}$ is a base for a Hausdorff topology τ' on the set $X = \omega_1 \cdot \omega_1$.

It is not difficult now to verify that the space (X, τ') is hereditarily meta-Lindelöf and it is not σ-discrete. $\square$

Example 2.4.1 shows that hereditary metalindelöfness alone is not sufficient to guarantee σ-discreteness of scattered topological spaces in general. Hence a natural question arises: is every Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space σ-discrete? See Problem 9.

3

Decompositions of function spaces

To see whether a space Z has a “nice" topological property it is often useful to split Z into subspaces which could possibly have this property. V. Tkachuk proved in [Tk2] that if $C_p(X)$ is a countable union of its subspaces with a property $\mathscr{P} \in \{$hereditary π-character $\leq \kappa$, pseudo-character $\leq \kappa$, Čech-completeness, tightness $\leq \kappa$, Fréchet-Urysohn property$\}$, then $C_p(X)$ has $\mathscr{P}$. We will see that in function spaces closure-preserving closed covers sometimes can be reduced to countable closed covers, so the following question arises naturally: given a topological property $\mathscr{P}$ assume that $C_p(X)$ is the union of a closure-preserving family $\mathscr{F}$ of closed subspaces and each element of $\mathscr{F}$ has $\mathscr{P}$. Does this imply that $C_p(X)$ has $\mathscr{P}$ or some related topological property?

3.1. Compact-like closure-preserving covers

In [Ar2] it is described how Velichko proved that if $C_p(X)$ is σ-compact if and only if X is finite and then Shakmatov and Tkachuk extended such result by proving that X is finite if and only if $C_p(X)$ is σ-countably compact. In a recent book by Kąkol, Kubiś and López Pellicer ([KaKuL]) there is an alternative approach to this matter. But what if the cover by compact-like subspaces of $C_p(X)$ is not countable? Can countability be replaced by some other more topological condition to still obtain the same conclusion:

X is finite? In [Gue] it is shown that if the elements of a closure-preserving cover of $C_p(X)$ are compact then X is finite. In the same paper it is also established that a compact space X is metrizable if and only if $C_p(X)$ admits a closure-preserving cover by separable subspaces and that if $C_p(X,[0,1])$ is a closure-preserving union of compact subspaces, then X is discrete.

Suppose that $\mathscr{F}$ is a closure-preserving cover of a space Z and every element of $\mathscr{F}$ is closed. Then every separable subspace of Z can be covered by a countable subfamily of $\mathscr{F}$. This simple observation has strong implications for function spaces.

Proposition 3.1.1. *[Gue, Proposition 2.1] Neither the space $\mathbb{R}^\omega$ nor $C_p(\mathbb{I})$ can be covered by a closure-preserving closed family of σ-pseudocompact subspaces.*

PROOF. In [ST, Theorem 3] Shakmatov and Tkachuk showed that $\mathbb{R}^\omega$ and $C_p(\mathbb{I})$ fail to be σ-pseudocompact, so it suffices to observe that they are both separable, because $d(C_p([0,1])) = nw(C_p(\mathbb{I})) = nw(\mathbb{I}) = \omega = w(\mathbb{R}^\omega)$. $\square$

Remark 3.1.2. *[Gue, Remark 2.2] Note that the word "closed" cannot be omitted in the formulation of Proposition 3.1.1 because $\mathbb{R}^\omega$ and $C_p(\mathbb{I})$ are both separable, and therefore each one admits a closure-preserving cover by countable subspaces. Indeed, take a countable dense set $A \subset C_p(\mathbb{I})$. Then the family $\{A \cup \{f\} : f \in C_p(\mathbb{I})\}$ is a closure-preserving cover of $C_p(\mathbb{I})$. Clearly every separable space has such a cover.*

Lemma 3.1.3. *[Gue Lemma 2.3] If X is a space such that $C_p(X) = \bigcup \mathscr{F}$ where the family $\mathscr{F}$ is closure-preserving and integrated by closed σ-pseudocompact subspaces of $C_p(X)$, then X is pseudocompact.*

PROOF. If the space X is not pseudocompact, then $C_p(X)$ contains a retract Y homeomorphic to $\mathbb{R}^\omega$. Let $r : C_p(X) \to Y$ be a retraction. The space $\mathbb{R}^\omega$ is separable so we can find a countable family $\mathscr{F}' \subset \mathscr{F}$ such that $Y \subset \bigcup \mathscr{F}'$. Then the equality $Y = \bigcup\{r(F) : F \in \mathscr{F}'\}$ shows that $Y = \mathbb{R}^\omega$ is σ-pseudocompact which is a contradiction. $\square$

Corollary 3.1.4. *[Gue, Corollary 2.4] If $C_p(X)$ is the union of a closure-preserving family of pseudocompact subsets, then X is pseudocompact.*

PROOF. It suffices to observe that the closure of a pseudocompact subspace of $C_p(X)$ is still pseudocompact and then apply Lemma 3.1.3. $\square$

Our purpose is to show that if a function space $C_p(X)$ is a closure-preserving union of closed σ-countably compact subspaces then X is finite. We will need the following facts.

Proposition 3.1.5. *[Gue, Proposition 2.5] If X is a space for which $C_p(X) = \bigcup \mathscr{F}$ and $\mathscr{F}$ is a closure-preserving family of closed σ-countably compact subspaces of $C_p(X)$, then $f(X)$ is finite for every $f \in C_p(X)$.*

PROOF. Let $f \in C_p(X)$ and $Y = f(X)$. The equality $\omega = w(Y) = d(C_p(Y))$ shows that $C_p(Y)$ is separable. The space X being pseudocompact by Lemma 3.1.3, the map f is $\mathbb{R}$-quotient so the dual map f^* embeds $C_p(Y)$ in $C_p(X)$ as a closed subspace. Therefore $C_p(Y)$ is covered by a closure-preserving family of its closed σ-countably compact subspaces and hence it is σ-countably compact. It follows from [ST, Theorem 3.11] that Y is finite. $\square$

The following lemma is a part of the folklore.

Lemma 3.1.6. *[Gue, Proposition 2.6] For any space X, if $f(X)$ is finite for every f in $C_p(X)$, then X is finite.*

PROOF. Suppose $|f(X)| < \omega$ for all $f \in C_p(X)$. If X is infinite, then it is possible to find a countable discrete subspace $D = \{x_n : n \in \omega\} \subset X$ and a countable family of open sets $\{U_n : n \in \omega\}$ such that $U_n \cap D = \{x_n\}$ for each $n \in \omega$ and $\overline{U_i} \cap \overline{U_j} = \varnothing$ if $i \neq j$. For every $n \in \omega$ there is $f_n \in C(X, \mathbb{I})$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) \subset \{0\}$; let $g_n = \frac{1}{(n+1)} f_n$. To see that the map $g = \sum g_n$ is continuous take an arbitrary $p \in X$. If $p \in U_n$ for some $n \in \omega$ and $V \in \tau(g(p), \mathbb{I})$, then $W = g^{-1}(V) \cap U_n$ is an open set containing p such that $g(W) \subset V$ and hence g is continuous at p. If $\varepsilon > 0$ and $p \notin \bigcup_{n \in \omega} U_n$ then $g(p) = 0$; let $m = min\{n \in \omega : \frac{1}{n+1} < \varepsilon\}$. If $p \notin \bigcup_0^m \overline{U_j}$ then $p \in W = X \setminus \bigcup_0^m \overline{U_j}$ and $g(W) \subset [0, \varepsilon)$ which implies that g is continuous at p. Otherwise suppose that $p \in \overline{U_k}$ for some $k \leq m$. The set $V = X \setminus \{\overline{U_j} : j \neq k \text{ and } j \leq m\}$, is in $\tau(p, X)$ since $\overline{U_i} \cap \overline{U_j} = \varnothing$ if $i \neq j$. If $W = \big(X \setminus g_k^{-1}([\varepsilon, \frac{1}{k+1}])\big) \cap V$, then $g(W) \subset [0, \varepsilon)$ and therefore the map g is continuous at

p. The map g is continuous and by construction the set $g(X)$ is infinite. This contradiction shows that the space X is finite. □

Corollary 3.1.7. *[Gue, Corollary 2.7] For a space X the following conditions are equivalent:*

(a) $C_p(X) = \bigcup \mathscr{F}$ *where* $\mathscr{F}$ *is a closure-preserving family of its compact subspaces.*

(b) $C_p(X) = \bigcup \mathscr{F}$ *where* $\mathscr{F}$ *is a closure-preserving family of its closed* σ*-compact subspaces.*

(c) $C_p(X) = \bigcup \mathscr{F}$ *where* $\mathscr{F}$ *is a closure-preserving family of its closed* σ*-countably compact subspaces.*

(d) The space X is finite.

PROOF. It is evident that (a) implies (b) and (b) implies (c). Apply Proposition 3.1.5 and Lemma 3.1.6 to verify that (c) implies (d). If $n \in \omega$ then $\mathbb{R}^n$ can be covered by an increasing countable family of compact balls which shows that (d) implies (a). □

If the hypothesis that the elements of a closure-preserving cover of $C_p(X)$ are closed is removed, we still have the following result.

Corollary 3.1.8. *[Gue, Corollary 2.8] If X is a space, and $C_p(X)$ admits a closure-preserving cover by countably compact subspaces then X is finite.*

PROOF. Let $\mathscr{C}$ be a closure-preserving cover of $C_p(X)$ by countably compact subspaces. Apply Corollary 3.1.4 to check that X is pseudocompact. So every $C \in \mathscr{C}$ is compact by [Ar2, Theorem III.4.23]. Applying Corollary 3.1.7(a) we conclude that X is finite. □

3.2. Closure-preserving closed covers of $C_p(X)$

Since $C_u(X)$ has a stronger topology than $C_p(X)$, every closure-preserving closed cover of $C_p(X)$ is also a closure-preserving cover of $C_u(X)$. It is well known that $C_u(X)$ is a Čech-complete space so it has the Baire property. In particular, if $\{F_n : n \in \omega\}$ is a closed cover of $C_u(X)$ then some F_n has non-empty interior in the space $C_u(X)$. To analyze closure-preserving covers of $C_p(X)$ we will use a fundamental result of Terada and Yajima [TY, Theorem 2.5] which implies that Čech-complete spaces have something like a Baire property with respect to closure-preserving covers.

We will extensively exploit the following Theorem proved by Terada and Yajima [TY, Theorem 2.5] to analyze closure-preserving covers of function spaces.

Theorem 3.2.1. *[TY, Theorem 2.5] No G_δ-subset of a countably compact space can have a closure-preserving cover by nowhere dense sets.*

PROOF. Let Z be countably compact and fix $X = \bigcap_{n\in\omega} G_n$ with $G_n \in \tau^*(Z)$ for each $n \in \omega$. Suppose $\mathscr{F}$ is a closure-preserving cover of X whose elements are nowhere dense subsets of X. We may assume that $\mathscr{F}$ is a closed cover of X. Take $F_0 \in \mathscr{F}$. Since $F_0 \neq X$, there exists $U_0 \in \tau^*(X)$ such that $\overline{U_0} \cap F_0 = \varnothing$ and $\overline{U_0}^Z \subset G_0$. The family $\mathscr{F}$ covers X, hence we can find $F_1 \in \mathscr{F}$ such that $U_0 \cap F_1 \neq \varnothing$. Moreover $U_0 \setminus F_1 \neq \varnothing$ implies that there exists $U_1 \in \tau^*(X)$ such that $\overline{U_1} \subset U_0$, and $\overline{U_1} \cap F_1 = \varnothing$ and $\overline{U_1}^Z \subset G_1$. Continuing this approach we can define two sequences $\{F_n : n \in \omega\}$ and $\{U_n : n \in \omega\}$ such that for every $n \in \omega$:

(a) $F_n \in \mathscr{F}$ and $U_n \in \tau(X)$;

(b) $\overline{U_{n+1}} \subset U_n$ and $\overline{U_n}^Z \subset G_n$;

(c) $\overline{U_n} \cap F_n = \varnothing$ and $U_n \cap F_{n+1} \neq \varnothing$.

By (c) we can take $x_n \in U_n \cap F_{n+1}$ for every $n \in \omega$. The space Z is countably compact, therefore the sequence $\{x_n : n \in \omega\}$ has a cluster point $z \in Z$. Then we have

$$z \in \bigcap_{n\in\omega} \overline{\{x_k : k \geq n\}} \subset \bigcap_{n\in\omega} \overline{U_n}^Z \subset \bigcap_{n\in\omega} G_n = X.$$

Thus, $z \in \bigcap_{n\in\omega} \overline{U_n}$. Apply (c) again to verify that $\bigcup_{n\in\omega} F_n$ does not meet $\bigcap_{n\in\omega} \overline{U_n}$ and therefore $z \notin \bigcup_{n\in\omega} F_n$. On the other hand we have $z \in \overline{\{x_n : n \in \omega\}} \subset \overline{\bigcup_{n\in\omega} F_n} = \bigcup_{n\in\omega} F_n$ a contradiction.

□

Proposition 3.2.2. *[GT, Proposition 2.1] Given a space X, a function $f \in C_p(X)$, and a number $\varepsilon > 0$, let $I(f,\varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| \leq \varepsilon$ for all $x \in X\}$. Then:*

(a) If $U \in \tau(C_u(X))$ then for any $f \in U$ there exists $\varepsilon > 0$ such that $I(f,\varepsilon) \subset U$.

(b) For any $f \in C_p(X)$ and $\varepsilon > 0$ there exists a homeomorphism $\varphi : C_p(X) \to C_p(X)$ such that $\varphi(C_p(X,\mathbb{I})) = I(f,\varepsilon)$.

(c) Every space $I(f,\varepsilon)$ is a retract of $C_p(X)$.

(d) Any space $I(f,\varepsilon)$ contains a homeomorphic copy of $C_p(X)$.

PROOF. (a) Take $f \in U \in \tau(C_u(X))$; there is $r > 0$ such that $B(f,r) \subset U$. Let $0 < \varepsilon < min\{1/2, r/2\}$. If $g \in I(f,\varepsilon)$ then $|f(x) - g(x)| \leq \varepsilon < min\{1/2, r/2\}$ for every $x \in X$ which implies that $\sup\{|f(x) - g(x)| : x \in X\} \leq \varepsilon < min\{1,r\}$; thus $d(f,g) < r$ and therefore $g \in B(f,r)$ and $I(f,\varepsilon) \subset U$.

(b) Let $f \in C_p(X)$ and $\varepsilon > 0$. Define $\varphi : C_p(X) \to C_p(X)$ by the formula

$$\varphi(g) = 2\varepsilon(g - \frac{1}{2}) + f.$$

It is easy to see that the function φ is a homeomorphism and $\varphi(C_p(X,\mathbb{I})) = I(f,\varepsilon)$.

(c) It follows from (b) that there exists a homeomorphism $\varphi : C_p(X) \to C_p(X)$ such that $\varphi(C_p(X,\mathbb{I})) = I(f,\varepsilon)$. Therefore it suffices to show that $C_p(X,\mathbb{I})$ is a retract of $C_p(X)$. Define $\theta : C_p(X) \to C_p(X,(-\infty,1])$ by the formula $\theta(f)(x) = min\{f(x),1\}$ for every $x \in X$. Also define the map

$$\psi : C_p(X) \to C_p(X,[0,+\infty)) \textit{ by the formula } \psi(f)(x) = max\{f(x),0\}$$

for every $x \in X$. It is clear that $\theta \circ \psi : C_p(X) \to C_p(X,\mathbb{I})$ is a continuous retraction.

(d) It suffices to apply (b) and to note that $C_p(X,\mathbb{I})$ contains the subspace $C_p(X,(0,1))$ homeomorphic to $C_p(X)$. □

Terada and Yajima established in [TY, Theorem 2.5] that if Z is a Čech-complete space and $\mathscr{F}$ is a closure-preserving closed cover of Z then some element $F \in \mathscr{F}$ must have non-empty interior. Since $C_u(X)$ is always Čech-complete, we have the following result which is crucial for understanding what happens when $C_p(X)$ has a closure-preserving cover by nice subspaces.

Proposition 3.2.3. *[GT, Proposition 2.2] For an arbitrary X, if $\mathscr{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X,\mathbb{I})$ then there exists $C \in \mathscr{C}$ such that $U \subset C$ for some non-empty open subset U of the space $C_u(X)$.*

PROOF. Indeed, $\mathscr{C}$ is also a closure-preserving closed cover of $C_u(X)$ or $C_u(X,\mathbb{I})$ so the above mentioned result of Terada and Yajima applies to conclude that the interior of some $C \in \mathscr{C}$ in $C_u(X)$ must be non-empty. $\square$

Corollary 3.2.4. *[GT, Corollary 2.3] For an arbitrary X, if $\mathscr{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X,\mathbb{I})$ then there exist $C \in \mathscr{C}$ and $f \in \mathscr{C}$ such that $I(f,\varepsilon) \subset \mathscr{C}$ for some $\varepsilon > 0$.*

PROOF. Apply Proposition 3.2.2 and Proposition 3.2.3. $\square$

Corollary 3.2.5. *[GT, Corollary 2.4] If X is a space and $\mathscr{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X,\mathbb{I})$ then some $C \in \mathscr{C}$ contains a homeomorphic copy of $C_p(X)$.*

PROOF. Apply Corollary 3.2.4 and Proposition 3.2.2. $\square$

The following Corollary gives a positive answer to Problem 4.5 of [Gue].

Corollary 3.2.6. *[GT, Corollary 2.5] Suppose that $\mathscr{P}$ is a hereditary topological property and either $C_p(X,\mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathscr{C}$ such that every $C \in \mathscr{C}$ has $\mathscr{P}$. Then $C_p(X)$ also has $\mathscr{P}$.*

PROOF. By Corollary 3.2.5, there exists $C \in \mathscr{C}$ such that some $I \subset C$ is homeomorphic to $C_p(X)$; since C has $\mathscr{P}$, the space I and hence $C_p(X)$ must have $\mathscr{P}$. $\square$

It turns out that Problem 4.8, Problem 4.7 and the second part of Problem 4.6 of [Gue] have a positive answer as can be seen from the remark below.

Remark 3.2.7. *[GT, Remark 2.6] Suppose that κ is an infinite cardinal. Notice that Corollary 3.2.5 applies, for instance, to the following properties: weight $\leq \kappa$, network weight $\leq \kappa$, i-weight $\leq \kappa$, diagonal number $\leq \kappa$, character $\leq \kappa$, pseudocharacter $\leq \kappa$, tightness $\leq \kappa$, spread $\leq \kappa$, hereditary Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$, κ-monolithicity, metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property, being perfect, being functionally perfect.*

If a property $\mathscr{P}$ is not hereditary and $C_p(X)$ has a closure-preserving closed cover by subspaces that have $\mathscr{P}$ then $C_p(X)$ does not necessarily have $\mathscr{P}$. Indeed, it was proved in [Tk3, Example 15] that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that the family $\{F_n : n \in \omega\}$ is closure-preserving, so countable π-weight is not preserved by closed closure-preserving unions. However, for the properties which are closed-hereditary we have the following result.

Theorem 3.2.8. *[GT, Theorem 2.7] Given a space X and a closed-hereditary property $\mathscr{P}$, if $C_p(X,\mathbb{I})$ has a closed closure-preserving cover $\mathscr{C}$ such that every $C \in \mathscr{C}$ has $\mathscr{P}$ then $C_p(X,\mathbb{I})$ also has the property $\mathscr{P}$.*

PROOF. By Corollary 3.2.5 there exists $C \in \mathscr{C}$ such that $I(f,\varepsilon) \subset C$ for some $f \in C$ and $\varepsilon > 0$. Proposition 3.2.2(b) completes the proof. $\square$

We point out that this result still embraces a number of properties.

Remark 3.2.9. *[GT, Remark 2.8] Let κ be an infinite cardinal. Theorem 3.2.8 is applicable to the following properties: Lindelöf number $\leq \kappa$, extent $\leq \kappa$, Nagami number $\leq \kappa$, K-analyticity, $\ell\Sigma \leq \kappa$, dm $\leq \kappa$, mg $\leq \kappa$, mi $\leq \kappa$ normality, sequentiality.*

Corollary 3.2.10. *[GT, Corollary 2.9] If $C_p(X,\mathbb{I})$ has a closure-preserving closed cover $\mathscr{C}$ such that every $C \in \mathscr{C}$ is realcompact then $C_p(X)$ is realcompact.*

PROOF. Since realcompactness is closed-hereditary, we can apply Theorem 3.2.8 to see that $C_p(X,\mathbb{I})$ has to be realcompact. Clearly, the space $C_p(X)$ is homeomorphic to the space $C_p(X,(0,1)) \subset C_p(X,\mathbb{I})$. It is easy to see that $C_p(X,(0,1))$ can be obtained from

$C_p(X,\mathbb{I})$ by throwing out a union of G_δ subsets of $C_p(X,\mathbb{I})$. Therefore $C_p(X,(0,1))$ and hence $C_p(X)$ is realcompact. □

Corollary 3.2.11. *[GT, Corollary 2.10] If X is a space for which $C_p(X,\mathbb{I})$ has a closure-preserving closed cover by Čech-complete subspaces, then X is discrete.*

PROOF. Apply Theorem 3.2.8 to see that $C_p(X,\mathbb{I})$ is Čech-complete and hence the space X is discrete (see Theorem 1.13 of [Tk3]). □

Theorem 3.2.12. *[GT, Theorem 2.11] If X is a space such that $C_p(X,\mathbb{I})$ has a closure-preserving closed cover by σ-countably compact subspaces, then $C_p(X,\mathbb{I})$ is countably compact.*

PROOF. Apply Theorem 3.2.8 to verify that $C_p(X,\mathbb{I})$ is σ-countably compact. By [Tk2, 1.5.3] the space $C_p(X,\mathbb{I})$ is countably compact. □

The following Corollary provides a positive answer to Problem 4.3 of [Gue].

Corollary 3.2.13. *[GT, Corollary 2.12] If $C_p(X,\mathbb{I})$ has a closure-preserving closed cover $\mathscr{C}$ such that every $C\in\mathscr{C}$ is σ-compact then X is discrete.*

PROOF. Since σ-compactness is closed-hereditary, we can apply Theorem 3.2.8 to conclude $C_p(X,\mathbb{I})$ is σ-compact. Therefore X is discrete by [Tk2, 1.5.2]. □

Theorem 3.2.14. *[GT, Theorem 2.13] Given a space X and a property $\mathscr{P}$ that is preserved by quotient images, if $C_p(X,\mathbb{I})$ has a closed closure-preserving cover $\mathscr{C}$ such that every $C\in\mathscr{C}$ has $\mathscr{P}$ then $C_p(X,\mathbb{I})$ also has the property $\mathscr{P}$.*

PROOF. By Corollary 3.2.4 there exists $C\in\mathscr{C}$ such that $I(f,\varepsilon)\subset C$ for some $f\in C$ and $\varepsilon>0$. Apply Proposition 3.2.2(c) to see that $I(f,\varepsilon)$ is a retract of $C_p(X)$; thus $I(f,\varepsilon)$ is also a retract of C which implies that it is a quotient image of C. Proposition 3.2.2(b) completes the proof. □

Remark 3.2.15. *[GT, Remark 2.14] Suppose that κ is an infinite cardinal. Observe that Theorem 3.2.14 applies to properties such as weak functional tightness $\leq \kappa$, functional tightness $\leq \kappa\}$. This theorem applies also to the property of κ-stability, but in this case not only $C_p(X,\mathbb{I})$ is κ-stable but the whole $C_p(X)$ is. Indeed, since $C_p(X,\mathbb{I})$ is κ-stable then $C_p(C_p(X,\mathbb{I}))$ is κ-monolithic by [Ar2, Theorem II.6.8] and X embeds in $C_p(C_p(X,\mathbb{I}))$, thus X is also κ-monolithic so $C_p(X)$ is κ-stable by [Ar2, Theorem II.6.9].*

In the rest of this section we consider closure-preserving closed covers of $C_p(X)$ whose elements are all either Lindelöf or Lindelöf Σ. It follows from Theorem 3.2.8 that $C_p(X,\mathbb{I})$ must have the respective property. However, we strongly suspect that in this case the whole space $C_p(X)$ must be Lindelöf or Lindelöf Σ respectively. It turned out not to be easy to verify, even for the spaces with a unique non-isolated point. We will prove some positive results in this direction; they are often generalizations of some well-known theorems about the properties of a space X for which $C_p(X)$ is either Lindelöf or has the Lindelöf Σ property.

Proposition 3.2.16. *[GT, Proposition 2.15] Given a space X and an infinite cardinal κ, suppose that $C_p(X)$ has a closure-preserving closed cover $\mathscr{C}$ such that $\ell(C) \leq \kappa$ for every $C \in \mathscr{C}$. Then any discrete family of non-empty open subsets of X has cardinality not greater that κ.*

PROOF. Fix a closure-preserving closed cover $\mathscr{C}$ of $C_p(X)$ such that $\ell(C) \leq \kappa$ for every $C \in \mathscr{C}$. Suppose that $\{U_\alpha : \alpha < \kappa^+\} \subset \tau^*(X)$ is a discrete family. If we choose a point $x_\alpha \in U_\alpha$ for each $\alpha < \kappa^+$ then the subspace $D = \{x_\alpha : \alpha < \kappa^+\}$ is closed and discrete. Fix a function $\varphi_\alpha \in C_p(X,\mathbb{I})$ such that $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(X \setminus U_\alpha) \subset \{0\}$ for every $\alpha < \kappa^+$. Given a function $f \in \mathbb{R}^D$ it is immediate that $u(f) = \sum\{f(x_\alpha) \cdot \varphi_\alpha : \alpha < \kappa^+\}$ is a continuous function on X such that $u(f)|D = f$. It is standard that $u : \mathbb{R}^D \to C_p(X)$ is a closed embedding; thus the set $E = u(\mathbb{R}^D)$ is homeomorphic to $\mathbb{R}^{\kappa^+}$.

The space $\mathbb{R}^{\kappa^+}$ has density not greater than κ so $d(E) \leq \kappa$ and we can find a family $\mathscr{C}' \subset \mathscr{C}$ such that $|\mathscr{C}'| \leq \kappa$ and $E \subset \bigcup \mathscr{C}'$. It is clear that $l(\bigcup \mathscr{C}') \leq \kappa$ and E is closed in $\bigcup \mathscr{C}'$ so $l(E) \leq \kappa$ and hence $l(\mathbb{R}^{\kappa^+}) \leq \kappa$, which is a contradiction. $\square$

Corollary 3.2.17. *[GT, Corollary 2.16] Suppose that κ is an infinite cardinal and X is a paracompact space such that $C_p(X)$ has a closure-preserving closed cover $\mathscr{C}$ with $l(C) \leq \kappa$ for every $C \in \mathscr{C}$. Then $l(X) \leq \kappa$; in particular, if X is metrizable then $w(X) \leq \kappa$.*

Our next step is to prove a positive result for the spaces with a unique non-isolated point and generalize an Asanov's theorem which states that the tightness of any finite power of X is countable whenever $C_p(X)$ is Lindelöf. Actually, Asanov's proof gives the same conclusion even if we only assume that $C_p(X,\mathbb{I})$ is Lindelöf. However, this was not even mentioned in Asanov's papers and numerous surveys that appeared afterwards. The proof differs very little from the proof of the original version as given in [Ar2, Theorem I.4.1] still we include it here.

Lemma 3.2.18. *[GT, Lemma 2.17] For an arbitrary space X and an infinite cardinal κ, if $l(C_p(X,\mathbb{I})) \leq \kappa$, then $t(X^n) \leq \kappa$ for every $n \in \omega$.*

PROOF. Take $x = (x_1, \ldots, x_n) \in X^n$ and $A \subset X^n$ with $x \in \overline{A}$. Pick $O_i \in \tau(x_i, X)$ such that $O_i \cap O_j = \varnothing$ if $x_i \neq x_j$ and $O_i = O_j$, if $x_i = x_j$. The set $O = O_1 \times \ldots \times O_n$ is a neighbourhood of x and $x \in \overline{A \cap O}$ therefore we can assume $A \subset O$. The set $\Phi = \{f \in C_p(X,\mathbb{I}) : f(x_i) = 1$ for every $i \leq n\}$ is closed in $C_p(X,\mathbb{I})$ and hence $l(\Phi) \leq \kappa$. Given $y = (y_1, \ldots, y_n) \in A$, define $W_y = \{g \in C_p(X,\mathbb{I}) : g(y_i) > 0$ for every $i \leq n\}$. Now, if $f \in \Phi$ then $U_i = f^{-1}((0,+\infty)) \in \tau(x_i, X)$ for every $i \leq n$. Since $x \in \overline{A}$, we can find a point $y = (y_1, \ldots, y_n) \in A \cap (U_1 \times \ldots \times U_n)$. It follows that $f(y_i) > 0$ for each $i \leq n$ thus $f \in W_y$. We have shown that $\Phi \subset \bigcup\{W_y : y \in A\}$.

Since $l(\Phi) \leq \kappa$, there exists $B \subset A$ such that $|B| \leq \kappa$ and $\Phi \subset \{W_y : y \in B\}$. We claim that $x \in \overline{B}$. Otherwise, $V \cap B = \varnothing$ for some $V = V_1 \times \ldots \times V_n$ where $x_i \in V_i \in \tau(O_i)$ for each $i \leq n$ and $V_i = V_j$ if $x_i = x_j$.

It is easy to see that there is a function $g \in \Phi$ such that $g(z) = 0$ for any $z \in X \setminus (\bigcup\{V_i : i \leq n\})$. We have $g \in W_y$ for some $y = (y_1, \ldots, y_n) \in B$. Fix any $i \in \{1, \ldots, n\}$; recall that $y \in A \subset O$ therefore $y_i \in O_i$. It follows from $g(y_i) > 0$ that $y_i \in \bigcup\{V_k : k \leq n\}$, so we can find $j \leq n$ such that $y_i \in V_j$. Recall that $V_j \subset O_j$, we can conclude $y_i \in O_i \cap O_j$ which implies that $O_i \cap O_j \neq \varnothing$ which shows that $O_i = O_j$ and hence $x_i = x_j$ because the election of O; this implies $V_i = V_j$. Therefore $y_i \in V_i$ for each $i \leq n$; consequently, $y \in (V_1 \times \ldots \times V_n) \cap B$ a contradiction. $\square$

The following Corollary generalizes the aforementioned result of Asanov.

Corollary 3.2.19. *[GT, Corollary 2.18] Given a space X, if $C_p(X)$ admits a closure-preserving closed cover $\mathscr{C}$ such that $l(C) \leq \kappa$ for every $C \in \mathscr{C}$, then $t(X^n) \leq \kappa$ for $n \in \mathbb{N}$.*

It is known (see e.g., [Le]) that for a space X with a unique non-isolated point, the space $C_p(X)$ is Lindelöf if and only if X is Lindelöf and $t(X^n) \leq \omega$ for any $n \in \mathbb{N}$. Our technique allows us to generalize this result in the following way.

Corollary 3.2.20. *[GT, Corollary 2.19] For a space X with a unique non-isolated point the following conditions are equivalent:*

(a) $C_p(X)$ is Lindelöf;

(b) $C_p(X)$ has a closure-preserving closed cover by Lindelöf subspaces;

(c) the space X is Lindelöf and $t(X^n) \leq \omega$ for any $n \in \mathbb{N}$.

PROOF. The implication $(a) \Longrightarrow (b)$ is trivial and $(c) \Longrightarrow (a)$ was proved in [Le]. To see that $(b) \Longrightarrow (c)$ observe first that $C_p(X,\mathbb{I})$ is Lindelöf by Remark 3.2.15. Therefore we can apply Lemma 3.2.18 to deduce that $t(X^n) \leq \kappa$ for $n \in \omega$. Let a be the unique non-isolated point of X. If $X \setminus U$ is uncountable for some $U \in \tau(a,X)$ then the family $\{\{x\} : x \in X \setminus U\}$ is discrete and consists of non-empty open subsets of X which contradicts Proposition 3.2.16. Therefore $|X \setminus U| \leq \omega$ for any $U \in \tau(a,X)$ so the space X is Lindelöf. □

In the rest of this section we will consider the situation when $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. It was asked in [Gue, Problem 4.11] whether $C_p(X)$ has to be Lindelöf Σ in this case. We conjecture that the answer should be positive, but we could not prove this even for a space X with a unique non-isolated point. Below we present several results to support our conjecture and generalize some well-known theorems on properties of spaces X such that $C_p(X)$ is Lindelöf Σ.

Proposition 3.2.21. *[GT, Proposition 2.20] If X is a Lindelöf Σ-space and $C_p(X)$ has a closure-preserving closed cover by Lindelöf Σ-subspaces then $C_p(X)$ is a Lindelöf Σ-space.*

PROOF. It follows from Remark 3.2.15 that $C_p(X,\mathbb{I})$ has to be a Lindelöf Σ-space. Therefore $C_p(X)$ is Lindelöf Σ by [Ar2, Proposition IV.9.17]. □

It is a non-trivial result of Arhangel'skii (see [Ar2, Theorem IV.9.8]) that if $C_p(X)$ is a Lindelöf Σ-space then it is ω-monolithic. Using our technique of dealing with closure-preserving covers, we generalize it as follows.

Theorem 3.2.22. *[GT, Theorem 2.21] Assume that X is a space and $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. Then $C_p(X)$ is ω-monolithic.*

PROOF. Let $\mathscr{F}$ be a closure-preserving closed family of Lindelöf Σ-subspaces of $C_p(X)$ such that $C_p(X) = \bigcup \mathscr{F}$. Take any countable set $A \subset C_p(X)$ and define the map $\varphi : X \to C_p(A)$ by the formula $\varphi(x)(f) = f(x)$ for each $f \in A$; let $Y = \varphi(X)$. Observe that we have the inequalities $w(Y) \leq w(C_p(A)) \leq \omega$. It is standard to see that $A \subset \varphi^*(C_p(Y))$.

There exists a space Z and continuous maps $\varphi' : X \to Z$ and $\xi : Z \to Y$ such that φ' is $\mathbb{R}$-quotient, ξ is injective and $\varphi = \xi \circ \varphi'$. The set $E = (\varphi')^*(C_p(Y))$ is closed in $C_p(X)$ and $A \subset \varphi^*(C_p(Y)) \subset E$ which shows that $\overline{A} \subset E$. The map ξ is a condensation of Z onto the second countable space Y; as a consequence, $d(C_p(Z)) = iw(Z) \leq w(Y) \leq \omega$; choose a countable set $Q \subset E$ such that $E = \overline{Q}$. For every $a \in Q$ choose a set $F_a \in \mathscr{F}$ such that $a \in F_a$; the family $\mathscr{G} = \{F_a : a \in Q\}$ is countable so the set $G = \bigcup \mathscr{G}$ is a Lindelöf Σ-space. Since G is closed in $C_p(X)$, we have $E = \overline{Q} \subset \overline{\bigcup \mathscr{G}} = \bigcup \mathscr{G} = G$ so E is a Lindelöf Σ-space. Therefore we can apply [Ar1, Theorem IV.9.8] to see that $C_p(Z)$ is ω-monolithic and hence E is ω-monolithic as well. As a consequence, $\overline{A} = cl_E(A)$ is a cosmic space. □

Answering a question of Arhangel'skii, Tkachuk proved in [Tk, Theorem 2.15] that if $C_p(X)$ is a Lindelöf Σ-space and ω_1 is a caliber of X then X is cosmic. It turns out that the Lindelöf Σ-property in this result can be substituted by existence of a closure-preserving closed cover by Lindelöf Σ-subspaces.

Corollary 3.2.23. *[GT, Corollary 2.22] If ω_1 is a caliber of a space X and $C_p(X)$ has a closed closure-preserving cover of $C_p(X)$ by Lindelöf Σ-subspaces, then X is cosmic.*

PROOF. Fix a closure-preserving closed cover $\mathscr{L}$ of the space $C_p(X)$ by Lindelöf Σ-subspaces. Since ω_1 is a caliber of X, the space $C_p(X)$ has a small diagonal according to [Tk, Theorem 1] and so does each element of the cover $\mathscr{L}$. Therefore if $L \in \mathscr{L}$ and K is a compact subspace of L then K has a small diagonal. As a consequence, the space K cannot contain convergent ω_1-sequences and so we have $t(K) \leq \omega$ by [JuS, Theorem 1.2] and [Ju, 3.12]. Apply Theorem 3.2.22 to verify that K is ω-monolithic; it is standard to see that any ω-monolithic compact space of countable tightness with a small diagonal is metrizable so K has to be metrizable.

Thus each $L \in \mathscr{L}$ has a small diagonal and a cover by compact metrizable subspaces for which there exists a countable network, so Theorem 2.1 of [Gr] allows us to conclude that the elements of $\mathscr{L}$ are cosmic. It follows from Remark 3.2.7 that $C_p(X)$ and X are cosmic. $\square$

It is a theorem of Arhangel'skii [Ar3, Theorem 10] that for any space X such that $C_p(X)$ is Lindelöf Σ, if $s(C_p(X)) \leq \omega$ then X is cosmic. This theorem can also be generalized in the same spirit as before.

Corollary 3.2.24. *[GT, Corollary 2.23] Assume that X is a space and $C_p(X)$ has a closed closure-preserving cover by its Lindelöf Σ-subspaces. If, additionally, the spread of $C_p(X)$ is countable then X is cosmic.*

PROOF. Note first that $s(X \times X) \leq s(C_p(X)) \leq \omega$. By Remark 3.2.9, the space $C_p(X, \mathbb{I})$ is Lindelöf Σ; since X embeds in $C_p(C_p(X, \mathbb{I}))$ it has to be monolithic. This shows that $X \times X$ is also monolithic; since every ω-monolithic space of countable spread is hereditarily Lindelöf (see [Ar1, Theorem 1.2.9]) we conclude that $hl(X \times X) \leq \omega$.

Therefore the diagonal of X is a G_δ-set; every Lindelöf space with a G_δ-diagonal has countable i-weight [Ar1, Theorem 2.1.8] so $d(C_p(X)) = iw(X) \leq \omega$. By Theorem 3.2.22, the space $C_p(X)$ is ω-monolithic and therefore $nw(X) = nw(C_p(X)) \leq \omega$. $\square$

3.3. General closure-preserving covers of $C_p(X)$

For most topological properties $\mathscr{P}$, if a space Z has a dense subspace Y which has $\mathscr{P}$ then Z is the closure-preserving union of subspaces with the property $\mathscr{P}$. To see this, it suffices to consider the family $\mathscr{F} = \{Y \cup \{x\} : x \in Z\}$; since all the elements of $\mathscr{F}$ are dense in Z, the family $\mathscr{F}$ is closure-preserving. It turns out that if $Z = C_p(X)$ for some X then the converse is true as well.

Theorem 3.3.1. *[GT, Theorem 3.1] Given a space X and a topological property $\mathscr{P}$ that is invariant under continuous images, if either $C_p(X)$ or $C_p(X, \mathbb{I})$ admits a closure-preserving (not necessarily closed) cover $\mathscr{C}$ such that each $C \in \mathscr{C}$ has $\mathscr{P}$ then $C_p(X, \mathbb{I})$ contains a dense subspace that has $\mathscr{P}$.*

PROOF. The family $\{\overline{C} : C \in \mathscr{C}\}$ is also closure-preserving cover of $C_p(X,\mathbb{I})$ (or $C_p(X)$ respectively. Apply Corollary 3.2.4 to find a function $f \in C_p(X,\mathbb{I})$ and $\varepsilon > 0$ such that $I(f,\varepsilon) \subset \overline{C}$ for some $C \in \mathscr{C}$. By Proposition 3.2.2 the set $R = I(f,\varepsilon)$ is a retract of $C_p(X)$ homeomorphic to $C_p(X,\mathbb{I})$. Consequently, there exists a retraction $r : \overline{C} \to R$. The set $r(C)$ is dense in $r(\overline{C}) = R$ and has the property $\mathscr{P}$. Since R is homeomorphic to $C_p(X,\mathbb{I})$, the latter also has a dense subspace with the property $\mathscr{P}$. □

Observe that if $\mathscr{C}$ is a closure-preserving (not necessarily closed) cover of $C_p(X)$ such that every $C \in \mathscr{C}$ has a property $\mathscr{P}$, there is no evident way to obtain a closure-preserving cover of $C_p(X,\mathbb{I})$ by subspaces with the property $\mathscr{P}$.

Theorem 3.3.2. *[GT, Theorem 3.2] Suppose that X is a space and $\mathscr{P}$ is a σ-additive topological property such that all singletons have $\mathscr{P}$ and $\mathscr{P}$ is invariant under continuous images. Then the following conditions are equivalent.*

(a) $C_p(X)$ admits a closure-preserving cover $\mathscr{C}$ such that each $C \in \mathscr{C}$ has $\mathscr{P}$.

(b) $C_p(X,\mathbb{I})$ admits a closure-preserving cover $\mathscr{C}$ such that each $C \in \mathscr{C}$ has $\mathscr{P}$.

(c) $C_p(X)$ has a dense subspace with the property $\mathscr{P}$.

(d) $C_p(X,\mathbb{I})$ has a dense subspace with the property $\mathscr{P}$.

PROOF. Since $C_p(X,\mathbb{I})$ is a continuous image of the space $C_p(X)$, it is immediate that $(c) \Longrightarrow (d)$. If D is a dense subspace of $C_p(X,\mathbb{I})$ with the property $\mathscr{P}$ then $D_n = n \cdot D$ is a dense subspace of $C_p(X,[-n,n])$ with the property $\mathscr{P}$ for any $n \in \mathbb{N}$. By σ-additivity of $\mathscr{P}$, the set $\bigcup\{D_n : n \in \mathbb{N}\}$ has the property $\mathscr{P}$; since it is dense in $C_p(X)$, this proves that $(d) \Longrightarrow (c)$ and hence the conditions (c) and (d) are equivalent. It follows from Theorem 3.3.1 that both (a) and (b) imply (d). At the beginning of this section we observed that $(c) \Longrightarrow (a)$ and $(d) \Longrightarrow (b)$ so all properties (a)-(d) are equivalent. □

Remark 3.3.3. *[GT, Remark 3.3] Observe that for an infinite cardinal κ Theorem 3.3.2 applies to the following properties: network weight $\leq \kappa$, spread $\leq \kappa$, Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$.*

The statement below gives a positive answer to Problem 4.4 of [Gue].

Corollary 3.3.4. *For any infinite cardinal κ consider the following list $\mathbb{M}_1$ of topological properties:*
$\mathbb{M}_1 = \{$*k-separability, caliber κ, point-finite cellularity $\leq \kappa$, density $\leq \kappa$*$\}$.
If a property $\mathscr{P}$ belongs to the list $\mathbb{M}_1$ then the following conditions are equivalent:

(a) $C_p(X)$ admits a closure-preserving cover $\mathscr{C}$ such that each $C \in \mathscr{C}$ has $\mathscr{P}$.

(b) $C_p(X,\mathbb{I})$ admits a closure-preserving cover $\mathscr{C}$ such that each $C \in \mathscr{C}$ has $\mathscr{P}$.

(c) $C_p(X)$ has the property $\mathscr{P}$.

(d) $C_p(X,\mathbb{I})$ has the property $\mathscr{P}$.

PROOF. Observe that every property $\mathscr{P}$ from the list $\mathbb{M}_1$ is σ-additive and invariant under continuous images; besides, all singletons have $\mathscr{P}$ and if a space Z has a dense subspace with the property $\mathscr{P}$ then Z has $\mathscr{P}$. By Theorem 3.3.2, the conditions (a) and (b) are equivalent and we have the implications $(c) \Longrightarrow (a)$ and $(d) \Longrightarrow (a)$.

Theorem 3.3.2 also shows that both (a) and (b) imply that $C_p(X,\mathbb{I})$ and $C_p(X)$ have a dense subspace with the property $\mathscr{P}$. By the above remark, both $C_p(X)$ and $C_p(X,\mathbb{I})$ must have $\mathscr{P}$ so all the properties (a)-(d) are equivalent. $\square$

The following result gives a positive answer to [Gue, Problem 4.1].

Corollary 3.3.5. *[GT, Corollary 3.4] For any space X, the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by pseudocompact subspaces.

(b) $C_p(X)$ has a closure-preserving cover by closed pseudocompact subspaces.

(c) $C_p(X)$ is σ-pseudocompact.

PROOF. Since the closure of a pseudocompact set is pseudocompact, it is immediate that the conditions (a) and (b) are equivalent. If $C_p(X) = \bigcup\{C_n : n \in \omega\}$ and every C_n is pseudocompact then consider the set $D_n = \overline{C}_0 \cup \ldots \cup \overline{C}_n$ for each $n \in \omega$. It is easy to see that $\{D_n : n \in \omega\}$ is a closure-preserving closed cover of $C_p(X)$ with pseudocompact elements so $(c) \Longrightarrow (b)$. If $C_p(X)$ has a closure-preserving cover by pseudocompact subspaces then apply Theorem 3.3.1 to see that $C_p(X,\mathbb{I})$ has a dense pseudocompact subspace and hence it is pseudocompact. Since X has to be pseudocompact by [Gue, Corollary 2.4], we have the equality $C_p(X) = \bigcup\{n \cdot C_p(X,\mathbb{I}) : n \in \omega\}$ which shows that $C_p(X)$ is σ-pseudocompact. □

The statement that follows shows that the answer to Problem 4.2 of [Gue] is positive.

Corollary 3.3.6. *[GT, Corollary 3.4] For any space X, the following conditions are equivalent:*

(a) $C_p(X,\mathbb{I})$ has a closure-preserving cover by pseudocompact subspaces.

(b) $C_p(X,\mathbb{I})$ has a closure-preserving cover by closed pseudocompact subspaces.

(c) $C_p(X,\mathbb{I})$ is pseudocompact.

PROOF. It is trivial that $(c) \Longrightarrow (b)$; since the closure of any pseudocompact space is pseudocompact, it is immediate that the conditions (a) and (b) are equivalent. Finally, if (a) holds, then Theorem 3.3.1 shows that $C_p(X,\mathbb{I})$ contains a dense pseudocompact subspace so it must be pseudocompact. □

The study of $C_p(K)$ spaces for K compact deserves a special treatment. In the rest of this section we will take a look at those spaces in the light of the closure-preserving covers they admit.

Before we proceed we should notice the following. If X is a space of density κ then X has a closure-preserving cover by subspaces of cardinality κ. Indeed, if we take subspace D dense in X such that $|D| = \kappa$, it is immediate that $\{D \cup \{x\} : x \in X\}$ is a closure-preserving cover of X by subspaces of cardinality κ. As a consequence we have that every separable space has a closure-preserving cover by countable subspaces.

Furthermore, assume that $\mathscr{F}$ is a closure-preserving cover of a space Z and that each element of $\mathscr{F}$ is closed. If X is a subspace of Z and the density of X is $\leq \kappa$, then there exists a subfamily $\mathscr{F}' \subset \mathscr{F}$ of cardinality $\leq \kappa$ that covers X. Indeed take a dense subspace $D \subset X$ such that $|D| \leq \kappa$. For every $d \in D$ there exists $F_d \in \mathscr{F}$ such that $d \in \mathscr{F}$. Put $\mathscr{F}' = \{F_d : d \in D\}$. Since $\mathscr{F}'$ is closure-preserving and closed we have $\bigcup \mathscr{F}' = \overline{\bigcup \mathscr{F}'} \supset \overline{D} = X$.

The following result appeared in [Gue] stated in a different, less convenient and not quite clear fashion. Also the proof presented in [Gue] is unnecessarily complicated. Here we state it clearly, so that its strength can be fully appreciated, and give a short proof.

Theorem 3.3.7. *[Gue, Lemma 2.10] Given a compact space K and a cardinal κ, if $C_p(K)$ admits a closure-preserving cover of subspaces of cardinality κ then $w(K) \leq \kappa$.*

PROOF. Suppose $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ is a closure-preserving cover of $C_p(K)$ by subspaces of cardinality κ. It is immediate that the density of the elements of $\mathscr{F}$ is not greater than κ. Therefore Corollary 3.3.4 applies to conclude $d(C_p(K)) \leq \kappa$ hence $w(K) \leq d(C_p(K) \leq \kappa$. □

Corollary 3.3.8. *[Gue, Lemma 2.12] Given a compact space K, if $C_p(K)$ admits a closure-preserving cover of subspaces of cardinality at most $\mathfrak{c}$ then $C_p(K)$ has cardinality not greater than $\mathfrak{c}$.*

PROOF. Suppose $\mathscr{C}$ is a closure-preserving cover of $C_p(K)$ by subspaces of cardinality at most $\mathfrak{c}$. The family $\mathscr{F} = \{\overline{C} : C \in \mathscr{C}\}$ is a closure-preserving closed cover of $C_p(K)$. By [Ar2, Lemma IV.11.3] we have $|F| \leq \mathfrak{c}$ for every $F \in \mathscr{F}$. From Theorem 3.3.7 it follows that $w(K) \leq \mathfrak{c}$ thus $d(C_p(K)) \leq \mathfrak{c}$, it is easy to see that this implies $\mathscr{F}$ has a subcover $\mathscr{G}$ of cardinality at most $\mathfrak{c}$ and therefore $|C_p(K)| \leq \mathfrak{c}$. □

Corollary 3.3.9. *[Gue, Lemma 2.13] For every infinite compact space K the following conditions are equivalent:*

(a) There exists a closure-preserving cover $\mathscr{F}$ of $C_p(K)$ such that the elements of $\mathscr{F}$ are cosmic.

(b) There exists a closure-preserving cover $\mathscr{F}$ of $C_p(K)$ such that the elements of $\mathscr{F}$ are separable.

(c) There exists a closure-preserving cover $\mathscr{F}$ of $C_p(K)$ such that the elements of $\mathscr{F}$ are countable.

(d) The space K is metrizable.

PROOF. It is clear that (a) implies (b), and that (c) implies (b). Now, if K is metrizable, then $C_p(K)$ is separable, hence we have

$$d(C_p(K)) = iw(K) = w(K) = nw(K) = nw(C_p(K))$$

which shows that (d) implies (a). To convince ourselves that (d) implies (c), take a countable subspace D dense in $C_p(K)$ and for each $f \in C_p(K)$ define the set $D_f = D \cup \{f\}$. It is clear that $\mathscr{D} = \{D_f : f \in C_p(X)\}$ is a closure-preserving cover of $C_p(K)$ whose elements are countable. It remains to prove that (b) implies (d). Suppose that K is not metrizable, that is $w(K) > \omega$ because K is compact. As a consequence, $d(C_p(K)) > \omega$ then by Theorem 3.3.7 we can conclude that (b) does not hold. □

Notice that in Theorem 3.3.9 it is not considered the case in which for a compact space K the space $C_p(K)$ admits a closure-preserving closed cover by second countable spaces. The reason is that in such particular situation we can say more about the compact space K.

Corollary 3.3.10. *[Gue, Lemma 2.14] If K is an infinite compact space then $C_p(K)$ admits a closure-preserving closed cover by second countable subspaces if and only if K is countable.*

Corollary 3.3.11. *[GT, Corollary 3.5] If X is a compact space then the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by σ-compact subspaces.

(b) $C_p(X, \mathbb{I})$ has a closure-preserving cover by σ-compact subspaces.

(c) X is Eberlein compact.

PROOF. Recall (see e.g. [Ar2, Theorem IV.1.7]) that X is Eberlein compact if and only if $C_p(X)$ (or equivalently, $C_p(X, \mathbb{I})$) has a dense σ-compact subspace and apply Theorem 3.3.1. □

Arhangel'skii [Ar2, Section IV.2] defined ω-perfect classes $\mathscr{P}$ as closed-hereditary, invariant under continuous images and such that $Z \in \mathscr{P}$ implies $(Z \times \omega)^\omega \in \mathscr{P}$. It turns out that ω-perfect classes are relevant to the topic of this chapter.

Proposition 3.3.12. *[GT, Corollary 3.6] If $\mathscr{P}$ is a ω-perfect class and X is a compact space then the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by subspaces that belong to $\mathscr{P}$.

(b) $C_p(X,\mathbb{I})$ *has a closure-preserving cover by subspaces that belong to* $\mathscr{P}$.

(c) $C_p(X)$ *belongs to* $\mathscr{P}$.

PROOF. The implications $(c) \Longrightarrow (a)$ and $(c) \Longrightarrow (b)$ are trivial. If (a) or (b) holds then we can apply Theorem 3.3.1 to convince ourselves that $C_p(X,\mathbb{I})$ has a dense subspace Z that belongs to $\mathscr{P}$. Therefore Z separates the points of X and hence we can apply [Ar2, Proposition IV.3.3] to conclude that $C_p(X)$ belongs to $\mathscr{P}$. □

Corollary 3.3.13. *Suppose that X is a compact space and $\mathscr{P}$ is either K-analyticity or $\ell\Sigma \leq \kappa$, or even $dm \leq \kappa$. Then the following conditions are equivalent:*

(a) $C_p(X)$ *has a closure-preserving cover by subspaces that have* $\mathscr{P}$.

(b) $C_p(X,\mathbb{I})$ *has a closure-preserving cover by subspaces that have* $\mathscr{P}$.

(c) $C_p(X)$ *has* $\mathscr{P}$.

PROOF. Observe that K-analyticity, $\ell\Sigma \leq \kappa$ and $dm \leq \kappa$ are ω-perfect properties and apply Proposition 3.3.12. □

For $\kappa = \omega$ in Theorem 3.3.9 we see that it holds because for any compact space X, if $C_p(X)$ has a dense Lindelöf Σ-subspace then the whole $C_p(X)$ is Lindelöf Σ. This well-known fact (see [Ar2, Corollary IV.2.11]) suggests a very natural question: could we find some natural extension $\mathscr{C}$ of the class of compact spaces such that for every space X from $\mathscr{C}$ if $C_p(X)$ has a dense Lindelöf Σ-subspace then $C_p(X)$ itself is Lindelöf Σ? The following example (constructed by Okunev [Ok, Example 2.7] for other purposes) shows that we cannot extend the property in question even to the class of σ-compact spaces.

Example 3.3.14. *[GT, Example 3.8] There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but some σ-compact set Q is dense in $C_p(X)$.*

PROOF. Consider the σ-product $S = \{x \in \{0,1\}^{\omega_1} : |x^{-1}(1)| < \omega\}$ in the space $\{0,1\}^{\omega_1}$ and let $a(\alpha) = 1$ for any $\alpha < \omega_1$. The space $X = S \cup \{a\}$ is as promised. Observe first that S is well known to be σ-compact so X is σ-compact itself. For every $\alpha < \omega_1$ let $U_\alpha = \{x \in X : x(\alpha) = 1\}$.

If U is a clopen subset of X then χ_U is the characteristic function of U, i.e., $\chi_U(x) = 1$ for all $x \in U$ and $\chi_U(x) = 0$ if $x \notin U$. The space X is zero-dimensional and $\psi(a,X) = \omega$ so we can find a family $\{W_n : n \in \omega\}$ of clopen subsets of X such that $W_{n+1} \subset W_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} W_n = \{a\}$. Let $\mathscr{W} = \{W_n : n \in \omega\}$ and $\mathscr{U}_n = \{U_\alpha \setminus W_n : \alpha < \omega_1\}$ for each $n \in \omega$. We omit an easy verification that every $\mathscr{U}_n$ is a point-finite family of clopen subsets of X and the family $\mathscr{V} = \mathscr{W} \cup \bigcup_{n\in\omega} \mathscr{U}_n$ is T_0-separating in X, i.e., for any distinct $x, y \in X$ there exists $V \in \mathscr{V}$ such that $V \cap \{x,y\}$ is a singleton. Let $u(x) = 0$ for any $x \in X$; it is standard to see that $K_n = \{\chi_U : U \in \mathscr{U}_n\} \cup \{u\}$ is a compact subset of $C_p(X)$ for any $n \in \omega$. Therefore the set $P = \{\chi_V : V \in \mathscr{V}\} \cup \{u\}$ is σ-compact and separates the points of X.

Let Q be the algebra generated by the set $\mathscr{P}$. Then Q is σ-compact and dense in $C_p(X)$. Finally observe that $t(X) > \omega$ because $a \in \overline{S}$ but no countable subset of S contains a in its closure. Since $t(X) \leq l(C_p(X))$ for any space X (see [Ar2, Theorem I.4.1]), we can conclude that $C_p(X)$ is not Lindelöf. $\square$

Corollary 3.3.15. *[GT, Corollary 3.9] There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but there exists a closure-preserving cover of $C_p(X)$ by its σ-compact subspaces.*

3.4. More compact-like covers of function spaces

As we mentioned in section 3.1, decomposing $C_p(X)$ spaces by compact-like spaces imply strong restrictions on X if the decomposition is countable or closure preserving. In both cases it seems crucial the fact that $\mathbb{R}^\omega$ does no embed in $C_p(X)$ as a closed subspace. Since the minimum amount of compact spaces needed to cover $\mathbb{R}^\omega$ is $\mathfrak{d}$ we decided to address the corresponding question in this section.

The following theorem summarizes some known results that characterize function spaces by expressing them as a union of compact-like subspaces (see for example [Gue, Corollary 2.7]).

Theorem 3.4.1. *For a space X the following conditions are equivalent:*

(a) The space X is finite.

(b) The space $C_p(X)$ is σ-compact.

(c) The space $C_p(X)$ is σ-countably compact.

(d) The space $C_p(X) = \bigcup \mathscr{F}$ where $\mathscr{F}$ is a closure-preserving closed σ-countably compact family.

To generalize $((a) \Leftrightarrow (b))$ we will use the following fact that will be helpful later.

Lemma 3.4.2. *If X is a pseudocompact infinite space then there is a closed subspace of $C_p(X)$ that maps continuously onto ω^ω.*

PROOF. Since X is infinite by Lemma 3.1.6 we can find $f \in C_p(X)$ such that $Y = f(X)$ is an infinite compact subspace of $\mathbb{R}$. It follows that there is a countable infinite compact $Z \subset Y$. We can identify $C_p(Y)$ with a closed subspace of $C_p(X)$. Apply [Tk6, Problem 152] to see that the restriction map $\pi_Z : C_p(Y) \longrightarrow C_p(Z)$ is continuous. Since Z is compact and countable the space $C_p(Z)$ is analytic but not σ-compact; by [RJ, Theorem 3.5.3] we can deduce that it contains a closed subspace T homeomorphic to ω^ω. It follows that $\pi_Z^{-1}(T)$ is homeomorphic to a closed subspace of $C_p(X)$ that maps continuously onto ω^ω. □

Recall that $\mathfrak{d}$ is the minimum amount of compact sets needed to cover ω^ω or equivalently $\mathbb{R}^\omega$.

Theorem 3.4.3. *Let $\kappa < \mathfrak{d}$. For a space X the following conditions are equivalent:*

(a) The space X is finite.

(b) The space $C_p(X) = \bigcup_{\alpha<\kappa} K_\alpha$ where K_α is a compact subspace of $C_p(X)$ for every $\alpha < \kappa$.

(c) The space $C_p(X) = \bigcup_{\alpha<\kappa} K_\alpha$ where K_α is a countably compact subspace of $C_p(X)$ for every $\alpha < \kappa$.

PROOF. We will show that $(c) \Leftrightarrow (a)$. By [Gue, Lemma 2.6] it suffices to show that every continuous real image of X is finite. Let $f \in C_p(X)$ and suppose that $f(X)$ is infinite.

Condition (c) implies that $\mathbb{R}^\omega$ does not embed in $C_p(X)$ as a closed subspace which means that X is pseudocompact (see [Tk6, S 186, Fact 1]). Thus, by Lemma 3.4.2 the space $C_p(X)$ contains a closed subspace Z that maps continuously onto ω^ω. It is clear that Z can be covered by κ many countably compact subsets and so can ω^ω. This contradiction shows that $f(X)$ is finite and so is X. $\square$

In [Gue, Corollary 3.8 and Corollary 2.4] it is proved that if $C_p(X)$ is equal to the union of a closure-preserving (not necessarily closed) family of countably compact subspaces then X is finite. Whereas in [GT, Theorem 2.11] the authors show that if $C_p(X,\mathbb{I})$ admits a closure-preserving closed cover by σ-countably compact subspaces then $C_p(X,\mathbb{I})$ is countably compact. The following example shows that it is not possible to replace $C_p(X)$ by $C_p(X,\mathbb{I})$ in the statement of [GT, Theorem 2.11]. Furthermore this example evinces that it is essential to assume that the elements of the closure-preserving cover in [GT, Theorem 2.11] are closed.

Example 3.4.4. *There exists a space X such that $C_p(X,\mathbb{I})$ contains a countably compact dense subspace but $C_p(X,\mathbb{I})$ is not countably compact.*

PROOF. By [Tk6, S.480 Fact 2] there exists a space X such that

(a) X condenses onto a P-space.

(b) X condenses onto $\mathbb{R}$ and every open subset of X has cardinality $\mathfrak{c}$, in particular, X does not contain isolated points.

We will show that the space X is the one we are looking for. By condition (a) we can find a P-space Y for which it is possible to find a condensation $r : X \to Y$. From [Tk6, Problem 397] it follows that $C_p(Y,\mathbb{I})$ is countably compact. The image of $C_p(Y,\mathbb{I})$ under the dual map $r^* : C_p(Y) \to C_p(X)$ is a dense subspace of $C_p(X,\mathbb{I})$. Besides, by [Tk6, Problem 133] we can see that $D = r^*(C_p(Y,\mathbb{I})))$ is a countably compact subspace of $C_p(X,\mathbb{I})$.

To verify that $C_p(X,\mathbb{I})$ is not countably compact it suffices to show that X is not a P-space (see [Tk6, Problem 397]). Indeed, condition (b) implies that there is a condensation $t : X \to \mathbb{R}$, thus the set $\{x\} = t^{-1}(t(x))$ is G_δ for every $x \in X$. If X were a P-space, then everyone of its points would be isolated which cannot happen by condition (b). $\square$

Corollary 3.4.5. *There is a space X such that $C_p(X,\mathbb{I}) = \bigcup \mathscr{F}$ where $\mathscr{F}$ is a closure-preserving family and each $F \in \mathscr{F}$ is countably compact, however $C_p(X,\mathbb{I})$ is not countably compact.*

PROOF. The space X from Example 3.4.4 has the property that the space $C_p(X,\mathbb{I})$ is not countably compact and contains a countably compact dense subspace F. The family $\{F \cup \{f\} : f \in C_p(X,\mathbb{I})\}$ is a closure-preserving cover of $C_p(X,\mathbb{I})$ by countably compact subspaces of $C_p(X,\mathbb{I})$. □

In [GT, Problem 4.1] the authors ask if the presence of a closure-preserving closed cover by Lindelöf subspaces of $C_p(X)$ implies that $C_p(X)$ is Lindelöf. We still do not know if this is so. However, recalling that $C_p(X)$ is paracompact if and only if it is Lindelöf we can provide a partial answer to the problem in [GT] for the case of locally finite covers that consist of paracompact subspaces of $C_p(X)$.

Proposition 3.4.6. *Suppose that $\mathscr{P}$ is a property preserved by subsets of type F_σ. If $C_p(X) = \bigcup \mathscr{F}$ and $\mathscr{F}$ is locally finite and each $F \in \mathscr{F}$ has $\mathscr{P}$ then $C_p(X)$ is the union of finitely many subspaces with $\mathscr{P}$.*

PROOF. It follows easily from the fact that $C_p(X)$ embeds as an F_σ subset of any of its non-empty open subspaces. □

Corollary 3.4.7. *Let $\mathbb{F} = \{$realcompleteness, monolithicity, paracompactness$\}$. If a property $\mathscr{P}$ is in the list $\mathbb{F}$ and $C_p(X)$ has a locally finite closed cover $\mathscr{C}$ such that $C \in \mathscr{C}$ has $\mathscr{P}$ then $C_p(X)$ also has $\mathscr{P}$.*

PROOF. Is an immediate consequence of Proposition 3.4.6 and Theorems 2.2, 2.5 and 2.9 of [Ca]. □

The most important problem yet unsolved in [Gue] and [GT] is to determine whether the space $C_p([0,1])$ can be covered by a closure-preserving family of second countable subspaces. It has already been shown that if $C_p(X)$ has a closure preserving cover $\mathscr{F}$ the space $C_p(X,[0,1])$ is contained in $\overline{F}$ for some $F \in \mathscr{F}$. Considering the case when $C_p(X,[-1,1])$ is contained in the closure of a second countable space $M \subset C_p(X)$ we have the following:

Proposition 3.4.8. *Suppose $C_p(X,[-1,1])$ is contained in the closure of a second countable space $M \subset C_p(X)$ and $M \cup \{f\}$ has a countable local base at f, then X is countable.*

PROOF. Take a countable local base $\mathscr{B} = \{U_n : n \in \omega\}$ of $M' = M \cup \{f\}$ at f. For each $U_n \in \mathscr{B}$ there is $V_n = ([(x_1,...,x_m),O_1,...O_m]) \cap M'$ such that $f \in V_n \subset U_n$. Let $A = \bigcup\{Supp(V_n) : n \in \omega\}$. Suppose there is $x \in X \setminus A$. We can find $O_x \in \tau(f(x),(-1,1))$ with the property that $\emptyset \neq O = (-1,1) \setminus cl(O_x)$. The function $f \in [x,O_x] \cap M'$ hence there is $n \in \omega$ such that $f \in V_n \subset [x,O_x] \cap M'$ with $V_n = ([(y_1,...,y_k),O_1,...O_k]) \cap M'$. Let $U = [(y_1,...,y_k,x),O_1,...O_k,O]$ and take $g \in C_p(X,,(-1,1))$ such that $g(y_i) = f(y_i)$ for $i = 1,...,k$ and such that $g(x) \in O$. The set $C_p(X,,(-1,1)) \cap U$ is not empty so there exists $h \in M \cap (C_p(X) \cap U)$ which implies $h(y_i) \in O_i$ for $i = 1,...,k$ hence $h \in V_n$. On the other hand $h(x) \in O$ implies $h \notin [x,O_x]$ a contradiction. $\square$

3.5. Topological games on $C_p(X)$ and $C_p(X,\mathbb{I})$

If a space Z has a compact closure-preserving cover then a topological game on Z can be defined in a natural manner; in this game the first player has a winning strategy. Therefore studying analogous games in function spaces gives a possibility to strengthen some results of the previous section. The following game is a slight variation of the one presented by R. Telgarsky in [Te]. It is worth to mention that studying properties of function spaces by means of topological games is a procedure that has already proven fruitful as shown in [NG]

Definition 3.5.1. *On a Tychonoff space Y, consider a family $\mathscr{C} \subset exp(Y)$. We define the game $\mathscr{G}(\mathscr{C},Y)$ of two players I and II who take turns in the following way: at the move number n, Player I chooses $C_n \in \mathscr{C}$ and Player II chooses a set $U_n \in \tau(C_n,Y)$. The game ends after the n-th move of each player has been made for every $n \in \omega$ and Player I wins if $X = \bigcup\{U_n : n \in \omega\}$; otherwise the winner is Player II.*

Definition 3.5.2. *A strategy t for the first Player in the game $\mathscr{G}(\mathscr{C},Y)$ on a space X is defined inductively in the following way. First the set $t(\varnothing) = F_0 \in \mathscr{C}$ is chosen. An open set $U_0 \in \tau(X)$ is legal if $F_0 \subset U_0$. For every legal set U_0 the set $t(U_0) = F_1 \in \mathscr{C}$ has to* *be defined. Let us assume that legal sequences* $(U_0,...,U_i)$ *and sets* $t(U_0,...U_i)$ *have been defined for each* $i \leq n$. *The sequence* $(U_0,...U_{n+1})$ *is legal if so is the sequence* $(U_0,...,U_i)$ *for each* $i \leq n$ *and* $F_{n+1} = t(U_0,...,U_n) \subset U_{n+1}$. *A strategy* t *for Player I is winning if it ensures victory for I in every play it is used.*

Definition 3.5.3. *A strategy* s *for Player II in the game* $\mathscr{G}(\mathscr{C},Y)$ *on a space* X *is simply a function that assigns to every finite sequence* $(F_0,\dots,F_n)$ *of elements of* $\mathscr{C}$ *an open set* $U \in \tau(F_n,X)$. *Such a strategy for Player II is winning if it ensures victory for II in every play it is used.*

Theorem 3.5.4. *[Gue, Theorem 3.4] Given a non-empty space X, if* $Y = C_p(X,\mathbb{I})$ *and the family* $\mathscr{F} = \{F \subset Y : F \text{ is nowhere dense in } C_u(X,\mathbb{I})\}$, *then Player II has a winning strategy in the game* $\mathscr{G}(\mathscr{F},Y)$.

PROOF. Recall that if $f \in C(X,\mathbb{I})$ and $\varepsilon \geq 0$ then the set $I(f,\varepsilon) = \{g \in Y : |g(x) - f(x)| \leq \varepsilon$ for all $x \in X\}$ is closed in the space Y. Define inductively a winning strategy s for Player II in the game $\mathscr{G}(\mathscr{F},Y)$ on Y in the following way: let $F_0 \in \mathscr{F}$ be the first move of Player I. If $B_0 \in \tau^*(C_u(X,\mathbb{I}))$ is an open ball of radius 1 in Y, then $\varnothing \neq (B_0 \setminus F_0) \in \tau^*(C_u(X,\mathbb{I}))$ and therefore there is a point $f_0 \in B_0 \setminus F_0$ and a positive real number $\varepsilon_0 < 1$ such that $I(f_0,\varepsilon_0) \subset B_0 \setminus F_0$; then $F_0 \subset (Y \setminus I(f_0,\varepsilon_0)) \in \tau^*(C_p(X,\mathbb{I}))$. Consequently, we can define $U_0 = s(F_0) = Y \setminus I(f_0,\varepsilon_0)$ as the first choice of Player II.

Assume that for each $i \leq j < n$ and every legal finite sequence $(F_0,\dots,F_j)$ of elements of the family $\mathscr{F}$ selected by Player I, we have defined the set $U_i = s(F_0,\dots,F_i)$ and the open ball B_i of radius at most 2^{-i}, together with a positive real number $\varepsilon_i < 2^{-i}$ as well as a function $f_i \in Y \setminus F_i$ with the following properties. If we fix $j < n$, then for all $k < i \leq j$ we have $I(f_i,\varepsilon_i) \subset B_i \subset I(f_k,\varepsilon_k) \subset B_k$ and $U_i = Y \setminus I(f_i,\varepsilon_i)$.

Let F_n be the n-th move of Player I. As above, if $B_n \in \tau^*(C_u(X,\mathbb{I}))$ is an open ball of radius not greater than 2^{-n} contained in $I(f_{n-1},\varepsilon_{n-1})$ then $\varnothing \neq B_n \setminus F_n \in \tau^*(C_u(X,\mathbb{I}))$, and hence, we can find a point $f_n \in B_n \setminus F_n$ and a positive real number $\varepsilon_n < 2^{-n}$ such that $I(f_n,\varepsilon_n) \subset B_n \setminus F_n$. The set F_n is contained in $Y \setminus I(f_n,\varepsilon_n)$ so we can take $U_n = s(F_n) = Y \setminus I(f_n,\varepsilon_n)$ to be the n-th move of Player II.

The definition of the strategy s is complete, let us convince ourselves that it is a winning one. Let $P = \{(F_n,U_n) : n \in \omega\}$ be a play in which Player II applies the strategy s. By definition of s we have the equality $U_n = s(F_n) = Y \setminus I(f_n,\varepsilon_n)$ where $\varepsilon_n < 2^{-n}$. The family $\{I(f_n,\varepsilon_n) : n \in \omega\}$ is a decreasing sequence of closed non-empty subsets of the complete metric space $C_u(X,\mathbb{I})$, and the corresponding sequence of diameters converges to zero. This means that $\bigcap\{I(f_n,\varepsilon_n) : n \in \omega\} \neq \varnothing$ and therefore $\bigcup\{U_n : n \in \omega\} \neq Y$. This shows that Player *II* wins whenever she (or he) applies the strategy s. □

Remark 3.5.5. *[Gue, Remark 3.5] It is possible to reformulate Theorem 3.5.4 for the set $Y = C_p(X)$ and the family $\mathscr{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$, applying the same method to prove that Player II has a winning strategy in the game $\mathscr{G}(\mathscr{F},Y)$.*

Remark 3.5.6. *[Gue, Remark 3.6] Given a space X consider the set $Y = C_p(X,\mathbb{I})$ (or $Y = C_p(X)$), and let $\mathscr{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X,\mathbb{I})\}$ (or $\mathscr{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$). If $\mathscr{C}$ is a family of non-empty closed subsets of Y for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{C},Y)$ then $\mathscr{C} \nsubseteq \mathscr{F}$.*

Lemma 3.5.7. *[Gue, Lemma 3.7] Let Y be a space, and define $\mathscr{C}$ to be the family of all non-void closed locally compact subspaces of Y; let $\mathscr{C}'$ be the family of all non-empty closed discrete unions of compact subspaces of Y. If there exists a closure-preserving compact cover $\mathscr{F}$ of the space Y, then Player I has a winning strategy in the games $\mathscr{G}(\mathscr{C},Y)$ and $\mathscr{G}(\mathscr{C}',Y)$.*

PROOF. For each $y \in Y$ let $K(y) = Y \setminus \bigcup\{F \in \mathscr{F} : y \notin F\}$. It is not difficult to verify that $K(y)$ is an open set which contains y, and if $x \in K(y)$, then $K(x) \subset K(y)$. Call a point $m \in Y$ maximal if $K(m)$ is not properly contained in the set $K(y)$ for any $y \in Y \setminus \{m\}$. Potoczny showed in [Po] that if $M(Y)$ is the set of all the maximal elements of Y then $M(Y)$ is a discrete union of compact subspaces of Y, and therefore $M(Y) \in \mathscr{C}' \subset \mathscr{C}$. Moreover in [PJ] it is established that if $\{U_n : n \in \omega\}$ is a family of open subsets of Y such that $M(Y) \subset U_0$, and for each $n \in \omega$ we have $M(Y \setminus \bigcup\{U_i : i = 0,\dots,n\}) \subset U_{n+1}$; then $Y = \bigcup\{U_n : n \in \omega\}$. Now apply results [Te, Theorem 10.1] and [Te, Theorem 10.2] to see that Player *I* has a winning strategy in the game $\mathscr{G}(\mathscr{C},Y)$. □

Corollary 3.5.8. *[Gue, Corollary 3.8] For every space X, if the space $Y = C_p(X,\mathbb{I})$ is covered by a closure-preserving family of compact subspaces, then X is discrete.*

PROOF. Let $\mathscr{C}$ be the family of all non-empty closed discrete unions of compact subspaces of Y. By Lemma 3.5.7, Player *I* has a winning strategy in the game $\mathscr{G}(\mathscr{C},Y)$. By Remark

3.5.6, not all members of $\mathscr{C}$ are nowhere dense in $C_u(X,\mathbb{I})$. Therefore there exists $C\in\mathscr{C}$ that contains an open ball $B(f,2r)\in\tau(C_u(X,\mathbb{I}))$ and hence $I(f,r)=\{g\in C(X,\mathbb{I}):|g(x)-f(x)|\le r$ for all $x\in X\}\subset C$. It is easy to see using connectedness of $I(f,r)$ that $I(f,r)$ is compact; therefore $C_p(X,\mathbb{I})$ is also compact being homeomorphic to $I(f,r)$ and hence X is discrete. $\square$

Corollary 3.5.9. *[Gue, Corollary 3.9] For every space X, let $Y=C_p(X,\mathbb{I})$ and let $\mathscr{C}$ be the family of all σ-compact subspaces of Y. If Player I has a winning strategy on Y for the game $\mathscr{G}(\mathscr{C},Y)$, then the space X is discrete.*

PROOF. Remark 3.5.6 states that not every element of $\mathscr{C}$ is nowhere dense in $C_u(X,\mathbb{I})$. Therefore, there is a σ-compact subspace F of Y such that F contains a closed subspace homeomorphic to $C_p(X,\mathbb{I})$. Thus $C_p(X,\mathbb{I})$ is σ-compact by [Te, Theorem 1.5.2] and hence X is discrete. $\square$

Lemma 3.5.10. *[Gue, Lemma 3.10] If $\mathscr{C}$ is the family of all σ-compact subspaces of $Y=\omega^\omega$, then Player II has a winning strategy for the game $\mathscr{G}(\mathscr{C},Y)$.*

PROOF. Suppose $(F_0,\dots,F_n)$ is any sequence of σ-compact subspaces of Y. For the set F_n it is possible to find a countable family $\{K^n_m : m\in\omega\}$ of compact subsets of Y such that $F_n=\bigcup_{m\in\omega}K^n_m$. For each $m\in\omega$, let $\pi_m : Y\to\omega$ be the natural projection from Y to the factor determined by m.

Define the finite set $C^m_n=\bigcup_{i=0}^{n}\pi_n(K^n_i)$, let $b^n_n=\sup C^n_n+1$. For each $m>n$ let $b^n_m=\sup\pi_m(K^n_m)+1$. For every $m\ge n$ let $V^n_m=\{0,\dots,b^n_m-1\}\times\omega^{\omega\setminus\{m\}}$. Observe that $K^n_i\subset V^n_n$ whenever $i\le n$ and $K^n_j\subset V^n_j$ for all $j>n$. Also if $y\in V^n_m$ then $y(m)<b^n_m$ for all $m\ge n$. Now, let $U_n=\bigcup_{m\ge n}V^n_m$. Define $t((F_0,\dots,F_n))=U_n$.

To see that t is a winning strategy, take a play $\mathscr{P}=\{(F_n,U_n)\}_{n\in\omega}$ in which Player II uses t. We will find a point in $Y\setminus\bigcup_{n\in\omega}U_n$. For every $n\in\omega$ there exists $x_n\in Y$ such that $x_n(m)=b^n_m$ for every $m\ge n$. Observe that $x_n\notin U_n$. Let $y_0(0)=x_0(0)+1$. Now, once the value $y(n)$ has been determined, let $y(n+1)=y(n)+\sum_0^{n+1}x_i(n+1)$. The point y has the property that for every $n \in \omega$, if $m \geq n$ then $y(m) > x_n(m)$ which implies that $y \notin U_n$. It has been verified that $Y \neq \bigcup_{n\in\omega} U_n$ and therefore Player *II* wins the play $\mathscr{P}$ proving that t is a winning strategy. $\square$

Note that, since the elements of $\mathscr{C}$ in Lemma 3.5.10 are not necessarily closed, such result does not follow from [Te].

Corollary 3.5.11. *[Gue, Corollary 3.11] If X is a non-σ-compact analytic space and $\mathscr{C}$ is the family of all σ-compact subsets of X, then Player II has a winning strategy in the game $\mathscr{G}(\mathscr{C},X)$.*

PROOF. Apply [RJ, Theorem 3.5.3] to see that X has a closed subspace Y homeomorphic to ω^ω. By Lemma 3.5.10 Player *II* has a winning strategy for the game $\mathscr{G}(\mathscr{C},Y)$. Finally apply [Te, Theorem 2.5] to conclude that Player *II* has a winning strategy in the game $\mathscr{G}(\mathscr{C},X)$. $\square$

Corollary 3.5.12. *[Gue, Corollary 3.12] For an arbitrary space X, let $\mathscr{C}$ be the family of all σ-compact closed subspaces of $C_p(X)$. If Player I has a winning strategy on $C_p(X)$ for the game $\mathscr{G}(\mathscr{C},C_p(X))$, then the space X is finite.*

PROOF. It is easy to see that if Y is closed in $C_p(X)$ then Player I has a winning strategy for the corresponding game on Y. Therefore if $\mathscr{C}'$ is the family of all σ-compact subspaces of $C_p(X,\mathbb{I})$ then Player I has a winning strategy for game $\mathscr{G}(\mathscr{C}',C_p(X,\mathbb{I}))$. Now apply Corollary 3.5.9 to verify that X is discrete. By Corollary 3.5.11 the space $\mathbb{R}^\omega$ does not embed into $C_p(X)$ as a closed subspace. Thus X is pseudocompact and hence finite. $\square$

We will continue to characterize function spaces by means of existing winning strategies for some topological games. In Corollary 3.5.12 it is proved that if $\mathscr{C}$ is the family of all the closed σ-compact subspaces of $C_p(X)$ and Player I has a winning strategy for the game $\mathscr{G}(\mathscr{C},C_p(X))$ then X is finite. In fact we can skip the assumption that the elements of $\mathscr{F}$ are closed in $C_p(X)$.

Theorem 3.5.13. *For a space X, let $\mathscr{C}$ be the family of all σ-compact (not necessarily closed) subspaces of $C_p(X)$. If Player I has a winning strategy on $C_p(X)$ for the game $\mathscr{G}(\mathscr{C},C_p(X))$, then the space X is finite.*

PROOF. By Lemma 3.1.6 it suffices to show that every continuous real image of X is finite. First let us observe that X is pseudocompact. Indeed, let $\mathscr{C}'$ be the family all σ-compact subspaces of ω^ω, if X is not pseudocompact then $\mathbb{R}^\omega$ embeds as a closed subspace of $C_p(X)$ and so does ω^ω. It is evident that this implies that there is a winning strategy for Player I in the game $\mathscr{G}(\mathscr{C}', \omega^\omega)$. This is a contradiction with Lemma 3.5.10. □

Corollary 3.5.14. *If X is a non-σ-compact analytic space and $\mathscr{C}$ is the family of all σ-compact subsets of X, then Player II has a winning strategy in the game $\mathscr{G}(\mathscr{C},X)$.*

PROOF. Apply [RJ, Theorem 3.5.3] to see that X has a closed subspace Y homeomorphic to ω^ω. By Lemma 3.5.10 Player *II* has a winning strategy for the game $\mathscr{G}(\mathscr{C},Y)$. Finally apply [Te, Theorem 2.5] to conclude that Player *II* has a winning strategy in the game $\mathscr{G}(\mathscr{C},X)$. □

Corollary 3.5.15. *For an arbitrary space X, let $\mathscr{C}$ be the family of all σ-compact subspaces of $C_p(X)$. If Player I has a winning strategy on $C_p(X)$ for the game $\mathscr{G}(\mathscr{C},C_p(X))$, then the space X is finite.*

PROOF. It is easy to see that if Y is closed in $C_p(X)$ then Player I has a winning strategy for the corresponding game on Y. Therefore if $\mathscr{C}'$ is the family of all σ-compact subspaces of $C_p(X,\mathbb{I})$ then Player I has a winning strategy for game $\mathscr{G}(\mathscr{C}',C_p(X,\mathbb{I}))$. Now apply Corollary 3.5.8 to verify that X is discrete. By Corollary 3.5.12 the space $\mathbb{R}^\omega$ does not embed into $C_p(X)$ as a closed subspace. Thus X is pseudocompact and hence finite. □

It is standard to verify that a space X is Lindelöf if and only if Player *I* has a winning strategy for the game $\mathscr{G}(\mathscr{L},X)$ where $\mathscr{F}$ is the family of all the Lindelöf not necessarily closed subspaces of X. Is it possible to characterize other topological properties of function spaces in an analogous way?

In Remark 3.5.6 it is established that if X is non-empty and $\mathscr{F} \subset \exp(C_p(X,\mathbb{I}))$ and player *I* has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ then there is $F \in \mathscr{F}$ that is not nowhere dense in $C_u(X,\mathbb{I})$. An analogous fact is also established for $C_p(X)$.

Proposition 3.5.16. *For a non-empty space X, if $\mathscr{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ and player I has a winning strategy for the game $\mathscr{G}(\mathscr{C},C_p(X)$ or the game $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ then there exists $C\in\mathscr{C}$ such that $U\subset C$ for some non-empty open subset U of the space $C_u(X)$.*

PROOF. Indeed, $\mathscr{C}$ is also a closed family of subsets of $C_u(X)$ or $C_u(X,\mathbb{I})$ so by Remark 3.5.6 the interior of some $C\in\mathscr{C}$ in $C_u(X)$ must be non-empty. □

Corollary 3.5.17. *For a non-empty X, if $\mathscr{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ and player I has a winning strategy for the game $\mathscr{G}(\mathscr{C},C_p(X)$ or $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ then there exist $C\in\mathscr{C}$ and $f\in\mathscr{C}$ such that $I(f,\varepsilon)\subset\mathscr{C}$ for some $\varepsilon>0$.*

PROOF. Apply [GT, Proposition 2.1] and Proposition 3.5.16. □

Corollary 3.5.18. *If X is a space and $\mathscr{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ such that every $C\in\mathscr{C}$ has $\mathscr{P}$. If player I has a winning strategy in the game $\mathscr{G}(\mathscr{C},C_p(X))$ or $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ then some $C\in\mathscr{C}$ contains a homeomorphic copy of $C_p(X)$.*

PROOF. Apply Corollary 3.5.17 and [GT, Proposition 2.1] □

Corollary 3.5.19. *Suppose that $\mathscr{P}$ is a hereditary topological property and if $\mathscr{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ such that every $C\in\mathscr{C}$ has $\mathscr{P}$. If player I has a winning strategy in the game $\mathscr{G}(\mathscr{C},C_p(X)$ or $\mathscr{G}(\mathscr{C},C_p(X,\mathbb{I}))$ then $C_p(X)$ also has $\mathscr{P}$.*

PROOF. By Corollary 3.5.17, there exists $C\in\mathscr{C}$ such that some $I\subset C$ is homeomorphic to $C_p(X)$; since C has $\mathscr{P}$, the space I and hence $C_p(X)$ must have $\mathscr{P}$. □

Remark 3.5.20. *Suppose that κ is an infinite cardinal. Notice that Corollary 3.5.19 applies, for instance, to the following properties: weight $\leq\kappa$, network weight $\leq\kappa$, i-weight $\leq\kappa$, diagonal number $\leq\kappa$, character $\leq\kappa$, pseudocharacter $\leq\kappa$, tightness $\leq\kappa$, spread $\leq\kappa$, hereditary Lindelöf number $\leq\kappa$, hereditary density $\leq\kappa$, κ-monolithicity, metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property, being perfect, being functionally perfect.*

In [Tk3, Example 15] it is proved that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that this implies that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{F}, C_p(K)$ where $\mathscr{F}$ is the family of all the closed subspaces of $C_p(K)$ with countable π-weight. We can conclude that if a property $\mathscr{P}$ is not hereditary and $\mathscr{F}$ is the family of all the subspaces of $C_p(X)$ that have $\mathscr{P}$, if Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F}, C_p(X)$ then $C_p(X)$ does not necessarily have $\mathscr{P}$.

Nevertheless, for properties that are inherited by closed subspaces we can proceed in a similar way as in [GT] and use Corollary 3.5.17 to observe the following.

Remark 3.5.21. *Given a non-empty space X and a closed-hereditary property $\mathscr{P}$, call $\mathscr{F}$ the family of all the closed subspaces of $C_p(X,\mathbb{I})$ that have $\mathscr{P}$. If Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F}, C_p(X,\mathbb{I}))$ then $C_p(X,\mathbb{I})$ also has the property $\mathscr{P}$. We can name some of these properties: extent $\leq \kappa$, Nagami number $\leq \kappa$, K-analyticity, $\ell\Sigma \leq \kappa$, dm $\leq \kappa$, mg $\leq \kappa$, mi $\leq \kappa$, normality, sequentiality.*

Again following the arguments presented in Section 3 of [GT] we notice that for some properties we can say even more.

Remark 3.5.22. *If $\mathscr{F}$ is a closed family of subsets of $C_p(X,\mathbb{I})$ for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X,\mathbb{I}))$ and every $F \in \mathscr{F}$ is realcompact then $C_p(X)$ is realcompact. If it is the case that every $F \in \mathscr{F}$ is Čech-complete subspaces, then X is discrete. Given a space X, if it happens that every $\mathscr{F}$ is integrated by σ-countably compact subspaces, then $C_p(X,\mathbb{I})$ is countably compact. Whereas if the elements of $\mathscr{F}$ are σ-compact then X is discrete.*

We will now proceed in the same spirit of Section 3 of [GT].

Theorem 3.5.23. *Given a space X and a property $\mathscr{P}$ that is preserved by quotient images, if $\mathscr{F}$ is a closed family of subsets of $C_p(X,\mathbb{I})$ for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X,\mathbb{I}))$ and every $F \in \mathscr{F}$ has $\mathscr{P}$ then $C_p(X,\mathbb{I})$ also has the property $\mathscr{P}$.*

Remark 3.5.24. *Suppose that κ is an infinite cardinal. Theorem 3.5.23 applies to properties such as weak functional tightness $\leq \kappa$, functional tightness $\leq \kappa$.*

Theorem 3.5.23 applies also to the property of κ-stability, but in this case not only $C_p(X,\mathbb{I})$ is κ-stable but the whole $C_p(X)$ is.

Remark 3.5.25. *Indeed, since $C_p(X,\mathbb{I})$ is κ-stable then $C_p(C_p(X,\mathbb{I}))$ is κ-monolithic by [Ar2, Theorem II.6.8] the space X embeds in $C_p(C_p(X,\mathbb{I}))$, thus X is also κ-monolithic so $C_p(X)$ is κ-stable by [Ar2, Theorem II.6.9].*

Theorem 3.5.26. *Given a space X and a topological property $\mathscr{P}$ that is invariant under continuous images, if $\mathscr{F}$ is a family of subsets (not necessarily closed) of either $C_p(X)$ or $C_p(X,\mathbb{I})$ for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F},C_p(X))$ or $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ and every $F\in\mathscr{F}$ has $\mathscr{P}$ then $C_p(X,\mathbb{I})$ contains a dense subspace that has $\mathscr{P}$.*

PROOF. Apply Corollary 3.5.17 to find a function $f\in C_p(X,\mathbb{I})$ and $\varepsilon>0$ such that $I(f,\varepsilon)\subset\overline{C}$ for some $C\in\mathscr{C}$. The set $R=I(f,\varepsilon)$ is a retract of $C_p(X)$ homeomorphic to $C_p(X,\mathbb{I})$. Consequently, there exists a retraction $r:\overline{C}\to R$. The set $r(C)$ is dense in $r(\overline{C})=R$ and has the property $\mathscr{P}$. Since R is homeomorphic to $C_p(X,\mathbb{I})$, the latter also has a dense subspace with the property $\mathscr{P}$. $\square$

Theorem 3.5.27. *Suppose that X is a space and $\mathscr{P}$ is a σ-additive topological property if $\mathscr{F}$ is a family of subsets (not necessarily closed) of either $C_p(X)$ or $C_p(X,\mathbb{I})$ for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F},C_p(X))$ or $\mathscr{G}(\mathscr{F},C_p(X,\mathbb{I}))$ and every $F\in\mathscr{F}$ has $\mathscr{P}$ then $C_p(X)$ contains a dense subspace that has $\mathscr{P}$.*

PROOF. Apply Theorem 3.5.26 to deduce that $C_p(X,\mathbb{I})$ has $\mathscr{P}$. Since the property $\mathscr{P}$ is σ-additive the space $\bigcup\limits_{n\in\mathbb{N}} C_p(X,[-n,n])$ has $\mathscr{P}$ and is dense in $C_p(X)$. $\square$

Remark 3.5.28. *For an infinite cardinal κ Theorem 3.5.27 applies to the following properties: network weight $\le\kappa$, spread $\le\kappa$, hereditary density $\le\kappa$. Furthermore when applying Theorem 3.5.27 to k-separability, caliber κ, point-finite cellularity $\le\kappa$, or density $\le\kappa$ then it is possible to ensure the presence of the corresponding property in $C_p(X)$ and in $C_p(X,\mathbb{I})$.*

In the case of pseudocompactness it is not possible to obtain this property for a non-empty $C_p(X)$ yet we obtain σ-pseudocompactness of $C_p(X)$ and $C_p(X,\mathbb{I})$ is pseudocompact.

Remark 3.5.29. *Assume $\mathscr{F}$ is a family of pseudocompact subspaces of $C_p(X)$ and Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X))$ then $C_p(X)$ is σ-pseudocompact and $C_p(X,\mathbb{I})$ is pseudocompact.*

PROOF. Since the closure of every element of $\mathscr{F}$ is pseudocompact we do not lose generality if we consider that $\mathscr{F}$ is a closed family. If the space X is not pseudocompact, then there is a retraction $C_p(X) \to \mathbb{R}^\omega$. Let $\mathscr{C} = \{r(F) : F \in \mathscr{F}\}$, it is standard to verify that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{C},\mathbb{R}^\omega)$. This is a contradiction with Corollary 3.5.14 which shows that X is pseudocompact. Apply now Theorem 3.5.26 to conclude that $C_p(X,\mathbb{I})$ has a dense pseudocompact subspace and therefore it is pseudocompact and therefore $C_p(X)$ is σ-pseudocompact. □

This method of studying topological games in function spaces can also provide characterizations of important classes of compact spaces.

Corollary 3.5.30. *If K is a compact space and $\mathscr{F}$ is the class of σ-compact spaces then the following conditions are equivalent:*

(a) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(K))$.

(b) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(K,\mathbb{I}))$.

(c) X is Eberlein compact.

PROOF. Recall (see e.g. [Ar2, Theorem IV.1.7]) that X is Eberlein compact if and only if $C_p(X)$ (or equivalently, $C_p(X,\mathbb{I})$) has a dense σ-compact subspace and apply Theorem 3.5.26. □

Arhangel'skii [Ar2, Section IV.2] defined ω-perfect classes $\mathscr{P}$ as in Chapter 1, Definition 1.4.4. It turns out as in the previous sections, that ω-perfect classes are also relevant to the topic of this section.

Proposition 3.5.31. *If K is a compact space and $\mathscr{P}$ is an ω-perfect class then the following conditions are equivalent:*

(a) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{P}, C_p(K))$.

(b) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{P}, C_p(K, \mathbb{I}))$.

(c) $C_p(X)$ belongs to $\mathscr{P}$.

PROOF. The implications $(c) \Longrightarrow (a)$ and $(c) \Longrightarrow (b)$ are trivial. If (a) or (b) holds then we can apply Theorem 3.5.26 to convince ourselves that $C_p(X, \mathbb{I})$ has a dense subspace Z that belongs to $\mathscr{P}$. Therefore Z separates the points of X and hence we can apply [Ar2, Proposition IV.3.3] to conclude that $C_p(X)$ belongs to $\mathscr{P}$. □

Corollary 3.5.32. *Suppose that X is a compact space and $\mathscr{P}$ is either K-analyticity or $\ell\Sigma \leq \kappa$, or even $dm \leq \kappa$. Then the following conditions are equivalent:*

(a) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{P}, C_p(K))$.

(b) Player I has a winning strategy for the game $\mathscr{G}(\mathscr{P}, C_p(K, \mathbb{I}))$.

(c) $C_p(X)$ belongs to $\mathscr{P}$.

PROOF. Observe that K-analyticity, $\ell\Sigma \leq \kappa$, and $dm \leq \kappa$, are ω-perfect properties and apply Proposition 3.5.31. □

Remark 3.5.33. *Suppose that K is a compact space and $\mathscr{F}$ is a family of subsets of $C_p(K, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(K, \mathbb{I}))$. If the elements of $\mathscr{F}$ are K-analytic then K is a Talagrand compact space whereas if every $F \in \mathscr{F}$ is Lindelöf Σ then K is Gul'ko compact.*

In the rest of this section we will consider the situation when Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$. We conjecture that in this case the space $C_p(X)$ has to be Lindelöf Σ and we present several results to support our conjecture and generalize some well-known theorems on properties of spaces X such that $C_p(X)$ is Lindelöf Σ.

Proposition 3.5.34. *If $dm(X) \leq \omega$ and Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F},C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then $C_p(X)$ is a Lindelöf Σ-space.*

PROOF. From Theorem 3.5.23 we obtain that $C_p(X,\mathbb{I})$ is Lindelöf Σ, since $dm(X) \leq \omega$ we can apply [COT, Proposition 2.14] to conclude that $C_p(X)$ Lindelöf Σ framed and therefore υX is Lindelöf Σ by [Ok, Theorem 3.6]. The space $C_p(\upsilon X,\mathbb{I}) = \pi^{-1}C_p(X,\mathbb{I})$ is Lindelöf Σ and therefore we can deduce $C_p(\upsilon X)$ is Lindelöf Σ by [Ok, Theorem 3.6], hence $C_p(X)$ is a Lindelöf Σ space for it is a continuous image of $C_p(\upsilon X)$. $\square$

Proposition 3.5.35. *Every Lindelöf space X with at most one isolated point is ω-stable.*

PROOF. If X is discrete then it is countable and there is nothing more to prove. If X has a non-isolated point a then for every continuous onto map $f : X \longrightarrow Y$ we have $|Y \setminus U| \leq \omega$ for every $U \in \tau(f(a),Y)$. Therefore if $\psi(Y) \leq \omega$ then Y is countable. Now if Z is a continuous image of X that condenses onto a second countable space M then M is a continuous image of X. It follows that M is countable and therefore so is Z. This shows that X is ω-stable. $\square$

Corollary 3.5.36. *If X is a space with at most one non-isolated point and Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F},C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then $C_p(X)$ is ω-monolithic.*

Theorem 3.5.37. *If X is a scattered space and Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F},C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then $C_p(X)$ is ω-monolithic.*

PROOF. Take any countable set $A \subset C_p(X)$ and define the map $\varphi : X \to C_p(A)$ by the formula $\varphi(x)(f) = f(x)$ for each $f \in A$; let $Y = \varphi(X)$. Observe that we have the inequalities $w(Y) \leq w(C_p(A)) \leq \omega$. It is standard to see that $A \subset \varphi^*(C_p(Y))$.

There exists a space Z and continuous maps $\varphi' : X \to Z$ and $\xi : Z \to Y$ such that φ' is $\mathbb{R}$-quotient, ξ is injective and $\varphi = \xi \circ \varphi'$. The set $E = (\varphi')^*(C_p(Y))$ is closed in $C_p(X)$ and $A \subset \varphi^*(C_p(Y)) \subset E$ which shows that $\overline{A} \subset E$. The map ξ is a condensation of Z onto the second countable space Y; as a consequence, $d(C_p(Z)) = iw(Z) \leq w(Y) \leq \omega$; also since Z is a continuous image of X we have that Z is scattered. It is standard to verify that every second countable image of Z is countable and therefore so is Z. It follows that $C_p(Z)$ is a cosmic space as well as E. As a consequence, $\overline{A} = cl_E(A)$ is a cosmic space. $\square$

Corollary 3.5.38. *If ω_1 is a caliber of a space X and Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then X is cosmic.*

PROOF. Let $\mathscr{L}$ be the family of all the Lindelöf Σ closed subspaces of the space $C_p(X)$. Since ω_1 is a caliber of X, the space $C_p(X)$ has a small diagonal according to [Tk2, Theorem 1] and so does each element of $\mathscr{L}$. Therefore if $L \in \mathscr{L}$ and K is a compact subspace of L then K has a small diagonal. As a consequence, the space K cannot contain convergent ω_1-sequences and so we have $t(K) \leq \omega$ by [JuS, Theorem 1.2] and [Ju, 3.12]. Apply Theorem 3.5.37 to verify that K is ω-monolithic; it is standard to see that any ω-monolithic compact space of countable tightness with a small diagonal is metrizable so K has to be metrizable.

Thus each $L \in \mathscr{L}$ has a small diagonal and a cover by compact metrizable subspaces for which there exists a countable network, so Theorem 2.1 of [Gr] allows us to conclude that the elements of $\mathscr{L}$ are cosmic. It follows from Corollary 3.5.17 that $C_p(X, [0,1])$ and X are cosmic. $\square$

The following is a generalization of the result by Arhangel'skii [Ar3, Theorem 10] which states that for any space X such that $C_p(X)$ is Lindelöf Σ, if $s(C_p(X)) \leq \omega$ then X is cosmic.

Corollary 3.5.39. *Assume that X is a space and Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}, C_p(X))$ where $\mathscr{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$. If, additionally, the spread of $C_p(X)$ is countable then X is cosmic.*

PROOF. Note first that $s(X \times X) \leq s(C_p(X)) \leq \omega$. By Theorem 3.5.23 the space $C_p(X, \mathbb{I})$ is Lindelöf Σ; since X embeds in $C_p(C_p(X, \mathbb{I}))$ it has to be monolithic. This shows that $X \times X$ is also monolithic; since every ω-monolithic space of countable spread is hereditarily Lindelöf (see [Ar1, Theorem 1.2.9]) we conclude that $hl(X \times X) \leq \omega$.

Therefore the diagonal of X is a G_δ-set; every Lindelöf space with a G_δ-diagonal has countable i-weight [Ar1, Theorem 2.1.8] so $d(C_p(X)) = iw(X) \leq \omega$. By Theorem 3.5.37, the space $C_p(X)$ is ω-monolithic and therefore $nw(X) = nw(C_p(X)) \leq \omega$. $\square$

3.6. The game $\mathscr{G}(\mathscr{F},X)$ on scattered spaces

In [Ha] R.W. Hansell proved, among other things, that for every compact K that is a continuous image of a Valdivia compact, the space $C_p(K)$ is hereditarily weakly θ-refinable. Thus if K is a continuous image of a Valdivia compact then each scattered subspace of $C_p(K)$ is σ-discrete. This implies in particular that if X is an Eberlein-Grothendieck scattered Lindelöf Σ-space then it is σ-discrete. Indeed, suppose that $X \subset C_p(Q)$ for some compact Q and let φ be the diagonal product of the elements of X. If $K = \varphi(Q)$ then the dual map φ^* embeds the space $C_p(K)$ into $C_p(Q)$ as a closed subspace that contains X (see [Tk6, Problem 163]). It follows that $C_p(K)$ contains a homeomorphic copy of X that is a Lindelöf Σ subspace of $C_p(K)$ which separates the points of K. This implies that $C_p(K)$ is Lindelöf Σ by [Ar2, Corollary IV.2.10] and therefore K is a Gul'ko (and hence Valdivia) compact space; thus X is hereditarily weakly θ-refinable and consequently σ-discrete. As a consequence, every Eberlein-Grothendieck K-analytic scattered space is σ-discrete.

Since every Lindelöf Čech-complete space X is Lindelöf Σ (see for example [Tk5, Theorem 1]) we already know that if X is scattered then it is σ-discrete. However, we can say more in this case; these spaces are known to be σ-compact but also they have interesting game-related properties, they are 1-like in the sense of [Te]. In this section we will apply the methods developed in [Te] to study topological games in the context of Eberlein-Grothendieck scattered spaces. In particular, we give a game-related proof that Lindelöf Čech-complete scattered spaces are σ-compact.

Recall that given a space X and $Y \subset X$, the set Y has countable character in X if there exists a countable family $\mathscr{U} \subset \tau(Y,X)$ such that for every $V \in \tau(Y,X)$ there exists $U \in \mathscr{U}$ such that $V \subset U$. A space X is of countable type if every compact subset of X is contained in a compact space of countable character in X. Notice that in a space of countable type X not necessarily every compact subset of X has countable character in X, therefore the following proposition does not follow from the results in [Te], however we will apply the ideas in [Te] to prove it.

Proposition 3.6.1. *[AG, Proposition 4.3] If a space X has countable type and $\mathscr{K}$ is the family of its compact subspaces, then the first player has a winning strategy for the game $\mathscr{G}(\mathscr{K},X)$ if and only if X is σ-compact.*

PROOF. It suffices to prove necessity. Suppose that the space X has countable type and that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{K},X)$. Let s be a winning strategy for the first Player in the game $\mathscr{G}(\mathscr{K},X)$. For every $F \in \mathscr{K}$ there is $K(F) \in \mathscr{K}$ such that $F \in K(F)$ and $\chi(K(F)) \leq \omega$ hence, for each $F \in \mathscr{K}$ it is possible to find a countable family $\mathscr{U}_F = \{U_n^F : n < \omega\} \subset \tau(K(F),X)$ such that $\bigcap \mathscr{U}_F = K(F)$. For the compact set $F_0 = s(\varnothing)$ define $A_0 = \{K(F_0)\}$ and for every $n \in \mathbb{N}$ define

$$A_n = \{K(s(U_{l_0}^{F_0},\dots,U_{l_{n-1}}^{F_{n-1}})) : l_0,\dots,l_{n-1} < \omega, F_i \in A_i$$

for each $i < n$ and

$$((F_0,U_{l_0}^{F_0}),\dots,(F_{n-1},U_{l_{n-1}}^{F_{n-1}}))$$

is an initial segment of a match of the game $\mathscr{G}(\mathscr{K},X)$ in which the first player applies the strategy $s\}$.

Observe that $|A_n| \leq \omega$ for every $n \in \omega$. Indeed, $|A_0| \leq \omega$. If we assume that $|A_n| \leq \omega$ then we have that $|A_{n+1}| \leq |A_n| \cdot |\mathscr{U}_{F_n}| \leq \omega^2 \leq \omega$. Therefore $|\bigcup\{A_n : n \in \omega\}| \leq \omega$.

For every $n \in \omega$ define $B_n = \bigcup A_n$ and $B = \bigcup B_n$. We will show that $X = B$. Suppose that $y \in X \setminus B$; this implies $y \neq K(F_0)$ thus there is $U_0 \in \mathscr{U}_{F_0}$ such that $y \notin U_0$. Let $F_1 = s(U_0)$. The set $K(F_1) \in A_1$ therefore $y \notin K(F_1)$ and there is $U_1 \in \mathscr{U}_{F_1}$ such that $y \notin U_1$. Let $F_2 = s(U_1,U_2)$. Suppose F_k, U_{k-1} have been defined by this procedure in such a way that $y \notin U_j$ with $j = 1,\dots,k-1$, it is then possible to find $U_k \in \mathscr{U}_{F_k}$ such that $y \notin U_k$. By the definition of $\{U_n : n \in \omega\}$ we have that $\mathscr{P} = \{(F_n,U_n) : n \in \omega\}$ is a match of $\mathscr{G}(\mathscr{K},X)$ in which the first player applies s, but $y \notin U_n$ for every $n \in \omega$. This contradiction shows that X is σ-compact. □

Let $\mathscr{S}$ be the family of singletons of a space Lindelöf scattered space X. In [Te, Theorem 9.3] Telgarsky proved that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{S},X)$. Thus we have the following corollary.

Corollary 3.6.2. *[AG, Corollary 4.4]*

Every Lindelöf scattered space of countable type is σ-compact.

The most important classes of countable type spaces are the first countable and the Čech-complete spaces, so we obtain the following known fact.

Theorem 3.6.3. *[AG, Theorem 4.5]*

Every Lindelöf first countable or Čech-complete scattered space is σ-compact.

Let $\mathscr{S}$ be the family of singletons of a compact space K. In [Te] it is proved, among other things, that the first player has a winning strategy for the game $\mathscr{G}(\mathscr{S},K)$ if and only if K is scattered. We can apply this fact in the context of Eberlein-Grothendieck spaces to obtain the following corollary.

Corollary 3.6.4. *[AG, Corollary 4.6]*

Let $\mathscr{S}$ be the family of all singletons of an Eberlein-Grothendieck Čech-complete space X. If the first player has a winning strategy for the game $\mathscr{G}(\mathscr{S},X)$ then X is σ-discrete.

PROOF. The space X being Čech-complete has countable type. Besides by [Te, Corollary 2.2], the first player also has a winning strategy for the game $\mathscr{G}(\mathscr{K},X)$, where $\mathscr{K}$ is the family of compact subsets of X. Apply Theorem 3.6.3 to conclude that X is σ-compact. We can write $X = \bigcup_{n\in\omega} K_n$ where each K_n is Eberlein compact. It is easy to see that for each $n \in \omega$ the first player has a winning strategy for the game $\mathscr{G}(\mathscr{S}_n,K_n)$, where $\mathscr{S}_n$ is the family of singletons of K_n implying that each K_n is scattered and consequently σ-discrete and so is X. □

Note that in Corollary 3.6.4 we do not assume as a hypothesis that X is scattered to begin with.

Another corollary of Theorem 3.6.3 and the results in [Te] is the following generalization of [Mu, Corolario 2.11.12]

Corollary 3.6.5. *For any K-analytic hereditarily Baire space Y the following conditions are equivalent:*

(I) *The fist player has a winning strategy for the game $\mathscr{G}(\mathscr{S},Y)$ where $\mathscr{S}$ is the family of all the singletons of Y.*

(II) *The space Y is scattered.*

(III) *The space Y is σ-scattered.*

(IV) *The space Y does not contain perfect compact subspaces.*

(V) *Every countable subset of $C_p(Y)$ has metrizable closure in $\mathbb{R}^Y$.*

(VI) *The space $C_p(Y)$ is Fréchet-Urysohn.*

(VII) *The space $C_p(Y)$ is sequential.*

(VIII) *The space $C_p(Y)$ is a K-space.*

(IX) *The space $C_p(Y)$ is a $K_{\mathbb{R}}$-space.*

Open Problems

As the authors of [COT] point out, the most important problem still unsolved about spaces dominated by second countable spaces is the following.

Problem 1. *Suppose that for a compact space K we have that $K^2 \setminus \Delta$ is dominated by a second countable space, does it imply that K is metrizable? The answer is affirmative if we assume that $K^2 \setminus \Delta$ is strongly dominated by a second countable one, but can we omit the hypothesis that the domination is strong?*

It is clear that the most evident remaining open problem on the topic of function spaces dominated by second countable ones is [COT, problem 3.10]. A positive result would come out of answering affirmatively the following.

Problem 2. *Suppose that a space $C_p(X)$ is strongly dominated by a second countable space. Does it have a countable network modulo a fundamental compact cover?*

For the case of Lindelöf-p spaces it would suffice to strengthen Theorem 1.4.24(VI).

Problem 3. *Suppose that $C_p(X)$ is strongly dominated by a second countable space and $K \subset X$ is compact. Must K be countable?*

Is it possible to generalize Theorem 1.4.22(1)? In other words:

Problem 4. *Let X be a space such that $sdm(C_p(X)) \leq \kappa$. Must $s(X) \leq \kappa$?*

In Chapter 1 it was explained why we can conjecture that the following question may have a positive answer.

Problem 5. *Is it true that $dm(C_p(X)) = \ell\Sigma(C_p(X))$ for every Tychonoff space X?*

It is already known that Eberlein compact scattered spaces are σ-discrete, the next evident step would be to consider the case of Eberlein-Grothendieck Lindelöf spaces. A positive answer would follow if $C_p(K)$ was hereditarily weakly θ-refinable whenever K is a compact subspace of $C_p(L)$ for a scattered Lindelöf space L.

Problem 6. *Is every Eberlein-Grothendick Lindelöf scattered space σ-discrete?*

Problem 7. *Suppose that L is a Lindelöf scattered space and K is a compact subspace of $C_p(L)$. Is $C_p(K)$ hereditarily weakly θ-refinable?*

In Chapter 2 we showed that Lindelöf Čech-complete scattered spaces are σ-discrete being σ-compact. What if we remove the hypothesis that the space is Lindelöf?

Problem 8. *Is it true that every Eberlein-Grothendieck Čech-complete scattered space has to be σ-discrete?*

As noticed in Chapter 2, if we consider spaces that are not Tychonoff, hereditarily metalindelöfness does not necessarily imply σ-discreteness of scattered spaces in general. Does it in Eberlein-Grothendieck spaces?

Problem 9. *Suppose X is an Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space. Must X be σ-discrete?*

Under Martin's Axiom, if the following question has a positive answer then Problem 9 will also be answered positively.

Problem 10. *Suppose that X is an Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space of height and cardinality equal to ω_1. For every point $x \in X$ there is an open set U_x that isolates x in its scattering level. There is a point-countable open refinement $\mathscr{V}$ of the cover $\{U_x : x \in X\}$. Let V_x be the union of all the elements of $\mathscr{V}$ that contain x. Define the partially ordered set $P = \{p \subset X : p$ is finite and $V_x \cap p = \{x\}$ for every $x \in p\}$ and $q < p$ if $p \subset q$. Has P got the countable chain condition?*

Take an Eberlein-Grothendieck right-separated transfinite sequence $X = \{x_\alpha : \alpha < \lambda\}$. In Chapter 2 we showed that X is hereditarily metalindelöf for $\lambda < \omega_2$. Moreover we proved that hereditarily metalindelöfness implies σ-discreteness of X for $\lambda < \omega_1 \cdot \omega_1$. It is not yet clear if hereditarily metalindelöfness implies σ-discreteness of X for $\lambda = \omega_1 \cdot \omega_1$.

Problem 11. *Suppose that $\omega_1 \cdot \omega_1 \le \lambda < \omega_2$ and $X = \{x_\alpha : \alpha < \lambda\}$ is an Eberlein-Grothendieck right-separated transfinite sequence. Is X σ-discrete?*

In the case when a space $C_p(X)$ admits a closure-preserving closed cover by its subspaces with a property $\mathscr{P}$ it often happens that $C_p(X,\mathbb{I})$ has $\mathscr{P}$. We found out that quite a few classical theorems about a property $\mathscr{P}$ in $C_p(X)$ do not extend automatically to the spaces X such that $C_p(X)$ has a closure-preserving closed cover whose elements have $\mathscr{P}$. In particular, it is not clear whether in these results we can substitute $C_p(X)$ by $C_p(X,\mathbb{I})$. If the respective question about $C_p(X,\mathbb{I})$ seems to be interesting in itself, we also formulate it here.

Problem 12. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf subspaces. We know that in this case $C_p(X,\mathbb{I})$ is a Lindelöf space. But must the whole $C_p(X)$ be Lindelöf?*

Problem 13. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. We know that in this case $C_p(X,\mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.*

The existence of a topological property in $C_p(C_p(X))$ usually implies stronger restrictions on X than having this property in $C_p(X)$. Therefore there is hope that the following question has a positive answer.

Problem 14. *Suppose that $C_p(C_p(X))$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must the space $C_p(C_p(X))$ be Lindelöf Σ?*

If $C_p(X)$ is a Lindelöf Σ-space and has the Baire property then X must be countable. This is the motivation for the following question.

Problem 15. *Suppose that X is a space such that $C_p(X)$ has the Baire property and can be represented as the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X be countable?*

If a space X has countable spread and $C_p(X)$ is a Lindelöf Σ-space then X must be cosmic. However it is not clear whether we could replace $C_p(X)$ by $C_p(X,\mathbb{I})$ in this result.

Problem 16. *Suppose that X is a space such that $s(X) \leq \omega$ and $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X have a countable network?*

Problem 17. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed K-analytic subspaces. We know that in this case $C_p(X,\mathbb{I})$ is a K-analytic space. But must the whole $C_p(X)$ be K-analytic?*

It is known that sequentiality and Fréchet-Urysohn property are equivalent in the spaces $C_p(X)$. However this is not clear for $C_p(X,\mathbb{I})$ so the following questions are obligatory.

Problem 18. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed sequential subspaces. We know that in this case $C_p(X,\mathbb{I})$ must be sequential. But must the whole $C_p(X)$ be sequential?*

Problem 19. *Suppose that X is a space such that $C_p(X,\mathbb{I})$ is sequential. Must $C_p(X,\mathbb{I})$ (or equivalently $C_p(X)$) be Fréchet-Urysohn?*

It is known that countable π-weight in $C_p(X)$ is preserved neither by countable unions nor by unions of closure-preserving closed families. The situation is not clear if we consider the weight of $C_p(X)$.

Problem 20. *Is the space $C_p(\mathbb{I})$ representable as the union of a closure-preserving family of its second countable subspaces?*

With respect of characterizing function spaces by means of the topological games described here the most important remaining problem on this topic is the following.

Problem 21. *Suppose that X is a space such that Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X))$ where $\mathscr{F}$ is the family of the closed Lindelöf Σ-subspaces of $C_p(X)$. We know that in this case $C_p(X,\mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.*

Not only we do not know the answer to Problem 21, but we do not even know if in that case the space $C_p(X)$ has any of the properties that it would have if it was a Lindelöf Σ space.

Problem 22. *Suppose that X is a space such that Player I has a winning strategy for the game $\mathscr{G}(\mathscr{F},C_p(X))$ where $\mathscr{F}$ is the family of the closed Lindelöf Σ-subspaces of $C_p(X)$. Must the whole $C_p(X)$ be ω-monolithic? The answer is yes if X has a unique non-isolated point.*

In Conclusion

As announced in the introduction, along this dissertation we dedicated ourselves to decompose topological and function spaces and then study those decompositions. First by M-ordered compact covers, then by discrete subsets, after that by closure-preserving covers and finally by open covers defined by matches of certain topological games. How fruitful has been our study? That would be for the reader to decide, nonetheless we can outline the following facts.

In Chapter 1 the implications of M-domination and strong M-domination for metric M on topological and function spaces is shown to be clearly closely related to the behaviour of the $\ell\Sigma$ number of the spaces considered. Specially in the case of function spaces both concepts M-domination and generalized K-determination seem to coincide although this has not been proven yet. Nevertheless in Section 2.1 we found sufficient evidence that suggests the conjecture $dm(C_p(X)) = \ell\Sigma(C_p(X))$ for every X. To establish whether this fact is true or not would help us to better understand the nature of the concept of M-domination and that of function spaces.

It is not always evident if it is possible to split a topological space of certain kind into subspaces of some special type. In Chapter 2 we study the case when Eberlein-Grothendieck scattered spaces can be representable as countable unions of discrete subspaces. Starting from the positive result, consequence of a result by K. Alster, that in the case of Eberlein compact spaces this decomposition can be obtained, we made an effort to extend the fact to the general case. The general result was not achieved so Section 2.2 presents quite a few positive results that allow us to conjecture an affirmative statement on this subject, at least for the case when the cardinality of the Eberlein-Grothendieck scattered space considered is ω_1.

The author of this thesis has obtained results on closure-preserving decompositions of function spaces that are collected in Chapter 3. In particular, some partial answers to open problems posed in [GT] are provided. It is also considered the subject of topological games on function spaces and subspaces of function spaces. Chapter 3 concludes showing that it is possible to characterize several topological properties in $C_p(X)$ spaces by means of topological games. Also we show in that chapter that topological games have a very deep and close relationship with the decomposition of Eberlein-Grothendieck scattered spaces into countably many discrete subspaces.

In Chapter 3 we answer 8 problems published in [Gue]. Whereas in Chapter 1 we solve a problem of Cascales, Orihuela and Tkachuk. Also, in the rest of chapters 1 and 3 we present some partial answers to questions posed in [COT] and [GT]. Some well known results of Muñoz, Alster, Tkachuk, Arhangel'skii, Casarrubias, Okunev, Cascales, and Orihuela are generalized.

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