# Chapter 1
# COMPACTNESS, OPTIMALITY AND RISK

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**Summary:** This is a survey about one of the most important achievements in optimization in Banach space theory, namely, James' weak compactness theorem, its relatives and its applications. We present here a good number of topics related to James' weak compactness theorem and try to keep the technicalities needed as simple as possible: Simons' inequality is our preferred tool. Besides the expected applications to measures of weak noncompactess, compactness with respect to boundaries, size of sets of norm-attaining functionals, etc., we also exhibit other very recent developments in the area. In particular we deal with functions and their level sets to study a new Simons' inequality on unbounded sets that appear as the epigraph of some fixed function $f$. Applications to variational problems for $f$ and to risk measures associated with its Fenchel conjugate $f^*$ are studied.

**Key words:** compactness, optimization, risk measures, measure of non weak-compactness, Simons' inequality, nonattaining functionals, I-generation, variational problems, reflexivity.
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*Dedicated to Jonathan Borwein on the occasion of his 60th Birthday*

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## 1.1 Introduction

In 1957 James proved that a separable Banach space is reflexive whenever each continuous and linear functional on it attains its supremum on the unit ball, see [82, Theorem 3]. This result was generalized in 1964 to the non separable case in [83, Theorem 5]: in what follows we will refer to it as *James' reflexivity theorem*. More generally (and we shall refer to it as to *James' weak compactness theorem*), the following characterization of weak compactness was obtained in [84, Theorem 5]:

**Theorem 1.1 (James).** A weakly closed and bounded subset $A$ of a real Banach space is weakly compact if, and only if, every continuous and linear functional attains its supremum on $A$.

This central result in Functional Analysis can be extended to complete locally convex spaces, as shown in [84, Theorem 6]. Note that it is not valid in the absence of completeness, as seen in [86]. Since a complex Banach space can be considered naturally as a real Banach space with the same weak topology, James' weak compactness theorem is easily transferred to the complex case. Nonetheless, and because of the strongly real nature of the optimization assumption, the setting for this survey will be that of real Banach spaces.

We refer to [85], [53] and [81] for different characterizations of weak compactness.

James' weak compactness theorem has two important peculiarities. The first one is that it has plenty of direct applications as well as it implies a number of important theorems in the setting of Banach spaces. Regarding the latter, we can say that this result is a sort of metatheorem within Functional Analysis. Thus, for instance, the Krein–Šmulian theorem (*i.e.*, the closed convex hull of a weakly compact subset of a Banach space is weakly compact) or the Milman–Pettis theorem (*i.e.*, every uniformly convex Banach space is reflexive) straightforwardly follow from it. Also, the Eberlein–Šmulian theorem, that states that a nonempty subset $A$ of a Banach space $E$ is relatively weakly compact in $E$ if, and only if, it is relatively weakly countably compact in $E$, can be easily derived from James' weak compactness theorem. Indeed, assume that $A$ is relatively weakly countably compact in $E$ and for a given continuous and linear functional $x^*$ on $E$, let $\{x_n\}_{n\geq 1}$ be a sequence in $A$ satisfying

$$\lim_n x^*(x_n) = \sup_A x^* \in (-\infty,\infty].$$

If a $x_0 \in E$ is a $w$-cluster point of the sequence $\{x_n\}_{n\geq 1}$, then

$$\sup_A x^* = x^*(x_0) < \infty.$$

The boundedness of $A$ follows from the Banach–Steinhaus theorem, and that $A$ is relatively weakly compact is then a consequence of James' weak compactness theorem.

The second singularity regarding James' weak compactness theorem is that this result not only has attracted the attention of many researchers due to the huge number of its different applications, but also that several authors in the last decades tried to find a reasonable simple proof for it. This search has produced plenty of new important techniques in the area.

Pryce, in [125], simplified the proof of James' weak compactness theorem by using two basic ideas. The first one was to use the Eberlein–Grothendieck double-limit condition, see for instance [53, pp. 11-18] or [135, Theorem 28.36], that states that a bounded subset $A$ of a Banach space $E$ is relatively weakly compact if, and only if,

$$\lim_m \lim_n x_m^*(x_n) = \lim_n \lim_m x_m^*(x_n) \tag{1.1}$$

for all sequences $\{x_n\}_{n\geq 1}$ in $A$ and all bounded sequences $\{x_m^*\}_{m\geq 1}$ in $E^*$ for which the above iterated limits do exist. Pryce's second idea was to use the following diagonal argument.

**Lemma 1.2 (Pryce).** Let $X$ be a nonempty set, $\{f_n\}_{n\geq 1}$ a uniformly bounded sequence in $\ell^\infty(X)$, and $D$ a separable subset of $\ell^\infty(X)$. Then there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that

$$\sup_X \left( f - \limsup_k f_{n_k} \right) = \sup_X \left( f - \liminf_k f_{n_k} \right),$$

for every $f \in D$.

We should stress here that from the lemma above it follows that for any further subsequence $\{f_{n_{k_j}}\}_{j\geq 1}$ of $\{f_{n_k}\}_{k\geq 1}$ we also have

$$\sup_X \left( f - \limsup_j f_{n_{k_j}} \right) = \sup_X \left( f - \liminf_j f_{n_{k_j}} \right),$$

for every $f \in D$. With the above tools, Pryce's proof of James' weak compactness theorem is done by contradiction: if a weakly closed and bounded subset $A$ of a Banach space $E$ is not weakly compact, then there exist sequences $\{x_n\}_{n\geq 1}$ and $\{x_m^*\}_{m\geq 1}$ for which (1.1) does not hold. Lemma 1.2 applied to $\{x_m^*\}_{m\geq 1}$ helped Pryce to derive the existence of a continuous linear functional that does not attain its supremum on $A$. In the text by Holmes [81, Theorem 19.A], one can find Pryce's proof for Banach spaces whose dual unit ball is $w^*$-sequentially compact: Pryce's original arguments are simplified in this case.

In 1972 Simons gave another simpler proof of James' weak compactness theorem in [137]. The proof by Simons uses an *"ad hoc"* minimax theorem (with optimization and convexity hypotheses) that follows from a diagonal argument different from that of Pryce above, together with a deep result known henceforth as *Simons' inequality*, see [136, Lemma 2], that we recall immediately below.

**Lemma 1.3 (Simons).** Let $\{f_n\}_{n\geq 1}$ be a uniformly bounded sequence in $\ell^\infty(X)$ and let $W$ be its convex hull. If $Y$ is a subset of $X$ with the property that for every sequence of non-negative numbers $\{\lambda_n\}_{n\geq 1}$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $y \in Y$ such that

$$\sum_{n=1}^{\infty} \lambda_n f_n(y) = \sup \left\{ \sum_{n=1}^{\infty} \lambda_n f_n(x) : x \in X \right\},$$

then

$$\inf \left\{ \sup_X g : g \in W \right\} \leq \sup_{y \in Y} \left\{ \limsup_n f_n(y) \right\}.$$

A converse minimax theorem, see [137, Theorem 15] (see also [139, Theorem 5.6] and [133, Lemma 18]) provides an easier proof of James' weak compactness theorem and a minimax characterization of weak compactness.

A different proof of James' weak compactness theorem, and even simpler than that in [84], was stated by James himself in [87]. He took into account ideas coming from Simons' inequality in his new proof. The result proved is: *A separable Banach space $E$ is reflexive if, and only if, there exists $\theta \in (0,1)$ such that for every sequence $\{x_n^*\}_{n\geq 1}$ in the unit ball of its dual space, either $\{x_n^*\}_{n\geq 1}$ is not weak$^*$-null or*

$$\inf_{x^* \in C} \|x^*\| < \theta,$$

*where $C$ is the convex hull of $\{x_n^* \colon n \geq 1\}$* –the characterization of weak compact subsets of a separable Banach spaces is easily guessed by analogy. If the assumption of separability on $E$ is dropped, a similar characterization is obtained, but perturbing the functionals in the convex hull of $\{x_n^* \colon\ n \geq 1\}$ by functionals in the annihilator of a nonreflexive separable subspace $X$ of $E$: *$E$ is reflexive if, and only if, there exists $\theta \in (0,1)$ such that for each subspace $X$ of $E$ and for every sequence $\{x_n^*\}_{n\geq 1}$ in the unit ball of the dual space of $E$, either $\{x_n^*\}_{n\geq 1}$ is not null for the topology in $E^*$ of pointwise convergence on $X$ or*

$$\inf_{x^* \in C,\ w \in X^\perp} \|x^* - w\| < \theta,$$

*whit $C$ being the convex hull of $\{x_n^* \colon n \geq 1\}$.*

It should be noted that the new conditions that characterize reflexivity above imply in fact that every continuous and linear functional attains the norm.

In 1974 De Wilde [152] stated yet another proof of James' weak compactness theorem, that basically uses as main tools the diagonal argument of Pryce and the ideas of Simons in [136] together with the Eberlein–Grothendieck double-limit condition.

More recently, Morillon [111] has given a different proof of James' reflexivity theorem, based on a previous result by her [112, Theorem 3.9] establishing, one the one hand, James' reflexivity theorem for spaces with a $w^*$-block compact dual unit ball by means of Simons' inequality and Rosenthal's $\ell^1$-theorem, and extending, on the other hand, the proof to the general case with an adaptation of a result of Hagler and Jonhson [72]. Along with these ideas another proof of James' reflexivity theorem has been given by Kalenda in [92]. Very recently, Pfitzner has gone a step further using the ideas above to solve the so-called *boundary problem* of Godefroy, [59, Question 2] –see Section 1.4, giving yet another approach to James' weak compactness theorem, [122].

Another approach to James' reflexivity theorem in the separable case is due to Rodé [129], by using his form of the minimax theorem in the setting of the so-called "superconvex analysis". Let us also point out that for separable Banach spaces, the proof in [45, Theorem I.3.2], directly deduced from the Simons inequality, can be considered an easy one. A completely different proof using Bishop–Phelps and Krein–Milman theorems is due to Fonf, Lindenstrauss and Phelps [56, Theorem 5.9], and an alternative approach is due to Moors [108, Theorem 4]. Nevertheless, the combinatorial principles involved (known in the literature as the (I)-formula) are equivalent to Simons' inequality, see [93, Lemma 2.1 and Remark 2.2] and [29, Theorem 2.2]. We refer the interested reader to the papers by Kalenda [93, 92], where other proofs for James' reflexivity theorem using (I)-envelopes in some special cases can be found.

The leitmotif in this survey is Simons' inequality, which is used, to a large extent, as the main tool for proving the results, most of them self-contained and different from the original ones. Section 1.2 is devoted to the discussion of a generalization of the Simons inequality, where the uniform boundedness condition is relaxed, together with its natural consequences as unbounded sup-limsup's and Rainwater–Simons' theorems. The first part of Section 1.3 is devoted to providing a proof of James' weak compactness theorem that, going back to the work of James, explicitly supplies nonattaining functionals in the absence of weak compactness; in the second part of Section 1.3 we study several measures of weak noncompactness and we introduce a new one that is very close to Simons' inequality. Section 1.4 deals with the study of boundaries in Banach spaces and some deep related results, that can be viewed as extensions of James' weak compactness theorem. Other extensions of James' weak compactness theorem are presented in Section 1.5, where we mainly focus our attention on those of perturbed nature, which have found some applications in mathematical finance and variational analysis, as seen in Section 1.6.

Let us note that each section of this paper concludes with a selected open problem.

### *1.1.1 Notation and terminology*

Most of our notation and terminology are standard, otherwise it is either explained here or when needed: unexplained concepts and terminology can be found in our standard references for Banach spaces [45, 49, 90] and topology [48, 95]. By letters $E,K,T,X$, etc. we denote sets and sometimes topological spaces. Our topological spaces are assumed to be completely regular.

All vector spaces $E$ that we consider in this paper are assumed to be real. Frequently, $E$ denotes a normed space endowed with a norm $\|\cdot\|$, and $E^*$ stands for its dual space. Given a subset $S$ of a vector space, we write $\operatorname{conv}(S)$ and $\operatorname{span}(S)$ to denote, respectively, the convex and the linear hull of $S$. If $S$ is a subset of $E^*$, then $\sigma(E,S)$ denotes the weakest topology for $E$ that makes each member of $S$ continuous, or equivalently, the topology of pointwise convergence on $S$. Dually, if $S$ is a subset of $E$, then $\sigma(E^*,S)$ is the topology for $E^*$ of pointwise convergence on $S$. In particular, $\sigma(E,E^*)$ and $\sigma(E^*,E)$ are the weak (denoted by $w$) and weak$^*$ (denoted by $w^*$) topologies, respectively. Of course, $\sigma(E,S)$ is always a locally convex topology, that is Hausdorff if, and only if, $E^* = \overline{\operatorname{span} S}^{w^*}$ (and similarly for $\sigma(E^*,S)$). Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ for the evaluation of $x^*$ at $x$. If $x \in E$ and $\delta > 0$, we denote by $B(x,\delta)$ (resp. $B[x,\delta]$) the open (resp. closed) ball centered at $x$ of radius $\delta$: we will simplify our notation and just write $B_E := B[0,1]$; the unit sphere $\{x \in E : \|x\| = 1\}$ will be denoted by $S_E$. Given a nonempty set $X$, and $f \in \mathbb{R}^X$ we write

$$S_X(f) := \sup_{x\in X} f(x) \in (-\infty, \infty].$$

$\ell^\infty(X)$ stands for the Banach space of real valued bounded functions defined on $X$, endowed with the supremum norm $S_X(|\cdot|)$.

## 1.2 Simons' inequality for pointwise bounded subsets of $\mathbb{R}^X$

The main goal of this section is to derive a generalized version of Simons' inequality, Theorem 1.5, in a pointwise bounded setting, as opposed to the usual uniform bounded context. As a consequence, we derive an unbounded version of the so-called Rainwater–Simons theorem, Corollary 1.7, that will provide us with some generalizations of James' weak compactness theorem, as well as new developments and applications in Sections 1.5 and 1.6. In addition, the aforementioned result will allow us to present the state of the art of a number of issues related to boundaries in Banach spaces in Section 1.4.

The inequality presented in Lemma 1.3, as Simons himself says in [136], is inspired by some of James' and Pryce's arguments in [84, 125], and contains the essence of the proof of James' weak compactness theorem in the separable case. As mentioned in the Introduction, James included later the novel contribution of Simons in his proof in [87]. We refer to [45, 61] for some applications of Simons' inequality, to [114, 43, 99, 29] for proper extensions, and to [115] for a slightly different proof.

Given a pointwise bounded sequence $\{f_n\}_{n\geq 1}$ in $\mathbb{R}^X$, we define

$$\operatorname{co}_{\sigma_p}\{f_n \colon n \geq 1\} := \left\{ \sum_{n=1}^{\infty} \lambda_n f_n \colon \lambda_n \geq 0 \text{ for every } n \geq 1 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\},$$

where a function of the form $\sum_{n=1}^{\infty} \lambda_n f_n \in \mathbb{R}^X$ is obviously defined by

$$\Big(\sum_{n=1}^{\infty} \lambda_n f_n\Big)(x) := \sum_{n=1}^{\infty} \lambda_n f_n(x)$$

for every $x \in X$.

Instead of presenting the results of Simons in [136] and [138], we adapt them to a pointwise but not necessarily uniformly bounded framework. This adaptation allows us to extend the original results of Simons and provides new applications, as we show below.

The next result follows by arguing as in the "Additive Diagonal Lemma" in [138]. Hereafter, any sum $\sum_{n=1}^{0}\dots$ is understood to be 0.

**Lemma 1.4.** If $\{f_n\}_{n\geq 1}$ is a pointwise bounded sequence in $\mathbb{R}^X$ and $\varepsilon>0$, then for every $m\geq 1$ there exists $g_m \in \mathrm{co}_{\sigma_p}\{f_n\colon n\geq m\}$ such that

$$S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}\right)\leq\left(1-\frac{1}{2^{m-1}}\right)S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\frac{\varepsilon}{2^{m-1}}.$$

*Proof.* It suffices to choose inductively, for each $m\geq 1$, $g_m\in\mathrm{co}_{\sigma_p}\{f_n\colon n\geq m\}$ satisfying

$$S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}+\frac{g_m}{2^{m-1}}\right)\leq\inf_{g\in\mathrm{co}_{\sigma_p}\{f_n\colon n\geq m\}}S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}+\frac{g}{2^{m-1}}\right)+\frac{2\varepsilon}{4^m}. \tag{1.2}$$

The existence of such $g_m$ follows from the easy fact that

$$\inf_{g\in\mathrm{co}_{\sigma_p}\{f_n\colon n\geq m\}}S_X(g)>-\infty,$$

according with the pointwise boundeness of our sequence $\{f_n\}_{n\geq 1}$. Since

$$2^{m-1}\sum_{n=m}^{\infty}\frac{g_n}{2^n}\in\mathrm{co}_{\sigma_p}\{f_n\colon n\geq m\},$$

then inequality (1.2) implies

$$S_X\left(\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}\right)+\frac{g_m}{2^{m-1}}\right)\leq S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\frac{2\varepsilon}{4^m}. \tag{1.3}$$

From the equality

$$\sum_{n=1}^{m-1}\frac{g_n}{2^n}=\sum_{k=1}^{m-1}\frac{1}{2^{m-k}}\left(\left(\sum_{n=1}^{k-1}\frac{g_n}{2^n}\right)+\frac{g_k}{2^{k-1}}\right),$$

and the help of (1.3) we finally derive that

$$
\begin{aligned}
S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}\right) &\le \sum_{k=1}^{m-1}\frac{1}{2^{m-k}}S_X\left(\left(\sum_{n=1}^{k-1}\frac{g_n}{2^n}\right)+\frac{g_k}{2^{k-1}}\right)\\
&\le \sum_{k=1}^{m-1}\frac{1}{2^{m-k}}\left(S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\frac{2\varepsilon}{4^k}\right)\\
&= \left(1-\frac{1}{2^{m-1}}\right)S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\left(1-\frac{1}{2^{m-1}}\right)\frac{2\varepsilon}{2^m}\\
&\le \left(1-\frac{1}{2^{m-1}}\right)S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\frac{\varepsilon}{2^{m-1}},
\end{aligned}
$$

and the proof is over.

□

We now arrive at the announced extension of Simons' inequality. Unlike the original work [136], we only assume pointwise boundedness of the sequence $\{f_n\}_{n\ge 1}$. Let us also emphasize that the extension of Simons' inequality stated in [114] is a particular case of the following non uniform version:

**Theorem 1.5 (Simons' inequality in $\mathbb{R}^{\mathbf{X}}$).** Let $X$ be a nonempty set, let $\{f_n\}_{n\ge1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ and let $Y$ be a subset of $X$ such that

$$\text{for every } g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\ge 1\} \text{ there exists } y\in Y \text{ with } g(y)=S_X(g).$$

Then

$$\inf_{g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\ge 1\}} S_X(g)\le S_Y\left(\limsup_n f_n\right).$$

*Proof.* It suffices to prove that for every $\varepsilon>0$ there exists $y\in Y$ and $g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\ge 1\}$ such that

$$S_X(g)-\varepsilon\le \limsup_n f_n(y).$$

Fix $\varepsilon>0$. Then Lemma 1.4 provides us with a sequence $\{g_m\}_{m\ge1}$ in $\mathbb{R}^X$ such that for every $m\ge1$, $g_m\in \mathrm{co}_{\sigma_p}\{f_n\colon n\ge m\}$ and

$$S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}\right)\le\left(1-\frac{1}{2^{m-1}}\right)S_X\left(\sum_{n=1}^{\infty}\frac{g_n}{2^n}\right)+\frac{\varepsilon}{2^{m-1}}. \tag{1.4}$$

Let us write $g:=\sum_{n=1}^{\infty}\frac{g_n}{2^n}\in \mathrm{co}_{\sigma_p}\{f_n\colon n\ge1\}$. Then by hypothesis there exists $y\in Y$ with

$$g(y)=S_X(g), \tag{1.5}$$

and so it follows from (1.4) and (1.5) that given $m\ge1$,

$$
\begin{aligned}
\left(1-\frac{1}{2^{m-1}}\right)g(y)+\frac{\varepsilon}{2^{m-1}} &\geq S_X\left(\sum_{n=1}^{m-1}\frac{g_n}{2^n}\right)\\
&\geq \sum_{n=1}^{m-1}\frac{g_n(y)}{2^n}\\
&= g(y)-\sum_{n=m}^{\infty}\frac{g_n(y)}{2^n}.
\end{aligned}
$$

Therefore,

$$
\inf_{m\geq 1} 2^{m-1}\sum_{n=m}^{\infty}\frac{g_n(y)}{2^n}\geq g(y)-\varepsilon. \tag{1.6}
$$

Since for every $m\geq 1$ we have $2^{m-1}\sum_{n=m}^{\infty}2^n=1$, we conclude that

$$
\sup_{n\geq m} f_n(y)\geq 2^{m-1}\sum_{n=m}^{\infty}\frac{g_n(y)}{2^n}.
$$

Now, with this last inequality in mind together with (1.5) and (1.6) we arrive at

$$
\begin{aligned}
\limsup_n f_n(y) &= \inf_{m\geq 1}\sup_{n\geq m} f_n(y)\\
&\geq \inf_{m\geq 1} 2^{m-1}\sum_{n=m}^{\infty}\frac{g_n(y)}{2^n}\\
&\geq g(y)-\varepsilon\\
&= S_X(g)-\varepsilon,
\end{aligned}
$$

as was to be shown.

□

Both in the original version of Simons' inequality and in the previous one, a uniform behavior follows from a pointwise one, resembling Mazur's theorem for continuous functions when $X$ is a compact topological space, see [146, Section 3, p.14]. Indeed, it turns out that Simons' inequality tell us that

$$
\inf\{\|g\|_\infty : g\in \mathrm{co}\{f_n : n\geq 1\}\}=0,
$$

whenever a uniformly bounded sequence of continuous functions $\{f_n\}_{n\geq 1}$ pointwise converges to zero on a compact space X.

As a consequence of the above version of Simons' inequality we deduce the following generalization of the sup-limsup theorem of Simons [136, Theorem 3] (see also [133, Theorem 7]). This result has recently been stated in [119, Corollary 1], but using the tools in [133].

**Corollary 1.6 (Simons' sup-limsup theorem in $\mathbb{R}^{\mathbf{X}}$).** Let $X$ be a nonempty set, let $\{f_n\}_{n\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ and let $Y$ be a subset of $X$ such that

for every $g\in \mathrm{co}_{\sigma_p}\{f_n : n\geq 1\}$ there exists $y\in Y$ with $g(y)=S_X(g)$.

Then

$$S_X\left(\limsup_n f_n\right) = S_Y\left(\limsup_n f_n\right).$$

*Proof.* Let us assume, arguing by *reductio ad absurdum*, that there exists $x_0 \in X$ such that

$$\limsup_n f_n(x_0) > S_Y\left(\limsup_n f_n\right).$$

We assume then, passing to a subsequence if necessary, that

$$\inf_{n\geq 1} f_n(x_0) > S_Y\left(\limsup_n f_n\right).$$

In particular,

$$\inf_{g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\geq 1\}} g(x_0) > S_Y\left(\limsup_n f_n\right),$$

and then, by applying Theorem 1.5, we arrive at

$$\begin{aligned} S_Y\left(\limsup_n f_n\right) &\geq \inf_{g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\geq 1\}} S_X(g) \\ &\geq \inf_{g\in \mathrm{co}_{\sigma_p}\{f_n\colon n\geq 1\}} g(x_0) \\ &> S_Y\left(\limsup_n f_n\right), \end{aligned}$$

a contradiction.

□

In the Banach space framework we obtain the sup-limsup's type result below, that also generalizes the so-called Rainwater–Simons theorem, see [136, Corollary 11] (see also [138, Sup-limsup Theorem], [101, Theorem 5.1] and [116, Theorem 2.2], the recent extension [108, Corollary 3] and for some related results [75]). It is a direct consequence of the Simons sup-limsup theorem in $\mathbb{R}^X$, Corollary 1.6, as in the uniform setting, see [50, Theorem 3.134]. In particular it generalizes the Rainwater theorem [127], which asserts that a sequence $\{x_n\}_{n\geq 1}$ in a Banach space $E$ is weakly null if it is bounded and for each extreme point $e^*$ of $B_{E^*}$,

$$\lim_n e^*(x_n) = 0.$$

Given a bounded sequence $\{x_n\}_{n\geq 1}$ in a Banach space $E$, we define

$$\mathrm{co}_\sigma\{x_n\colon n\geq 1\} := \left\{\sum_{n=1}^{\infty} \lambda_n x_n\colon \text{ for all } n\geq 1,\ \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty}\lambda_n = 1\right\}$$

Note that series are clearly norm-convergent and that

$$\mathrm{co}_\sigma\{x_n\colon n\geq 1\} = \mathrm{co}_{\sigma_p}\{x_n\colon n\geq 1\}$$

when for the second set we look at the $x_n$'s as functions defined on $B_{E^*}$.

**Corollary 1.7 (Unbounded Rainwater–Simons' theorem).** If $E$ is a Banach space, $C$ is a subset of $E^*$, $B$ is a nonempty subset of $C$ and $\{x_n\}_{n\geq 1}$ is a bounded sequence in $E$ such that

$$\text{for every } x \in \mathrm{co}_\sigma\{x_n:\ n\geq 1\} \text{ there exists } b^* \in B \text{ with } b^*(x) = S_C(x),$$

then

$$S_B\left(\limsup_n x_n\right) = S_C\left(\limsup_n x_n\right).$$

As a consequence

$$\sigma(E,B)\text{-}\lim_n x_n = 0 \;\Rightarrow\; \sigma(E,C)\text{-}\lim_n x_n = 0.$$

The unbounded Rainwater–Simons theorem (or the Simons inequality in $\mathbb{R}^X$) not only gives as special cases those classical results that follow from Simons's inequality (some of them are discussed here, besides the already mentioned [45, 61]), but it also provides new applications whose discussion we delay until the next sections. We only remark here that Moors has recently obtained a particular case of the unbounded Rainwater–Simons theorem, see [108, Corollary 1], which leads him to a proof of James' weak compactness theorem for Banach spaces whose dual unit ball are $w^*$-sequentially compact.

A very interesting consequence of Simons' inequality in the bounded case is the (I)-formula (1.7) of Fonf and Lindenstrauss, see [55] and [29]:

**Corollary 1.8 (Fonf–Lindenstrauss' theorem).** Let $E$ be a Banach space, $B$ a bounded subset of $E^*$ such that for every $x \in E$ there exists some $b_0^* \in B$ satisfying $b_0^*(x) = \sup_{b^*\in B} b^*(x)$. Then we have that, for every covering $B \subset \bigcup_{n=1}^\infty D_n$ by an increasing sequence of $w^*$-closed convex subsets $D_n \subset \overline{\mathrm{co}(B)}^{w^*}$, the following equality holds true

$$\overline{\cup_{n=1}^\infty D_n}^{\|\cdot\|} = \overline{\mathrm{co}(B)}^{w^*}. \tag{1.7}$$

*Proof.* Here is the proof given in [29, Theorem 2.2]. We proceed by contradiction assuming that there exists $z_0^* \in \overline{\mathrm{co}(B)}^{w^*}$ such that $z_0^* \notin \overline{\cup_{n=1}^\infty D_n}^{\|\cdot\|}$. Fix $\delta > 0$ such that

$$B[z_0^*,\delta]\cap D_n = \emptyset, \text{ for every } n\geq 1.$$

The separation theorem in $(E^*,w^*)$, when applied to the $w^*$-compact set $B[0,\delta]$ and the $w^*$-closed set $D_n - z_0^*$, provides us with a norm-one $x_n \in E$ and $\alpha_n \in \mathbb{R}$ such that

$$\inf_{v^*\in B[0,\delta]} x_n(v^*) > \alpha_n > \sup_{y^*\in D_n} x_n(y^*) - x_n(z_0^*).$$

But

$$-\delta = \inf_{v^*\in B[0,\delta]} x_n(v^*),$$

and consequently the sequence $\{x_n\}_{n\geq 1}$ in $B_E$ satisfies

$$x_n(z_0^*) - \delta > x_n(y^*) \tag{1.8}$$

for each $n \geq 1$ and $y^* \in D_n$. Fix a $w^*$-cluster point $x^{**} \in B_{E^{**}}$ of the sequence $\{x_n\}_{n\geq 1}$ and let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ such that $x^{**}(z_0^*) = \lim_k x_{n_k}(z_0^*)$. We can and do assume that for every $k \geq 1$,

$$x_{n_k}(z_0^*) > x^{**}(z_0^*) - \frac{\delta}{2}. \tag{1.9}$$

Since $B \subset \cup_{n=1}^{\infty} D_n$ and $\{D_n\}_{n\geq 1}$ is an increasing sequence of sets, given $b^* \in B$ there exists $k_0 \geq 1$ such that $b^* \in D_{n_k}$ for each $k \geq k_0$. Now inequality (1.8) yields

$$x^{**}(z_0^*) - \delta \geq \limsup_k x_{n_k}(b^*), \quad \text{for every } b^* \in B, \tag{1.10}$$

and, on the other hand, inequality (1.9) implies that

$$w(z_0^*) \geq x^{**}(z_0^*) - \frac{\delta}{2}, \quad \text{for every } w \in \mathrm{co}_\sigma\{x_{n_k} : k \geq 1\}. \tag{1.11}$$

Now Theorem 1.5 can be applied to the sequence $\{x_{n_k}\}_{k\geq 1}$, to deduce

$$\begin{aligned} x^{**}(z_0^*) - \delta &\overset{(1.10)}{\geq} \sup_{b^* \in B} \limsup_k x_{n_k}(b^*) \geq \\ \geq \inf\Big\{ \sup\{w(z^*) : z^* \in \overline{\mathrm{co}(B)}^{w^*}&, w \in \mathrm{co}_\sigma\{x_{n_k} : k \in \mathbb{N}\}\}\Big\} \\ \geq \inf\big\{w(z_0^*) : w \in \mathrm{co}_\sigma\{x_{n_k} : k \in \mathbb{N}\}\big\} &\overset{(1.11)}{\geq} x^{**}(z_0^*) - \frac{\delta}{2}. \end{aligned}$$

From the inequalities above we obtain $0 \geq \delta$, which is a contradiction that completes the proof.

□

To conclude this section, let us emphasize that in [29, Theorem 2.2] the equivalence between Simons' inequality, the sup-limsup theorem of Simons and the (I)-formula of Fonf and Lindenstrauss was established in the bounded case. However, in the unbounded case we propose the following question:

**Question 1.** Are the unbounded versions of Simons' inequality and sup-limsup theorem of Simons equivalent to some kind of *I*-formula for the unbounded case?

## 1.3 Nonattaining functionals

This section is devoted to describe how to obtain nonattaining functionals in the absence of weak compactness. Simons' inequality provides us a first way of doing it in a wide class of Banach spaces, which includes those whose dual unit balls are $w^*$-sequentially compact. We introduce a new measure of weak noncompactness, tihgtly connected with Simons' inequality, and we relate it with recent quantification results of classical theorems about weakly compact sets.

When Simons' inequality in $l^\infty(\mathbb{N})$ holds for a $w^*$-null sequence $\{x_n^*\}_{n\geq 1}$ in a dual Banach space $E^*$, it follows that the origin belongs to the norm-closed convex hull of the sequence, $\overline{\mathrm{co}\{x_n^*:n\geq 1\}}^{\|\cdot\|}$. Therefore every time we have a $w^*$-null sequence $\{x_n^*\}_{n\geq 1}$ with $0\notin \overline{\mathrm{co}\{x_n^*:n\geq 1\}}^{\|\cdot\|}$ we will have some $x_0^*\in \mathrm{co}_\sigma\{x_n^*:n\geq 1\}$ such that $x_0^*$ doest not attain its supremum on $B_E$.

We note that just Simons' inequality, or its equivalent sup-limsup theorem, provides us with the tools to give a simple proof of James' weak compactness theorem for a wide class of Banach spaces. We first recall the following concept:

**Definition 1.9.** Let $E$ be a vector space, and let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be sequences in $E$. We say that $\{y_n\}_{n\geq 1}$ is a *convex block sequence* of $\{x_n\}_{n\geq 1}$ if for a certain sequence of nonempty finite subsets of integers $\{F_n\}_{n\geq 1}$ with

$$\max F_1<\min F_2\leq \max F_2<\min F_3\leq\cdots\leq \max F_n<\min F_{n+1}\leq\cdots$$

and adequate sets of positive numbers $\{\lambda_i^n : i\in F_n\}\subset (0,1]$ we have that

$$\sum_{i\in F_n}\lambda_i^n=1 \quad \text{and} \quad y_n=\sum_{i\in F_n}\lambda_i^n x_i.$$

For a Banach space $E$, its dual unit ball $B_{E^*}$ is said to be *$w^*$-convex block compact* provided that each sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ has a convex block $w^*$-convergent sequence.

It is clear that if the dual unit ball $B_{E^*}$ of a Banach space $E$ is $w^*$-sequentially compact, then it is $w^*$-convex block compact. This happens, for example, when $E$ is a weakly Lindelöf determined (in short, WLD) Banach space, see [74]. Let us emphasize that both kinds of compactness do not coincide. Indeed, on the one hand, an example of a Banach space with a non $w^*$-sequentially compact dual unit ball and not containing $\ell^1(\mathbb{N})$ is presented in [73]. On the other hand, it is proved in [23] that if a Banach space $E$ does not contain an isomorphic copy of $\ell^1(\mathbb{N})$, then $B_{E^*}$ is $w^*$-convex block compact. This last result was extended for spaces not containing an isomorphic copy of $\ell^1(\mathbb{R})$ under Martin Axiom and the negation of the Continuum Hypothesis hypothesis in [80].

For a bounded sequence $\{x_n^*\}_{n\geq 1}$ in a dual Banach space $E^*$, we denote by $L_{E^*}\{x_n^*\}$ the set of all cluster points of the given sequence in the $w^*$-topology, and when no confusion arises, we just write $L\{x_n^*\}$.

**Lemma 1.10.** Suppose that $E$ is a Banach space, $\{x_n\}_{n\geq 1}$ is a bounded sequence in $E$ and $x_0^{**}$ in $E^{**}$ is a $w^*$-cluster point of $\{x_n\}_{n\geq 1}$ with $d(x_0^{**},E)>0$. Then for every $\alpha$ with $d(x_0^{**},E)>\alpha>0$ there exists a sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ such that

$$\langle x_n^*, x_0^{**}\rangle > \alpha \tag{1.12}$$

whenever $n \geq 1$, and
$$\langle x_0^*, x_0^{**} \rangle = 0 \tag{1.13}$$
for any $x_0^* \in L\{x_n^*\}$.

*Proof.* The Hahn–Banach theorem applies to provide us with $x^{***} \in B_{E^{***}}$ satisfying $x^{***}_{|E} = 0$ and $x^{***}(x_0^{**}) = d(x_0^{**}, E)$. For every $n \geq 1$ the set

$$V_n := \left\{ y^{***} \in E^{***} : y^{***}(x_0^{**}) > \alpha,\ |y^{***}(x_i)| \leq \frac{1}{n},\ i = 1, 2, \dots, n \right\}$$

is a $w^*$-open neighborhood of $x^{***}$, and therefore, by Goldstein's theorem, we can pick up $x_n^* \in B_{E^*} \cap V_n$. The sequence $\{x_n^*\}_{n\geq 1}$ clearly satisfies

$$\lim_n \langle x_n^*, x_p \rangle = 0, \quad \text{for all } p \in \mathbb{N},$$

and for each $n \geq 1$,
$$\langle x_n^*, x_0^{**} \rangle > \alpha.$$
Fix an arbitrary $x_0^* \in L\{x_n^*\}$. For every $p \geq 1$ we have that

$$\langle x_0^*, x_p \rangle = 0,$$

and thus
$$\langle x_0^*, x_0^{**} \rangle = 0,$$
because $x_0^{**} \in \overline{\{x_p : p = 1, 2, \cdots\}}^{w^*}$.

□

**Theorem 1.11.** Let $E$ be a Banach space with a $w^*$-convex block compact dual unit ball. If a bounded subset $A$ of $E$ is not weakly relatively compact, then there exists a sequence of linear functionals $\{y_n^*\}_{n\geq 1} \subset B_{E^*}$ with a $w^*$-limit point $y_0^*$, and some $g^* \in \mathrm{co}_\sigma\{y_n^* : n \geq 1\}$, such that $g^* - y_0^*$ doest not attain its supremum on $A$.

*Proof.* Assume that $A$ is not weakly relatively compact, which in view of the Eberlein–Šmulian theorem is equivalent to the existence of a sequence $\{x_n\}_{n\geq 1}$ in $A$ and a $w^*$-cluster point $x_0^{**} \in E^{**} \setminus E$ of it. Then Lemma 1.10 applies to provide us with a sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ and $\alpha > 0$ satisfying (1.12) and (1.13).

Let $\{y_n^*\}_{n\geq 1}$ be a convex-block sequence of $\{x_n^*\}_{n\geq 1}$ and let $y_0^* \in B_{E^*}$ such that $w^*\text{-}\lim_n y_n^* = y_0^*$. It is clear that (1.12) and (1.13) are valid when replacing $\{x_n^*\}_{n\geq 1}$ and $x_0^*$ with $\{y_n^*\}_{n\geq 1}$ and $y_0^*$, respectively. Then

$$\begin{aligned} S_{\overline{A}^{w^*}}\left(\limsup_n (y_n^* - y_0^*)\right) &\geq \limsup_n (y_n^* - y_0^*)(x_0^{**}) \\ &\geq \alpha \\ &> 0 \\ &= S_A\left(\limsup_n (y_n^* - y_0^*)\right), \end{aligned}$$

so in view of the Rainwater–Simons theorem, Corollary 1.7, there exists $g^* \in \mathrm{co}_\sigma\{y_n^* : n \geq 1\}$ such that $g^* - y_0^*$ does not attain its supremum on $A$, as announced.

□

In Section 1.5.2 we shall show a nonlinear extension of this result, with the use of the (necessarily unbounded) Rainwater–Simons theorem, Corollary 1.7. For the space $\ell^1(\mathbb{N})$, James constructed in [82] a continuous linear functional $g : \ell^1(\mathbb{N}) \to \mathbb{R}$ such that $g$ can be extended to $\hat{g} \in E^*$ on any Banach space $E$ containing $\ell^1(\mathbb{N})$, but $\hat{g}$ doest not attain its supremum on $B_E$. Rosenthal's $\ell^1(\mathbb{N})$-theorem, together with Theorem 1.11, provides another approach for James' reflexivity theorem. These ideas, developed by Morillon in [111], are the basis for new approaches to the weak compactness theorem of James, as the very successful one due to Pfitzner in [122].

We now deal with the general version of Theorem 1.11, that is, James' weak compactness theorem with no additional assumptions on the Banach space. If $E$ is a Banach space and $A$ is a bounded subset of $E$, we denote by $\|\cdot\|_A$ the seminorm on the dual space $E^*$ given by the Minkowski functional of its polar set, *i.e.,* the seminorm of uniform convergence on the set $A$. If $A = -A$, given a bounded sequence $\{x_n^*\}_{n\geq 1}$ in $E^*$ and $h^* \in L\{x_n^*\}$, Simons' inequality for the sequence $\{x_n^* - h^*\}_{n\geq 1}$ in $\ell^\infty(A)$ reads as follows: Under the assumption that every element in $\mathrm{co}_{\sigma_p}\{x_n^* - h^* : n \geq 1\}$ attains its supremum on $A$,

$$\mathrm{dist}_{\|\cdot\|_A}(h^*, \mathrm{co}\{x_n^* : n \geq 1\}) \leq S_A\left(\limsup_n x_n^* - h^*\right).$$

Therefore,

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) \leq \inf_{h^*\in L\{x_n^*\}} S_A\left(\limsup_n x_n^* - h^*\right).$$

We state the following characterization:

**Proposition 1.12.** Let $A$ be a bounded subset of a Banach space $E$. Then $A$ is weakly relatively compact if, and only if, for every bounded sequence $\{x_n^*\}_{n\geq 1}$ in $E^*$ we have

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) = 0. \tag{1.14}$$

*Proof.* We first prove that if $A$ is weakly relatively compact then equality (1.14) holds for any bounded sequence $\{x_n^*\}_{n\geq 1}$ in $E^*$. To this end, we note that, since $\overline{\mathrm{co}(A)}^{\|\cdot\|}$ is weakly compact by the Krein–Šmulian theorem, the seminorm $\|\cdot\|_A = \|\cdot\|_{\overline{\mathrm{co}(A)}^{\|\cdot\|}}$ is continuous for the Mackey topology $\mu(E^*, E)$. Hence we have the inclussions

$$L\{x_n^*\} \subset \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{w^*} = \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{\mu(E^*,E)} \subset \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{\|\cdot\|_A},$$

that clearly explain the validity of (1.14).

To prove the converse we will show that if $A$ is not weakly relatively compact in $E$, then there exists a sequence $\{x_n^*\}_{n\geq 1} \subset B_{E^*}$ such that

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\},\mathrm{co}\{x_n^*: n\geq 1\})>0.$$

Let us assume that $A$ is not relatively weakly compact in $E$. Then the Eberlein–Šmulian theorem guarantees the existence of a sequence $\{x_n\}_{n\geq 1}$ in $A$ with a $w^*$-cluster point $x_0^{**}\in E^{**}\setminus E$. If $d(x_0^{**},E)>\alpha>0$, an appeal to Lemma 1.10 provides us with a sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ satisfying

$$\langle x_n^*, x_0^{**}\rangle>\alpha$$

whenever $n\geq 1$ and

$$\langle x_0^*, x_0^{**}\rangle=0$$

for any $x_0^*\in L\{x_n^*\}$. Therefore we have that

$$\|\sum_{i=1}^n \lambda_i x_{n_i}^* - x_0^*\|_A \geq \left\langle \sum_{i=1}^n \lambda_i x_{n_i}^* - x_0^*, x_0^{**}\right\rangle>\alpha$$

for any convex combination $\sum_{i=1}^n \lambda_i x_{n_i}^*$, and consequently

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\},\mathrm{co}\{x_n^*: n\geq 1\})\geq\alpha>0, \tag{1.15}$$

and the proof is over.

□

Pryce's diagonal procedure is used in the proof of the following result:

**Proposition 1.13.** Let $E$ be a Banach space, $A$ a bounded subset of $E$ with $A=-A$, $\{x_n^*\}_{n\geq 1}$ a bounded sequence in the dual space $E^*$ and $D$ its norm-closed linear span in $E^*$. Then there exists a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ such that

$$S_A\left(x^*-\liminf_k x_{n_k}^*\right)=S_A\left(x^*-\limsup_k x_{n_k}^*\right)=\mathrm{dist}_{\|\cdot\|_A}(x^*,L\{x_{n_k}^*\}) \tag{1.16}$$

for all $x^*\in D$.

*Proof.* Lemma 1.2 implies the existence of a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ such that

$$S_A\left(x^*-\liminf_k x_{n_k}^*\right)=S_A\left(x^*-\limsup_k x_{n_k}^*\right)$$

for all $x^*\in D$. Since for any $h^*\in L\{x_{n_k}^*\}$ we have

$$\liminf_k x_{n_k}^*(a)\leq h^*(a)\leq \limsup_k x_{n_k}^*(a)$$

for all $a\in A$, it follows that

$$S_A\left(x^*-\liminf_k x_{n_k}^*\right)=\|x^*-h^*\|_A=S_A\left(x^*-\limsup_k x_{n_k}^*\right).$$

Therefore

$$S_A\left(x^* - \liminf_k x^*_{n_k}\right) = S_A\left(x^* - \limsup_k x^*_{n_k}\right) = \mathrm{dist}_{\|\cdot\|_A}(x^*, L\{x^*_{n_k}\})$$

for all $x^* \in D$, and the proof is finished.

□

Equality (1.16) will be in general the source to look for nonattaining linear functionals whenever we have

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x^*_{n_k}\}, \mathrm{co}\{x^*_{n_k} : k \geq 1\}) > 0,$$

which means, in view of Proposition 1.12, whenever $A$ is a non relatively weakly compact subset of $E$. Until now all such constructions depend on this fact, which is called the *technique of the undetermined function*. The next result is so far the most general perturbed version for the existence of nonattaining functionals, see [133, Corollary 8]:

**Theorem 1.14.** Let $X$ be a nonempty set, $\{h_j\}_{j\geq 1}$ a bounded sequence in $\ell^\infty(X)$, $\varphi \in \ell^\infty(X)$ with $\varphi \geq 0$ and $\delta > 0$ such that

$$S_X\left(h - \limsup_j h_j - \varphi\right) = S_X\left(h - \liminf_j h_j - \varphi\right) \geq \delta,$$

whenever $h \in \mathrm{co}_\sigma\{h_j : \ j \geq 1\}$. Then there exists a sequence $\{g_i\}_{i\geq 1}$ in $\ell^\infty(X)$ with

$$g_i \in \mathrm{co}_\sigma\{h_j : \ j \geq i\}, \quad \text{for all } i \geq 1,$$

and there exists $g_0 \in \mathrm{co}_\sigma\{g_i : \ i \geq 1\}$ such that for all $g \in \ell^\infty(X)$ with

$$\liminf_i g_i \leq g \leq \limsup_i g_i \quad \text{on } X,$$

the function $g_0 - g - \varphi$ doest not attain its supremum on $X$ .

The proof given in [133] for the above result involves an adaptation of the additive diagonal lemma we have used for Simons' inequality in $\mathbb{R}^X$, Theorem 1.5. Let us include here a proof for the following consequence, that was stated first in this way by James in [87, Theorems 2 and 4].

**Theorem 1.15 (James).** Let $A$ be a nonempty bounded subset of a Banach space $E$ which is not weakly relatively compact. Then there exists a sequence $\{g^*_n\}_{n\geq 1}$ in $B_{E^*}$ and some $g_0 \in \mathrm{co}_\sigma\{g^*_n : \ n \geq 1\}$ such that, for every $h \in \ell^\infty(A)$ with

$$\liminf_n g^*_n \leq h \leq \limsup_n g^*_n \quad \text{on } A,$$

we have that $g_0 - h$ does not attain its supremum on $A$.

*Proof.* Without loss of generality we can assume that $A$ is convex and that $A = -A$. Proposition 1.12 gives us a sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ such that $\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) > 0$. By Proposition 1.13 there exists a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ that verifies the hypothesis of Theorem 1.14 with $\varphi = 0$. So we find a sequence $\{g_n^*\}_{n\geq 1}$ with $g_n^* \in \mathrm{co}_\sigma\{x_{n_k}^* : k \geq n\}$, for every $n \in \mathbb{N}$, and $g_0 \in \mathrm{co}_\sigma\{g_n^* : n \geq 1\}$ such that $g_0 - h$ doest not attain its supremum on $A$, where $h$ is any function in $\ell^\infty(A)$ with $\liminf_n g_n^* \leq h \leq \limsup_n g_n^*$ on $A$.

□

In particular we have seen how to construct linear functionals $g_0 - g$ that do not attain their supremum on $A$, whenever $g$ is a $w^*$-cluster point of the sequence $\{g_n^*\}_{n\geq 1}$ in $B_{E^*}$.

We finish this section with a short visit to the so-called *measures of weak noncompactness* in Banach spaces: the relationship of these measures with the techniques already presented in this survey will be plain clear when progressing in our discussion below.

We refer the interested reader to [14, 105], where measures of weak noncompactness are axiomatically defined. A measure of weak noncompactness is a non-negative function $\mu$ defined on the family $\mathscr{M}_E$ of bounded subsets of a Banach space $E$, with the following properties:

(i) $\mu(A) = 0$ if, and only if, $A$ is weakly relatively compact in $E$,
(ii) if $A \subset B$ then $\mu(A) \leq \mu(B)$,
(iii) $\mu(\mathrm{conv}(A)) = \mu(A)$,
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$.

Inspired by Proposition 1.12, we introduce the following:

**Definition 1.16.** For a bounded subset $A$ of a Banach space $E$, $\sigma(A)$ stands for the quantity

$$\sup_{\{x_n^*\}_{n\geq 1} \subset B_{E^*}} \mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}).$$

Observe that $\sigma$ satisfies properties (i), (ii), (iii), (iv) and (vi), and therefore $\sigma$ can be considered as a measure of weak noncompactness. Beyond the formalities we will refer *in general* to measures of weak noncompactness to quantities as above fulfilling property (i), and sometimes a few of the others. These measures of noncompactness or weak noncompactness have been successfully applied to the study of compactness, operator theory, differential equations and integral equations, see for instance [10, 11, 12, 20, 31, 33, 51, 65, 64, 69, 103, 105, 104].

The next definition collects several measures of weak noncompactness that appeared in the aforementioned literature. If $A$ and $B$ are nonempty subsets of $E^{**}$, then $d(A,B)$ denotes the *usual* inf *distance* (associated to the bidual norm) between $A$ and $B$, and the *Hausdorff non-symmetrized distance* from $A$ to $B$ is defined by

$$\widehat{\mathrm{d}}(A,B) = \sup\{d(a,B) : a \in A\}.$$

Notice that $\widehat{\mathrm{d}}(A,B)$ can be different from $\widehat{\mathrm{d}}(B,A)$, and that $\max\{\widehat{\mathrm{d}}(A,B),\widehat{\mathrm{d}}(B,A)\}$ is the *Hausdorff distance* between $A$ and $B$. Notice further that $\widehat{\mathrm{d}}(A,B)=0$ if, and only if, $A\subset\overline{B}$ (norm-closure) and that

$$\widehat{\mathrm{d}}(A,B)=\inf\{\varepsilon>0: A\subset B+\varepsilon B_{E^{**}}\}.$$

**Definition 1.17.** Given a bounded subset $A$ of a Banach space $E$ we define:

$$\omega(A):=\inf\{\varepsilon>0: A\subset K_\varepsilon+\varepsilon B_E \text{ and } K_\varepsilon\subset E \text{ is } w\text{-compact}\},$$

$$\gamma(A):=\sup\{|\lim_n\lim_m x_m^*(x_n)-\lim_m\lim_n x_m^*(x_n)|:\{x_m^*\}_{m\geq 1}\subset B_{E^*},\{x_n\}_{n\geq 1}\subset A\},$$

assuming the involved limits exist,

$$\mathrm{ck}_{\mathrm{E}}(A):=\sup_{\{x_n\}_{n\geq 1}\subset A} d(L_{E^{**}}\{x_n\},E),$$

$$\mathrm{k}(A):=\widehat{\mathrm{d}}(\overline{A}^{w^*},E)=\sup_{x^{**}\in\overline{A}^{w^*}} d(x^{**},E),$$

and

$$\begin{aligned}\mathrm{Ja}_{\mathrm{E}}(A):=\inf\{\varepsilon>0:\ &\text{for every } x^*\in E^*, \text{ there exists } x^{**}\in\overline{A}^{w^*}\\ &\text{such that } x^{**}(x^*)=S_A(x^*) \text{ and } d(x^{**},E)\leq\varepsilon\}.\end{aligned}$$

The function $\omega$ was introduced by de Blasi [20] as a measure of weak noncompactness that is somehow the counterpart for the weak topology of the classical Kuratowski measure of norm noncompactness. Properties for $\gamma$ can be found in [11, 12, 33, 51, 105] and for $\mathrm{ck}_{\mathrm{E}}$ in [11] –note that $\mathrm{ck}_{\mathrm{E}}$ is denoted as ck in that paper. The quantity k has been used in [11, 33, 51, 64]. A thorough study for $\mathrm{Ja}_{\mathrm{E}}$ has been done in [31] to prove, amongst other things, a quantitative version of James' weak compactness theorem, whose statement is presented as part of Theorem 1.18 bellow. This theorem tells us that *all classical approaches used to study weak compactness in Banach spaces (Tychonoff's theorem, Eberlein–Šmulian's theorem, Eberlein–Grothendieck double-limit criterion and James' weak compactness theorem) are qualitatively and quantitatively equivalent.*

**Theorem 1.18.** For any bounded subset $A$ of a Banach space $E$ the following inequalities hold true:

$$\begin{array}{ccccccccc} & & & & \sigma(A) & \leq & 2\omega(A) & & \\ & & & \nearrow & & & \vee\!\mid & & \\ \frac{1}{2}\gamma(A) & \leq & \mathrm{Ja}_{\mathrm{E}}(A)\leq \mathrm{ck}_E(A) & \leq & k(A) & \leq & \gamma(A). & & \end{array} \tag{1.17}$$

Moreover, for any $x^{**}\in\overline{A}^{w^*}$ there exists a sequence $\{x_n\}_{n\geq 1}$ in $A$ such that

$$\|x^{**}-y^{**}\|\leq\gamma(A) \tag{1.18}$$

for any $w^*$-cluster point $y^{**}$ of $\{x_n\}_{n\geq 1}$ in $E^{**}$.

Furthermore, $A$ is weakly relatively compact in $E$ if, and only if, one (equivalently, all) of the numbers $\gamma(A), \mathrm{Ja_E}(A), \mathrm{ck}_E(A), \mathrm{k}(A), \sigma(A)$ and $\omega(A)$ is zero.

*Proof.* A full proof with references to prior work for the inequalities

$$\frac{1}{2}\gamma(A) \leq \mathrm{ck}_E(A) \leq k(A) \leq \gamma(A) \leq 2\omega(A)$$

and (1.18) is provided in [11, Theorem 2.3]. The inequalities

$$\frac{1}{2}\gamma(A) \leq \mathrm{Ja_E}(A) \leq \mathrm{ck}_E(A)$$

are established in Theorem 3.1 and Proposition 2.2 of [31].

To prove $\mathrm{ck}_E(A) \leq \sigma(A)$ we proceed as follows. If $0 = ck_E(A)$, the inequality is clear. Assume that $0 < \mathrm{ck}_E(A)$ and take an arbitrary $0 < \alpha < \mathrm{ck}_E(A)$. By the very definition of $\mathrm{ck}_E(A)$ there exist a sequence $\{x_n\}_{n\geq 1}$ in $A$ and a $w^*$-cluster point $x_0^{**} \in E^{**}$ with $d(x_0^{**}, E) > \alpha > 0$. If we read now the second part of the proof of Proposition 1.12, we end up producing a sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ that according to inequality (1.15) satisfies

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) \geq \alpha.$$

Since $\alpha$ with $0 < \alpha < \mathrm{ck}_E(A)$ is arbitrary, the above inequality yields $\mathrm{ck}_E(A) \leq \sigma(A)$.

To complete the chain of inequalities we establish $\sigma(A) \leq 2\omega(A)$. Let $\omega(A) < \varepsilon$ and take a weakly compact subset $K_\varepsilon$ of $E$ such that $A \subset K_\varepsilon + \varepsilon B_E$. This inclusion leads to the inequality

$$\|\cdot\|_A \leq \|\cdot\|_{K_\varepsilon} + \varepsilon\|\cdot\|. \tag{1.19}$$

Fix an arbitrary sequence $\{x_n^*\}_{n\geq 1}$ in $B_{E^*}$ and now take a $w^*$-cluster point $x_0^* \in L\{x_n^*\}$. Since $K_\varepsilon$ is weakly compact we know that $x_0^* \in \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{\|\cdot\|_{K_\varepsilon}}$. Hence, for an arbitrary $\eta > 0$ we can find a convex combination $\sum_{i=1}^n \lambda_i x_{n_i}^*$ with $\|x_0^* - \sum_{i=1}^n \lambda_i x_{n_i}^*\|_{K_\varepsilon} < \eta$. Thus, inequality (1.19) allows us to conclude that

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) \leq \left\|x_0^* - \sum_{i=1}^n \lambda_i x_{n_i}^*\right\|_A \leq$$
$$\leq \left\|x_0^* - \sum_{i=1}^n \lambda_i x_{n_i}^*\right\|_{K_\varepsilon} + \varepsilon\left\|x_0^* - \sum_{i=1}^n \lambda_i x_{n_i}^*\right\| \leq \eta + 2\varepsilon.$$

Since $\varepsilon, \eta$ and $\{x_n^*\}_{n\geq 1}$ are arbitrary, we conclude $\sigma(A) \leq 2\omega(A)$.

Finally, recall a well-known result of Grothendieck [46, Lemma 2, p. 227] stating that $\omega(A) = 0$ if, and only if, $A$ is weakly relatively compact in $E$. Observe that, as a consequence of (1.17), one of the numbers $\gamma(A), \mathrm{Ja_E}(A), \mathrm{ck}_E(A), \mathrm{k}(A)$ is zero if, and only if, all of them are zero. Clearly, $\mathrm{k}(A) = 0$ if, and only if, $\overline{A}^{w^*} \subset E$, that is equivalent to the fact that $A$ is weakly relatively compact by Tychonoff's theorem. To establish $\sigma(A)=0$ if, and only if, $A$ is weakly relatively compact either use Proposition 1.12 or the comments above for $\omega$ and $\mathrm{ck}_E$, together with the inequalities $\mathrm{ck}_E(A)\le\sigma(A)\le\omega(A)$. The proof is over.

□

It is worth noticing that the inequalities

$$\mathrm{ck}_E(A)\le k(A)\le 2\,\mathrm{ck}_E(A),$$

that follow from (1.17), offer a quantitative version (and imply) of the Eberlein–Šmulian theorem saying that weakly relatively countably compact sets in Banach spaces are weakly relatively compact. Note also that (1.18) implies that points in the weak closure of a weakly relatively compact set of a Banach space are reachable by weakly convergent sequences from within the set (summing up, the inequalities are a *quantitative* version of the angelicity of weakly compact sets in Banach spaces, see Definition 1.19). In a different order of ideas the inequality

$$\frac{1}{2}\gamma(A)\le \mathrm{Ja_E}(A) \tag{1.20}$$

implies James' weak compactness theorem, Theorem 1.1, and since $\mathrm{Ja_E}(A)\le \mathrm{ck}_E(A)$ as well, we therefore know that James' weak compactness theorem can be derived and implies the other classical results about weak compactness in Banach spaces. We should mention that the proof of inequality (1.20) in [31, Theorem 3.1] follows the arguments by Pryce in [125] suitably adapted and strengthened for the occasion: assuming that $0<r<\gamma(A)$, two sequences $\{x_n\}_{n\ge1}\subset A$ and $\{x_m^*\}_{m\ge1}\subset B_{E^*}$ are produced satisfying

$$\lim_m\lim_n x_m^*(x_n)-\lim_n\lim_m x_m^*(x_n)>r.$$

Then Lemma 1.2 is applied to the sequence $\{x_m^*\}_{m\ge1}$, and after some twisting and fine adjustments in Pryce's original arguments, for arbitrary $0<r'<r$ a sequence $\{g_n^*\}_{n\ge1}$ in $B_{E^*}$ and $g_0\in\mathrm{co}_\sigma\{g_n^*:\ n\ge1\}$ are produced with the property that for any $w^*$-cluster point $h\in B_{E*}$ of $\{g_n^*\}_{n\ge1}$, if $x^{**}\in\overline{A}^{w^*}$ is such that

$$x^{**}(g_0-h)=S_A(g_0-h)$$

then $d(x^{**},E)\ge\frac{1}{2}r'$. Since $0<r<\gamma(A)$ and $r'\in(0,r)$ are arbitrary the inequality (1.20) follows. Of course, $g_0-h\in E^*$ does not attain its supremum on $A$ but we moreover know how far from $E$ in $\overline{A}^{w^*}$ we need to go in order that $g_0-h$ might attain it: compare with Theorem 1.15.

The aforementioned references contain examples showing when the inequalities in (1.17) are sharp, as well as sufficient conditions of when the inequalities become equalities. An example of the latter is given in the theorem below, where we use the notion of angelic space that follows.

**Definition 1.19 (Fremlin).** A regular topological space $T$ is *angelic* if every relatively countably compact subset $A$ of $T$ is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from $A$.

In angelic spaces the different concepts of compactness and relative compactness coincide: the (relatively) countably compact, (relatively) compact and (relatively) sequentially compact subsets are the same, as seen in [53]. Examples of angelic spaces include $C(K)$ endowed with the topology $t_p(K)$ of pointwise convergence on a countably compact space $K$ ([71, 96]) and all Banach spaces in their weak topologies. Another class of angelic spaces are dual spaces of weakly countably $K$-determined Banach spaces, endowed with their $w^*$-topology, [117].

**Theorem 1.20 ([31, Theorem 6.1]).** Let $E$ be a Banach space such that $(B_{E^*}, w^*)$ is angelic. Then for any bounded subset $A$ of $E$ we have

$$\frac{1}{2}\gamma(A) \leq \gamma_0(A) = \mathrm{Ja_E}(A) = \mathrm{ck_E}(A) = \mathrm{k}(A) \leq \gamma(A),$$

where

$$\gamma_0(A) := \sup\{|\lim_i \lim_j x_i^*(x_j)| : \{x_j\}_{j\geq 1} \subset A, \{x_i^*\}_{i\geq 1} \subset B_{E^*}, x_i^* \stackrel{w^*}{\to} 0\}.$$

A moment of thought and the help of Riesz's lemma suffice to conclude that for the unit ball $B_E$ we have that

$$k(B_E) = \sup_{x^{**}\in B_{E^{**}}} d(x^{**}, E) \in \{0, 1\}.$$

Reflexivity of $E$ is equivalent to $k(B_E) = 0$ and non reflexivity to $k(B_E) = 1$. Note then that, when $(B_{E^*}, w^*)$ is angelic, reflexivity of $E$ is equivalent to $\mathrm{Ja_E}(B_E) = 0$, and non reflexivity to $\mathrm{Ja_E}(B_E) = 1$. In other words, James' reflexivity theorem can be strengthened to: *If there exists $0 < \varepsilon < 1$ such that for every $x^* \in E^*$ there exists $x^{**} \in B_{E^{**}}$ with $d(x^{**}, E) \leq \varepsilon$ and $S_{B_E}(x^*) = x^{**}(x^*)$, then $E$ is reflexive.* Indeed, the above comments provide a proof of this result when $(B_{E^*}, w^*)$ is angelic; for the general case we refer to [67].

With regard to convex hulls, the quantities in Theorem 1.18 behave quite differently. Indeed, if $A$ is a bounded set of a Banach space $E$, then the followings statements hold:

$$\begin{aligned}
&\gamma(\mathrm{co}(A)) = \gamma(A), \quad \mathrm{Ja_E}(\mathrm{co}(A)) \leq \mathrm{Ja_E}(A);\\
&\mathrm{ck}_E(\mathrm{co}(A)) \leq 2\,\mathrm{ck}_E(A), \quad \mathrm{k}(\mathrm{co}(A)) \leq 2\,\mathrm{k}(A);\\
&\sigma(\mathrm{co}(A)) = \sigma(A), \quad \omega(\mathrm{co}(A)) = \omega(A).
\end{aligned}$$

Constant 2 for $\mathrm{ck}_E$ and $r$ is sharp, [31, 64, 65], and it is unknown if $\mathrm{Ja_E}$ might really decrease when passing to convex hulls. The equality $\gamma(A) = \gamma(\mathrm{co}(A))$ is a bit delicate and has been established in [33, 51].

Last, but not least, we present yet another measure of weak noncompactness inspired by James' ideas in [85]. Following [105], for a given bounded sequence

$\{x_n\}_{n\geq 1}$ in a Banach space we define

$$\mathrm{csep}(\{x_n\}_{n\geq 1}) := \inf\{\|u_1 - u_2\| : (u_1, u_2) \in \mathrm{scc}(\{x_n\}_{n\geq 1})\},$$

where

$$\mathrm{scc}(\{x_n\}_{n\geq 1}) := \{(u_1, u_2) : u_1 \in \mathrm{conv}\{x_i\}_{1\leq i\leq m}, u_2 \in \mathrm{conv}\{x_i\}_{i\geq m+1}, m \in \mathbb{N}\}.$$

**Definition 1.21 ([105, Definition 2.2]).** If $A$ is a bounded subset of a Banach space, we define

$$\alpha(A) := \sup\{\mathrm{csep}(\{x_n\}_{n\geq 1}) : \{x_n\}_{n\geq 1} \subset A\}.$$

It is proved in [105] that the relationship of $\alpha$ with the measures of weak noncompactness already presented are given by the formulas:

$$\alpha(A) = \sup\left\{d(x^{**}, \mathrm{conv}\{x_n : n \geq 1\} : \{x_n\}_{n\geq 1} \subset A, x^{**} \in L_{E^{**}}\{x_n\}\right\}$$

and

$$\gamma(A) = \alpha(\mathrm{conv}(A)).$$

For the measure of weak noncompactness $\sigma$ introduced in Definition 1.16, and in view of Theorem 1.18, the following question naturally arises:

**Question 2.** With regard to the measure of weak noncompactness $\sigma$, are the derived estimates sharp? Is it equivalent to the others (except $\omega$)?

## 1.4 Boundaries

Given a $w^*$-compact subset $C$ of $E^*$, a *boundary* for $C$ is a subset $B$ of $C$ with the property that

$$\text{for every } x \in E \text{ there exists some } b^* \in B \text{ such that } b^*(x) = \sup\{c^*(x) : c^* \in C\}.$$

Note that if $C$ is moreover convex, then the Hahn-Banach theorem shows that $\overline{\mathrm{co}(B)}^{w^*} = C$. In addition, the set $\mathrm{ext}(C)$ of the extreme points of $C$ is a boundary for $C$, thanks to Bauer's maximum principle, see [53, p. 6], and therefore also satisfies $C = \overline{\mathrm{co}(\mathrm{ext}(C))}^{w^*}$. Note that Milman's theorem [46, Corollary IX.4] tells us that $\mathrm{ext}(C) \subset \overline{B}^{w^*}$. Nonetheless, in general, boundaries can be disjoint of the set of extreme points as the following example shows: let $\Gamma$ be a non countable set and consider $\left(\ell^1(\Gamma), \|\cdot\|_1\right)$ and

$$B := \left\{(x_\gamma)_{\gamma\in\Gamma} : x_\gamma \in \{-1, 0, 1\} \text{ and } \{\gamma \in \Gamma : x_\gamma \neq 0\} \text{ is countable}\right\}.$$

A moment of thought suffices to conclude that $B$ is a boundary for the dual unit ball $B_{\ell^\infty(\Gamma)}$ that is clearly disjoint from $\mathrm{ext}\big(B_{\ell^\infty(\Gamma)}\big)$, see [136, Example 7].

If $B$ is a boundary for $B_{E^*}$ we will say that $B$ is a *boundary for $E$*.

Two problems regarding boundaries in Banach spaces have attracted the attention of a good number of authors during the years, namely:

The study of *strong* boundaries. The goal here is to find conditions under which a boundary $B$ for the $w^*$-compact convex $C$ is *strong*, i.e., $\overline{\mathrm{co}(B)}^{\|\cdot\|} = C$.

The boundary problem. Let $E$ be a Banach space, let $B$ be a boundary for $E$ and let $A$ be a bounded and $\sigma(E,B)$-compact subset of $E$. Is $A$ weakly compact? (Godefroy, [59, Question V.2]).

At first glance, the two questions above may look unrelated. They are not. Indeed, on the one hand, the boundary problem has an easy and positive answer for all strong boundaries $B$ in $B_{E^*}$. On the other hand, many studies about strong boundaries and several partial answers to the boundary problem use Simons' inequality as a tool. Regarding strong boundaries, the following references are a good source for information [34, 29, 39, 45, 50, 56, 55, 59, 61, 77, 78, 88, 123, 130, 149]. At the end of this section we will provide some recent results on strong boundaries.

Let us start by considering the boundary problem. It has been recently solved in full generality in the paper [122]. It is interesting to recall the old roots and the long history of the problem.

The first result that provided a partial positive result to the boundary problem (before its formulation as such a question) was the following characterization of weak compactness in continuous function spaces, due to Grothendieck, see [71, Théorème 5]:

**Theorem 1.22.** If $K$ is a Hausdorff and compact topological space and $A$ is a subset of $C(K)$, then $A$ is weakly compact if, and only if, it is bounded and compact for the topology of the pointwise convergence on $K$.

More generally, Theorem 1.22 was generalized by Bourgain and Talagrand [24, Théorème 1] in the following terms:

**Theorem 1.23.** Let $E$ be a Banach space, $B = \mathrm{ext}(B_{E^*})$ and let $A$ be a bounded and $\sigma(E,B)$-compact subset of $E$. Then $A$ is weakly compact.

Note that the result of Bourgain and Talagrand is far from being a full solution to the boundary problem, because as presented above there are examples of boundaries of Banach spaces that do not contain any extreme point.

Bearing in mind the Rainwater–Simons theorem, Corollary 1.7, it is easy to give another partial solution to the boundary problem.

**Corollary 1.24.** For any separable Banach space $E$ and any boundary for $E$, the boundary problem has positive answer.

*Proof.* Let $B$ be a boundary for $E$ and let $A$ be a bounded and $\sigma(E,B)$-compact subset of $E$. Since $E$ is separable, the unit ball $(B_{E^*}, w^*)$ is metrizable and separable. It follows that $B$ is $w^*$-separable. Take $D$ a countable and $w^*$-dense subset of $B$. The topology $\sigma(E,D)$ is then Hausdorff, metrizable and coarser than $\sigma(E,B)$. Consequently we obtain that $\sigma(E,D)$ and $\sigma(E,B)$ coincide when restricted to $A$ and we conclude that $(A,\sigma(E,B))$ is sequentially compact. An application of Corollary 1.7 taking into account the Eberlein-Šmulian theorem gives us that $A$ is weakly compact, which concludes the proof.

$\square$

A first approach to the next result appears implicitly in [136, Theorem 5]. Using the ideas of Pryce in [125] and those of Rodé on the so-called "superconvex analysis" in [129], H. Konig formulated it in [101, Theorem 5.2, p. 104]. We present here our approach based on the criteria given by Theorem 1.14.

**Theorem 1.25.** Let $E$ be a Banach space and $B(\subset B_{E^*})$ a boundary for $E$. If $A$ is a bounded convex subset of $E$ such that for every sequence $\{a_n\}_{n\geq 1}$ in $A$ there exists $z\in E$ such that

$$\liminf_n \langle a_n,b^*\rangle \leq \langle z,b^*\rangle \leq \limsup_n \langle a_n,b^*\rangle \tag{1.21}$$

for every $b^*\in B$, then $A$ is weakly relatively compact.

*Proof.* Let us proceed by contradiction and assume that $A$ is not weakly relatively compact in $E$. Then the Eberlein-Šmulian theorem says that there exists a sequence $\{a_n\}_{n\geq 1}\subset A$ without weak cluster points in $E$. According to Pryce's diagonal argument, Lemma 1.2, we can and do assume that

$$S_B\left(a-\liminf_n a_n\right)=S_B\left(a-\liminf_k a_{n_k}\right)=S_B\left(a-\limsup_k a_{n_k}\right)=S_B\left(a-\limsup_n a_n\right)$$

for every $a\in \mathrm{co}_\sigma\{a_n : n\geq 1\}$ and every subsequence of integers $n_1<n_2<\cdots$.

Let us fix $x_0\in E$ such that for every $b^*\in B$

$$\liminf\langle a_n,b^*\rangle \leq \langle x_0,b^*\rangle \leq \limsup\langle a_n,b^*\rangle.$$

Keeping in mind that $A$ is $w^*$-relatively compact in $E^{**}$, we know that $\{a_n\}_{n\geq 1}$ has a $w^*$-cluster point $x_0^{**}\in E^{**}\setminus E$. Let us fix $h^*\in B_{E^*}$ and $\xi\in\mathbb{R}$ such that

$$h^*(x_0)<\xi<h^*(x_0^{**}).$$

Since $h^*(x_0^{**})$ is a cluster point of the sequence $\{h^*(a_n)\}_{n\geq 1}$, then there exists a subsequence $\{a_{n_k}\}_{k\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $h^*(a_{n_k})>\xi$ for every $k\geq 1$. Thus we also have $h^*(a)\geq\xi$ for every $a\in \mathrm{co}_\sigma\{a_{n_k} : k\geq 1\}$. Consequently we have that

$$\begin{aligned} S_B\left(a-\liminf_n a_n\right)=S_B\left(a-\liminf_k a_{n_k}\right)=S_B\left(a-\limsup_k a_{n_k}\right)=\\ =S_B\left(a-\limsup_n a_n\right)=S_B\left(a-x_0\right)=S_{B_{E^*}}\left(a-x_0\right)\geq\\ \geq h^*(a)-h^*(x_0)\geq\xi-h^*(x_0)>0\end{aligned}$$

for every $a\in \mathrm{co}_\sigma\{a_{n_k} : k\geq 1\}$. We can apply now Theorem 1.14 with $X:=B$, $\varphi=0$ and $\{h_j\}_{j\geq 1}$ being $\{a_{n_k}\}_{k\geq 1}$ to get a sequence $\{y_i\}_{i\geq 1}$ such that for all $i\geq 1$,

$y_i \in \mathrm{co}_\sigma\{a_{n_j} : j \geq i\}$, together with some $y_0 \in \mathrm{co}_\sigma\{y_i : i \geq 1\}$, in such a way that $y_0 - y$ doest not attain its supremum on $B$ for any $y$ with

$$\liminf_i y_i(b^*) \leq y(b^*) \leq \limsup_i y_i(b^*), \quad \text{for all } b^* \in B.$$

Given $i \geq 1$, since $y_i \in \overline{\mathrm{co}}^{\|\cdot\|}\{a_{n_j} : j \geq i\}$ we can pick up $z_i \in \mathrm{co}\{a_{n_j} : j \geq i\}$ with $\|y_i - z_i\|_\infty < 2^{-i}$. Note that the convexity of $A$ implies $z_i \in A$ for every $i \geq 1$. But our hypothesis provide us with some $z \in E$ such that

$$\liminf_i y_i(b^*) = \liminf_i z_i(b^*) \leq z(b^*) \leq \limsup_i z_i(b^*) = \limsup_i y_i(b^*)$$

for every $b^* \in B$. Thus we have that $y_0 - z \in E$ doest not attain its norm on $B$, which contradicts that $B$ is a boundary for $E$ and the proof is over.

□

The following result straightforwardly follows from Theorem 1.25.

**Theorem 1.26.** Let $E$ be a Banach space and $B(\subset B_{E^*})$ a boundary for $E$. If $A$ is a convex bounded and $\sigma(E,B)$-relatively countably compact subset of $E$, then it is weakly relatively compact.

*Proof.* It suffices to note that if $A$ is $\sigma(E,B)$-relatively countably compact in $E$, then for any given sequence $\{a_n\}_{n\geq 1}$ in $A$ and each $\sigma(E,B)$-cluster point $z \in E$ of it, $z$ satisfies the inequalities in (1.21). Then Theorem 1.25 applies and the proof is over.

□

A different proof for Theorem 1.26, even in a more general setting, can be found in [53, Corollary 3, p. 78]: the arguments for this proof go back to the construction of norm-nonattaining functionals in Pryce's proof of James' weak compactness theorem. A different proof by Godefroy appeared in [60, Proposition II.21] (this proof has been rewritten in [50, Theorem 3.140]).

Theorem 1.26 opens another door for positive answers to the boundary problem as long as for the given boundary $B(\subset B_{E^*})$ for $E$ and the norm-bounded $\sigma(E,B)$-compact set $A(\subset E)$ we have that $\overline{\mathrm{co}(A)}^{\sigma(E,B)} \subset E$ is $\sigma(E,B)$-compact. In other words, the boundary problem would have a positive answer subject to the locally convex space $(E,\sigma(E,B))$ satisfies Krein-Šmulian's property just mentioned. Note though, that the classical Krein-Šmulian theorem only works for locally convex topologies in between the weak and the norm-topology of $E$ and that $\sigma(E,B)$ can be strictly coarser than the weak topology, [102, §24]. Positive results along this direction were established in [36, 32] and [35].

Recall that a subset $B$ of $B_{E^*}$ is said to be *norming* (resp. 1-*norming*) if

$$|||x||| = \sup\{|b^*(x)| : b^* \in B\}$$

is a norm in $E$ equivalent (resp. equal) to the original norm of $E$. Particularly, if $B(\subset B_{E^*})$ is a boundary for $E$ then $B$ is 1-norming.

The three results that follow are the set up to address the boundary problem from the point of view of the existence of isomorphic copies of the basis of $\ell^1(\mathbb{R})$. A proof for these results can be found in [32] (see also [35]).

**Theorem 1.27 (Krein-Šmulian type result).** Let $E$ be a Banach space and let $B$ be a norming subset of $B_{E^*}$. If $E$ does not contain an isomorphic copy of $\ell^1(\mathbb{R})$, then the $\sigma(E,B)$-closed convex hull of every bounded $\sigma(E,B)$-relatively compact subset of $X$ is $\sigma(E,B)$-compact.

**Corollary 1.28.** Let $E$ be a Banach space which does not contain an isomorphic copy of $\ell^1(\mathbb{R})$ and let $B(\subset B_{E^*})$ be a boundary for $E$. Then, every bounded $\sigma(E,B)$-compact subset of $E$ is weakly compact.

**Theorem 1.29.** Let $E$ be a Banach, $B(\subset B_{E^*})$ a boundary for $E$ and let $A$ be a bounded subset of $E$. Then, the following statements are equivalent:

(i) $A$ is weakly compact,
(ii)$A$ is $\sigma(E,B)$-compact and does not contain a family $(x_\alpha)_{\alpha\in\mathbb{R}}$ equivalent to the usual basis of $\ell^1(\mathbb{R})$.

Note that Theorems 1.27 and 1.26 straightforwardly imply Corollary 1.28. Theorem 1.27 is of interest by itself. The original proof for this result in [32] uses techniques of Pettis integration together with fine subtleties about independent families of sets in the sense of Rosenthal. Other proofs are available as for instance in [35, 68], where it is established that if for the Banach space $E$ the Krein-Šmulian property in Theorem 1.27 holds true for any norming set $B(\subset B_{E^*})$ then $E$ cannot contain isomorphically $\ell^1(\mathbb{R})$ (see also [21] for related results).

It is worth mentioning a few things about the class of Banach spaces not containing isomorphic copies of $\ell^1(\mathbb{R})$. Good references for this class of Banach spaces are [106], [79] and [145]. On the one hand, a Banach space $E$ does not contain isomorphically $\ell^1(\mathbb{R})$ if, and only if, $\ell^\infty(\mathbb{N})$ is not a quotient of $E$, [120, Lemma 4.2]. On the other hand, $E$ does not admit $\ell^\infty(\mathbb{N})$ as a quotient if, and only if, the dual unit ball $(B_{X^*},w^*)$ does not contain a homeomorphic copy of the Stone-Čech compactification of the natural numbers, $\beta\mathbb{N}$, [145]. In particular each one of the following classes of Banach spaces are made up of spaces which do not contain isomorphically $\ell^1(\mathbb{R})$:

a) Banach spaces with a weak$^*$-sequentially compact dual unit ball,
b) Banach spaces which are Lindelöf for their weak topologies, or more in general, Banach spaces with the property $(\mathscr{C})$ of Corson.

Recall that $E$ has *property* $(\mathscr{C})$, see [124], if every family of convex closed subsets of it with empty intersection has a countable subfamily with empty intersection.

Finally, the positive answer to the boundary problem due to Pfitzner, see [122, Theorem 9], is formulated as follows:

**Theorem 1.30 (Pfitzner).** Let A be a bounded set in a Banach space $E$ and let $B(\subset E^*)$ be a boundary of a $w^*$-compact subset $C$ of $E^*$. If $A$ is $\sigma(E,B)$-countably compact then A is $\sigma(E,C)$-sequentially compact. In particular, if $B$ is a boundary for $E$, then a bounded subset of $E$ is weakly compact if, and only if, it is $\sigma(E,B)$-compact.

In the proof of this fine result, Pfitzner does a localized analysis on $A$ that goes beyond Theorem 1.29 and involves the quantitative version of Rosenthal's $\ell^1$-theorem in [17], Simons' inequality, and a modification of a result of Hagler and Johnson in [72].

Although Theorem 1.30 answers in full generality the boundary problem a few open problems still remain. For instance, it is unknown if given a boundary $B$ ($\subset B_{E^*}$) for $E$, the topology $\sigma(E,B)$ is angelic on bounded subsets of $E$. A few comments are needed here. We first note that since in angelic spaces compact subsets are sequentially compact, [53], when $\sigma(E,B)$ is angelic on bounded subsets of $E$, a positive answer to the boundary problem is easily given as a consequence of Rainwater–Simons' theorem, Corollary 1.7 –see Corollary 1.24 as illustration. In general it is not true that $(E,\sigma(E,B))$ is angelic, see [141, Theorem 1.1 (b)]: an $L^1$-predual $E$ is constructed together with a $\sigma(E,\text{ext}(B_{E^*})$-countably compact set $A \subset E$ for which not every point $x \in \overline{A}^{\sigma(E,\text{ext}(B_{E^*})}$ is the $\sigma(E,\text{ext}(B_{E^*})$-limit of a sequence in $A$ (see also [110]). Nonetheless there are cases where angelicity of $\sigma(E,B)$ (or $\sigma(E,B)$ on bounded sets) is known, and therefore for these cases a stronger positive answer to the boundary problem is provided. One of this cases is presented in [24] where it is proved that for any Banach space $E$ the topology $\sigma(E,\text{ext}(B_{E^*}))$ is angelic on bounded sets –compared with [141, Theorem 1.1 (b)]. Two more of these positive cases are presented below in Theorem 1.33 and Theorem 1.34.

The proof of Theorem 1.33 needs the two lemmas that follow. The first one, see [35, Lemma 4.5], that implicitly appears in a particular case in [30], can be consider as a kind of strong version of an "Angelic Lemma" in the spirit of [53, Lemma in p. 28].

**Lemma 1.31.** Let $X$ be a nonempty set and $\tau$, $\mathfrak{T}$ two Hausdorff topologies on $X$ such that $(X,\tau)$ is regular and $(X,\mathfrak{T})$ is angelic. Assume that for every sequence $\{x_n\}_{n\geq 1}$ in $X$ with a $\tau$-cluster point $x \in X$, $x$ is $\mathfrak{T}$-cluster point of $\{x_n\}_{n\geq 1}$. The following assertions hold true:

(i) If $L$ is a $\tau$-relatively countably compact subset of $X$, then $L$ is $\mathfrak{T}$-relatively compact.
(ii) If $L$ is a $\tau$-compact subset of $X$, then $L$ is $\mathfrak{T}$- compact.
(iii) $(X,\tau)$ is an angelic space.

The lemma below, see [30, Lemma 1] and [35, Lemma 4.7], evokes properties of the real-compactification (also called the *repletion*) of a topological space, cf. [53, §4.6].

**Lemma 1.32.** Let $K$ be a compact space and $B(\subset B_{C(K)^*})$ a boundary for the Banach space $(C(K),\|\cdot\|_\infty)$. If $\{f_n\}_{n\geq 1}$ is an arbitrary sequence in $C(K)$ and $x \in K$, then there exists $\mu \in B$ such that

$$f_n(x) = \int_K f_n d\mu$$

for every $n \geq 1$.

*Proof.* If we define the continuous function $g : K \to [0,1]$ by the expression

$$g(t) := 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(t) - f_n(x)|}{1 + |f_n(t) - f_n(x)|}, \qquad (t \in K),$$

then

$$F := \bigcap_{n=1}^{\infty} \{y \in K : f_n(y) = f_n(x)\} = \{y \in K : g(y) = 1 = \|g\|_\infty\}. \tag{1.22}$$

Since $B$ is a boundary, there exists $\mu \in B$ such that $\int_K g d\mu = 1$. So we arrive at

$$1 = \|\mu\| = |\mu|(K) \geq \int_K g d|\mu| \geq \int_K g d\mu = 1, \tag{1.23}$$

in other words,

$$0 = |\mu|(K) - \int_K g d|\mu| = \int_K (1-g) d|\mu|.$$

Since $1 - g \geq 0$ we obtain $|\mu|(\{y \in K : 1 - g(y) > 0\}) = 0$, that is $|\mu|(K \setminus F) = 0$. Therefore, for every $n \in \mathbb{N}$ we have

$$\int_K f_n d\mu = \int_F f_n d\mu = \int_F f_n(x) d\mu = f_n(x)$$

because $\mu(F) = \int_F g d\mu = \int_K g d\mu = 1$ by the equalities (1.22) and (1.23) (note that $\mu$ is actually a probability!).

□

We are ready to proof the next result that appeared in [30, 35]:

**Theorem 1.33.** Let $K$ be a compact space and $B(\subset B_{C(K)^*})$ a boundary for the Banach space $(C(K), \|\cdot\|_\infty)$. Then the following statements hold true:

(i) $(C(K), \sigma(C(K),B))$ is angelic;
(ii) If a subset $A$ of $C(K)$ is $\sigma(C(K),B)$-relatively countably compact in $C(K)$, then $A$ is $\sigma(C(K),B)$-relatively sequentially compact.
(iii) If $A$ is a norm-bounded and $\sigma(C(K),B)$-compact subset of $C(K)$, then $A$ is weakly compact.

*Proof.* Let us fix the notation $X := C(K)$, $\tau := \sigma(C(K),B)$ and $\mathfrak{T} := t_p(K)$ the topology of pointwise convergence on $C(K)$. Then Lemma 1.32 implies that the hypotheses in Lemma 1.31 are fulfilled. On the one hand, let $\{f_n\}_{n\geq 1}$ be a sequence in $C(K)$ that has $\tau$-cluster point $f_0 \in C(K)$ and take an arbitrary $\mathfrak{T}$-open neighborhood of $f_0$

$$V(f_0, x_1, x_2, \ldots, x_m, \varepsilon) := \{g \in C(K) : \sup_{1\leq i\leq m} |g(x_i) - f_0(x_i)| < \varepsilon\},$$

with $\varepsilon > 0$, $x_1, x_2, \ldots, x_m \in K$. Use Lemma 1.32 to pick $\mu_i \in B$ associated to each $x_i$ and the sequence $\{f_n\}_{n\geq 1} \cup \{f_0\}$, $1 \leq \cdots \leq i \leq \cdots \leq m$. Since $\{f_n\}_{n\geq 1}$ visits frequently the $\tau$-open neighborhood of $f_0$

$$V(f_0, \mu_1, \mu_2, \ldots, \mu_m, \varepsilon) := \Big\{ g \in C(K) : \sup_{1\leq i\leq m} \Big| \int_K g d\mu_i - \int_K f_0 d\mu_i \Big| < \varepsilon \Big\},$$

we conclude that $\{f_n\}_{n\geq 1}$ visits frequently $V(f_0, x_1, x_2, \ldots, x_m, \varepsilon)$, hence $f_0$ is also a $\mathfrak{T}$-cluster point of $\{f_n\}_{n\geq 1}$. On the other hand, the space $(C(K), t_p(K))$ is angelic, [71, 96] (see also [53]). Therefore $(C(K), \sigma(C(K), B))$ is angelic by Lemma 1.31 that explains (i). Since in angelic spaces relatively countably compactness implies relatively sequentially compactness, statement (ii) follows from (i). Finally (iii) follows from (ii) and the Rainwater–Simons theorem, Corollary 1.7 –we have no need here for the general solution given in Theorem 1.30 for the boundary problem.

□

Given a topological space $X$ we denote by $C_b(X)$ the Banach space of bounded continuous real valued functions on $X$ endowed with the supremum norm $\|\cdot\|_\infty$. $\mathscr{M}(X)$ stands for the dual space $(C_b(X), \|\cdot\|_\infty)^*$, for which we adopt the Alexandroff representation as the space of finite, finitely-additive zero-set regular Baire measures on $X$, [150, Theorem 6].

The following result was published in [36]:

**Theorem 1.34.** Let $E$ be a Banach space whose dual unit ball $B_{E^*}$ is $w^*$-angelic and let $B$ be a subset of $B_{E^*}$.

(i) If $B$ is norming and $A$ is a bounded and $\sigma(E,B)$-relatively countably compact subset of $E$, then $\overline{\mathrm{co}(A)}^{\sigma(E,B)}$ is $\sigma(E,B)$-compact.
(ii)If $B$ if a boundary for $E$, then every bounded $\sigma(E,B)$-relatively countably compact subset of $E$ is weakly relatively compact. Therefore the topology $\sigma(E,B)$ is angelic on bounded sets of $E$.

*Proof.* It is clear that (ii) follows from (i) when taking into account Theorem 1.26.

Here is a proof for (i). We note first that is not restrictive to assume that $B$ is 1-norming and in this case $\overline{\mathrm{co}(B)}^{w^*} = B_{E^*}$. Consider $X := \overline{A}^{\sigma(E,B)}$ endowed with the topology induced by $\sigma(E,B)$. Now we will state that every Baire probability $\mu$ on $X$ has a barycentre $x_\mu$ in $X$. Since $A$ is $\sigma(E,B)$-relatively countably compact, every $\sigma(E,B)$-continuous real function on $X$ is bounded, which means that $X$ is a pseudocompact space. For pseudocompact spaces $X$, the space $\mathscr{M}(X)$ is made up of countably additive measures defined on the Baire $\sigma$-field $\mathscr{B}a$ of $X$, [58] and [150, Theorem 21]. Take a Baire probability $\mu$ on $X$ and $x^* \in B_{E^*}$. On the one hand, since $(B_{E^*}, w^*)$ is angelic, for every $x^* \in B_{E^*}$ there exists a sequence in co$(B)$ that $w^*$-converges to $x^*$, and therefore $x^*|_X$ is $\mathscr{B}a$-measurable. On the other hand, $X$ is norm-bounded and thus $x^*|_X$ is also bounded, hence $\mu$-integrable. Since $x^* \in E^*$ is arbitrary, for the given $\mu$ we can consider the linear functional $T_\mu : E^* \to \mathbb{R}$ given for each $x^* \in E^*$ by the formula

$$T_\mu(x^*) := \int_X x^*|_X d\mu.$$

We claim that $T_\mu|_{B_{E^*}}$ is $w^*$-continuous. To this end it is enough to prove that for any subset $C$ of $B_{E^*}$ we have that

$$T_\mu(\overline{C}^{w^*}) \subset \overline{T_\mu(C)}. \tag{1.24}$$

Take $y^* \in \overline{C}^{w^*}$ and use the angelicity of $(B_{E^*}, w^*)$ to pick up a sequence $\{y_n^*\}_{n\geq 1}$ in $C$ with $y^* = w^*\text{-}\lim_n y_n^*$; in particular we have that considered as functions, the sequence $\{y_n^*|_X\}_{n\geq 1}$ converges pointwise to $y^*|_X$ and it is uniformly bounded on $X$. The Lebesgue convergence theorem gives us that $T_\mu(y^*) = \lim_n T_\mu(y_n^*)$ and this proves (1.24). Now Grothendieck's completeness theorem, [102, §21.9.4], applies to conclude the existence of an element $x_\mu$ in $E$ such that $T_\mu(x^*) = x^*(x_\mu)$ for every $x^* \in E^*$. $x_\mu$ is the barycentre of $\mu$ that we are looking for. Now we define the map $\phi : \mu \to x_\mu$ from the $\sigma(\mathscr{M}(X), C_b(X))$-compact convex set $\mathscr{P}(X)$ of all Baire probabilities on $X$ into $E$. It is easy to prove that $\phi$ is $\sigma(\mathscr{M}(X), C_b(X))$-to-$\sigma(E,B)$ continuous and its range $\phi(\mathscr{P}(X))$ is a $\sigma(E,B)$-compact convex set that contains $X$. The proof is concluded.

□

A particular class of angelic compact spaces is that of the Corson compact spaces: a compact space $K$ is said to be *Corson compact* if for some set $\Gamma$ it is (homeomorphic to) a compact subset of $[0,1]^\Gamma$ such that for every $x = (x(\gamma))$ in $K$ the set $\{\gamma : x(\gamma) \neq 0\}$ is countable, see [40]. If we assume that $(B_{E^*}, w^*)$ is Corson compact, techniques of Radon–Nikodým compact spaces introduced in [113] can be used to prove that (i) in Theorem 1.34 can be completed by proving that $A$ is also $\sigma(E,B)$-relatively sequentially compact. Let us remark that many Banach spaces have $w^*$-angelic dual unit ball as for instance the weakly compactly generated or more general the weakly countably $K$-determined Banach spaces, see [117, 144].

We finish this section with a few brief comments regarding strong boundaries. If $B$ is a norm-separable boundary for a $w^*$-compact subset $C$ in $E^*$, then $B$ is a *strong boundary* of $C$, in the sense that $C$ is the norm-closed convex hull of $B$. This result was first stated in [130], and later, with techniques based on (I)-generation in [56, 55] –note that it straightforwardly follows from Corollary 1.8. If the boundary $B$ is weakly Lindelöf it is an open problem to know if it is strong. When $B$ is weakly Lindelöf determined, the angelic character of $C_p((B,w))$, see [117], tell us that every $x^{**} \in B_{E**}$ is the pointwise limit of a sequence of elements in $B_E$ and Simons' inequality implies that $B$ is a strong boundary, see [59, Theorem I.2]. If $C$ is a $w^*$-compact and weakly Lindelöf subset of $E^*$ we also have that every boundary of $C$ is strong, see [34, Theorem 5.7]. For separable Banach spaces $E$ without isomorphic copies of $\ell^1(\mathbb{N})$ we also have that every boundary of any $w^*$-compact set is a strong boundary, [59]. In the non separable case the same is true if the boundary is assumed to be $w^*$-K-analytic as established in the result below that can be found in [29, Theorem 5.6]:

**Theorem 1.35.** A Banach space $E$ doest not contain isomorphic copies of $\ell^1(\mathbb{N})$ if, and only if, each $w^*$-K-analytic boundary of any $w^*$-compact subset $C$ of $E^*$ is strong.

In particular, $w^*$-analytic boundaries are always strong boundaries in the former situation. We note that recently Theorem 1.35 has been extended to $w^*$-$K$-countably determined boundaries in [66]. In a different order of ideas, let us remark here that the sup-limsup theorem can be extended to more general functions in this situation, see [29, Theorem 5.9]:

**Theorem 1.36.** Let E be a Banach space without isomorphic copies of $\ell^1(\mathbb{N})$, $C$ a $w^*$-compact subset in $E^*$ and $B$ a boundary of $C$. Let $\{z_n^{**}\}_{n\geq 1}$ be a sequence in $E^{**}$ such that for all $n \geq 1$, $z_n^{**} = w^*\text{-}\lim_m z_m^n$, for some $\{z_m^n\}_{m\geq 1} \subset E$. Then we have:

$$\sup_{b^*\in B}\{\limsup_n z_n^{**}(b^*)\} = \sup_{x^*\in C}\{\limsup_n z_n^{**}(x^*)\}.$$

When the boundary is built up by using a measurable map, it is always strong:

**Theorem 1.37.** Let $E$ be a Banach space, and let $C$ be a $w^*$-compact subset of $E^*$. Assume that $f : E \to C$ is a norm-to-norm Borel map such that $\langle x, f(x)\rangle = S_C(x)$ for every $x \in E$. Then

$$\overline{\text{co}(f(X))}^{\|\cdot\|} = C.$$

*Proof.* [34, Corollary 2.7] says that we are in conditions to apply [29, Theorem 4.3] to get the conclusion.

□

Borel maps between complete metric spaces send separable sets to separable ones, see [142, Theorem 4.3.8]. This fact implies that a $w^*$-compact set $C$ as in Theorem 1.37 is going to be fragmented by the norm of $E^*$. Indeed, for every separable subspace $S$ of $E$ we have that $f(S)$ is a separable boundary of the $w^*$-compact set $C_{|S}(\subset S^*)$, thus $C_{|S} = \overline{\text{co} f(S)_{|S}}^{\|\cdot\|_{S^*}}$ is a separable subset of $S^*$, and therefore $C$ is fragmented by the norm of $E^*$, see [113]. If $C = B_{E^*}$ the space $E$ must be an Asplund space. With these results in mind, strong boundaries of an Asplund space are characterized in terms of the following concept, introduced in [29], A subset $C$ of the dual of a Banach space $E$ is said to be *finitely self-predictable* if there is a map $\xi : \mathscr{F}_E \longrightarrow \mathscr{F}_{\text{co}(C)}$ from the family of all finite subsets of $E$ into the family of all finite subsets of $\text{co}(C)$ such that for each increasing sequence $\{\sigma_n\}_{n\geq 1}$ in $\mathscr{F}_E$ with

$$\Sigma = \bigcup_{n=1}^{\infty} \sigma_n, \qquad D = \bigcup_{n=1}^{\infty} \xi(\sigma_n),$$

we have that

$$C_{|\Sigma} \subset \overline{\text{co}}^{\|\cdot\|}(D_{|\Sigma}).$$

The characterization of strong boundaries in Asplund spaces is stated in the following terms, see [29, Theorem 3.9]:

**Theorem 1.38.** For a boundary $B$ of an Asplund space, $B$ is a strong boundary if, and only if, it is finitely self-predictable.

In particular, Asplund spaces are those Banach spaces for which the above equivalence holds, see [29, Theorem 3.10]. A procedure for generating finitely self-predictable subsets is also provided in [29, Corollary 4.4], as the range of $\sigma$-fragmented selectors, (see [88] for the definition) of the duality mapping, which leads to another characterization of Asplund spaces, see [29, Corollary 4.5].

In a different order of ideas, the paper [94] contains a good number of interesting results of how to transfer topological properties from a boundary $B$ of $C$ to the whole set $C$ (in particular fragmentability) as well as how to embed a Haar system in an analytic boundary of a separable non-Asplund space. Other results about $w^*$-K-analytic boundaries non containing isomorphic copies of the basis of $\ell^1(\mathbb{R})$ can be found in [66] –see also Theorem 1.29.

We finish this section with the following open question:

**Question 3.** Let $E$ be a Banach space and $B$ a boundary of it. Is $\sigma(E,B)$ an angelic topology on bounded sets of $E$?

## 1.5 Extensions of James' weak compactness theorem

Since its appearance, James' weak compactness theorem has become the subject of much interest for many researchers. As discussed in the Introduction, one of the concerns about it has been to obtain proofs which are simpler than the original one. Another, and we deal with it in this section, is to generalize it, which in particular has led to new applications that we will show in Section 1.6. Clearly the commented developments on boundaries represent a first group of results along these lines. The other extensions that we present fall into two kind of results. On the one hand, we can have those that for a Banach space $E$ guarantee reflexivity, whenever the set $\mathrm{NA}(E)$ of the continuous and linear functionals that attain their norms,

$$\mathrm{NA}(E) := \{x^* \in E^* : \text{ there exists } x_0 \in B_E \text{ such that } x^*(x_0) = \|x^*\|\},$$

is large enough. On the other hand, we have James' type results but considering more general optimization problems.

### *1.5.1 Size of the set of norm attaining functionals*

Roughly speaking, the basic question we are concerned with here is whether the reflexivity of a Banach space $E$ follows from the fact that the set of norm attaining functionals $\mathrm{NA}(E)$ is not small in some sense. Most of these results are based on a suitable meaning for being topologicaly big.

With regard to the norm-topology, the concrete question is to know whether a Banach space $E$ is reflexive provided that the set $\mathrm{NA}(E)$ has nonempty norm-interior. The space $\ell^1(\mathbb{N})$ shows that the answer is negative, and in addition it is easily proven in [7, Corollary 2] that every Banach space admits an equivalent norm for which the set of norm attaining functionals has nonempty norm-interior. For this very reason we cannot assume an isomorphic hypothesis on the space when studying the question above. Some geometric properties have been considered. Before collecting some results in this direction, let us say something more from the isomorphic point of view. In 1950 Klee proved that a Banach space $E$ is reflexive provided that for every space isomorphic to $E$, each functional attains its norm [100]. Latter, in 1999 Namioka asked whether a Banach space $E$ is reflexive whenever the set $\mathrm{NA}(X)$ has nonempty norm-interior for each Banach space $X$ isomorphic with $E$. In [5, Theorem 1.3], Acosta and Kadets provided a positive answer (see also [6]).

In order to state the known results for the norm-topology, let us recall that a Banach space $E$ has the *Mazur intersection property* when each bounded, closed and convex subset of $E$ is an intersection of closed balls ([107]). This is the case of a space with a Fréchet differentiable norm ([45, Proposition II.4.5]). Another different geometric condition is this one: a Banach space $E$ is *weakly Hahn–Banach smooth* if each $x^* \in \mathrm{NA}(E)$ has a unique Hahn–Banach extension to $E^{**}$. It is clear that if $E$ is *very smooth* (its duality mapping is single valued and norm-to-weak continuous [140]), then it is weakly Hahn–Banach smooth. Examples of very smooth spaces are those with a Fréchet differentiable norm and those which are an $M$-ideal in its bidual [76, 151] –for instance $c_0$ or the space of compact operators on $\ell^2$. The following statement, shown in [89, Proposition 3.3] and [8, Theorem 1], provides a first generalization of James' reflexivity theorem for the above classes of Banach spaces:

**Theorem 1.39.** Suppose that $E$ is a Banach space that has the Mazur intersection property or is weakly Hahn–Banach smooth. Then $E$ is reflexive if, and only if, $\mathrm{NA}(E)$ has nonempty norm-interior.

The above result is a consequence of James' reflexivity theorem applied to an adequate renorming, in the Mazur intersection property case, and of the Simons inequality after a sequential reduction, for weakly Hahn–Banach smooth spaces.

Note that Theorem 1.39 fails when the space is *smooth* (norm Gâteaux differentiable). Indeed, any separable Banach space is isomorphic to another smooth Banach space whose set of norm attaining functional has nonempty norm-interior, see [7, Proposition 9].

For some concrete Banach spaces we can say something better. For instance, the sequence space $c_0$ satisfies that the set $\mathrm{NA}(c_0)$ is of the first Baire category, since it is nothing more than the subset of sequences in $\ell^1(\mathbb{N})$ with finite support. Bourgain and Stegall generalized it for any separable Banach space whose unit ball is not dentable. As a matter of fact, they established the following result in [25, Theorem 3.5.5]:

**Theorem 1.40.** If $E$ is a Banach space and $C$ is a closed, bounded and convex subset of $E$ that is separable and nondentable, then the set of functionals in $E^*$ that attain their supremum on $C$ is of the first Baire category in $E^*$.

When $C$ is the unit ball of the continuous functions space on a infinite Hausdorff and compact topological space $K$, Kenderov, Moors and Sciffer proved in [97] that $\mathrm{NA}(C(K))$ is also of the first Baire category. However we do not know whether or not Theorem 1.40 is valid if $C$ is nonseparable. However, Moors has provided us (private communication) with the proof of the following unpublished result wich follows from Lemma 4.3 in [109]: Suppose that a Banach space $E$ admits an equivalent weakly midpoint LUR norm and that $E$ has the Namioka property, i.e., every weakly continuous mapping acting from a Baire space into $E$ is densely norm continuous. Then every closed, bounded and convex subset $C$ of $E$ for which the set of functionals in $E^*$ attaining their supremum on $C$ is of the second Baire category in $E^*$ has at least one strongly exposed point. In particular, $C$ is dentable.

Now we present a group of results whose hypotheses involve the weak topology of the dual space. Jiménez-Sevilla and Moreno showed a series of results, from which we emphasize the following consequence of Simons' inequality [89, Proposition 3.10]:

**Theorem 1.41.** Let $E$ be a separable Banach space such that the set $\mathrm{NA}(E) \cap S_{E^*}$ has nonempty relative weak-interior in $S_{E^*}$. Then $E$ is reflexive.

Regarding the $w^*$-topology in the dual space, the first result was obtained, also applying Simons' inequality, by Deville, Godefroy and Saint Raymond [44, Lemma 11] and is the version for the $w^*$-topology of the preceding theorem. Later, an adequate use of James' reflexivity theorem for a renorming of the original space implies the same assertion, but removing the separability assumption [89, Proposition 3.2]:

**Theorem 1.42.** A Banach space is reflexive if, and only if, the set of norm-one norm attaining functionals contains a nonempty relative $w^*$-open subset of its unit sphere.

This result has been improved for a certain class of Banach spaces, for instance for *Grothendieck spaces*, *i.e.,* those Banach spaces for which the sequential convergence in its dual space for the $w$-topology is equal to that of the $w^*$-topology. It is clear that any reflexive space is a Grothendieck space and the converse is true when the space does not contain $\ell^1(\mathbb{N})$, see [148, 63]. Moreover, the Eberlein–Šmulian theorem guarantees that a Banach space with a $w^*$-sequentially compact dual unit ball is reflexive whenever is a Grothendieck space.

**Theorem 1.43.** If $E$ is a Banach space $E$ that is not Grothendieck, then $\mathrm{NA}(E)$ is not a $w^*$-$G_\delta$ subset of $E^*$.

This result has been stated in [5, Theorem 2.5], although it previously appeared in [44, Theorem 3] for separable spaces. Finally, a characterization of the reflexivity in terms of the $w^*$-topology, and once again by means of the Simons inequality but with other kind of assumptions, was obtained in [2, Theorem 1]:

**Theorem 1.44.** Assume that $E$ is a Banach space that does not contain $\ell^1(\mathbb{N})$ and that for some $r > 0$

$$B_{E^*} = \overline{\text{co}}^{w^*}\{x^* \in S_{E^*} : x^* + rB_{E^*} \subset \text{NA}(E)\}.$$

Then $E$ is reflexive.

A similar result is stated in [2, Proposition 4], but replacing the assumption of non containing $\ell^1(\mathbb{N})$ with that of the norm of the space is not *rough*, *i.e.,* there exists $\varepsilon > 0$ such that for all $x \in E$

$$\limsup_{h\to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} \geq \varepsilon.$$

Here we have emphasized some extensions of James' reflexivity theorem in connection to the size of the set of norm attaining functionals, but there are other ways of measuring such size. For example, one can look for linear subspaces into $\text{NA}(E)$.

The first of these results was obtained by Petunin and Plichko in [121]. To motivate it, let us observe that for a dual space $E = F^*$ we have that $F$ is a closed and $w^*$-dense subspace of $E^*$ with $F \subset \text{NA}(E)$. Their result deals with the converse:

**Theorem 1.45.** A separable Banach space $E$ is isometric to a dual space provided that there exists a Banach space $F$ which is $w^*$-dense in $E^*$ and satisfies $F \subset \text{NA}(E)$.

There are some recent results that provide conditions implying that the set of norm attaining functionals contains an infinite dimensional linear subspace. See [15, 1, 57] and the references therein. For instance, in [57] the following renorming result is stated:

**Theorem 1.46.** Every Banach space that admits an infinite dimensional separable quotient is isomorphic to another Banach space whose set of norm attaining functionals contains an infinite dimensional linear subspace.

However, some questions still remain to be studied. For instance, whether for every infinite dimensional Banach space $E$, the set $\text{NA}(E)$ contains a linear subspace of dimension 2 is an irritating open problem, posed in [15, Question 2.24].

### *1.5.2 Optimizing other kind of functions*

In the past several years, some extensions of James' weak compactness theorem appeared. A common thing for these results is that the optimization condition – each continuous and linear functional attains its supremum on a weakly closed and bounded subset of the space– is replaced by another one: the objective function is more general. We present some of them here, when considering either polynomials or perturbed functionals.

For a Banach space $E$ and $n \geq 1$, let us consider the space $\mathscr{P}(^nE)$ of all continuous $n$-homogeneous polynomials on $E$, endowed with its usual sup norm. Recall that a polynomial in $\mathscr{P}(^nE)$ *attains the norm* when the supremum defining its norm is a maximum. It is clear that if for some $n$ each polynomial in $\mathscr{P}(^nE)$ attains its norm, then every functional attains the norm and thus James' reflexivity theorem implies the reflexivity of $E$. So the polynomial version of James' reflexivity theorem should be stated in terms of a subset of $\mathscr{P}(^nE)$. This is done in the following characterization, see [131, Theorem 2], when dealing with weak compactness of a bounded, closed and convex subset of $E$:

**Theorem 1.47.** A bounded, closed and convex subset $A$ of a Banach space $E$ is weakly compact if, and only if, there exist $n \geq 1$ and $x_1^*, \ldots, x_n^* \in E^*$ such that for all $x^* \in E^*$, the absolute value of the continuous $(n+1)$-homogeneous polynomial

$$x \mapsto x_1^*(x) \cdots x_n^*(x) x^*(x), \quad (x \in E),$$

when restricted to $A$, attains its supremum and

$$A \not\subset \cup_{j=1}^n \ker x_j^*.$$

Similar results for symmetric multilinear forms, including some improved versions for the case $A = B_E$, can be found in [4, 131].

A related question to that of "norm attaining" (or "sup attaining") is that of "numerical radius attaining". More specifically, the *numerical radius* of a continuous and linear operator $T : E \longrightarrow E$ is the real number $v(T)$ given by

$$v(T) := \sup\{|x^* T x| : (x, x^*) \in \Pi(E)\},$$

where $\Pi(E) := \{(x, x^*) \in S_E \times S_{E^*} : x^*(x) = 1\}$ and such an operator $T$ is said to *attain the numerical radius* when there exists $(x_0, x_0^*) \in \Pi(E)$ with $|x_0^* T x_0| = v(T)$.

The following sufficient condition for reflexivity was stated in [9, Theorem 1] (see also [132, Corollary 3.5] for a more general statement about weak compactness), and was obtained by applying the minimax theorem [137, Theorem 5].

**Theorem 1.48.** A Banach space such that every rank-one operator on it attains its numerical radius is reflexive.

Surprisingly enough, the easy-to-prove part in the classical James' reflexivity theorem does not hold. Indeed, a Banach space is finite dimensional if, and only if, in any equivalent norm each rank-one operator attains its numerical radius, as seen in [9, Example] and [3, Theorem 7].

However, the James type result that seems to be more applied nowadays, see Section 1.6, is a perturbed version: there exists a fixed function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ such that

$$\text{for every } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Let us note that this optimization condition generalizes that in the classical James' weak compactness theorem. Indeed, $x^* \in E^*$ attains its supremum on the set $A (\subset E)$

if, and only if, $x^* - \delta_A$ attains its supremum on $E$, where $\delta_A$ denotes the *indicator function* of $A$ defined as

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A \\ \infty, & \text{otherwise} \end{cases}.$$

The first result along these lines was stated in [52, 27] by Calvert and Fitzpatrick:

**Theorem 1.49.** A Banach space is reflexive whenever its dual space coincides with the range of the subdifferential of an extended real-valued coercive, convex and lower semicontinuous function whose effective domain has nonempty norm-interior.

The erratum [27] makes [52] more difficult to follow, since the main addendum requires to correct non-written proofs of some statements in [52], which are adapted from [84]. A complete and more general approach was presented in Theorems 2, 5 and 7 of [118].

Let us point out that, for a Banach space $E$ and a proper function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$, *coercive* means

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty,$$

and that the *effective domain* of $f$, $\operatorname{dom}(f)$, is the set of those $x \in E$ with $f(x)$ finite.

Taking into account that for a function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ which is *proper* ($\operatorname{dom}(f) \neq \emptyset$), and $x \in \operatorname{dom}(f)$, we have that the subdifferential of $f$ at $x$ is given by

$$\partial f(x) = \{x^* \in E^* : \; x^* - f \text{ attains its supremum on } E \text{ at } x\},$$

then the surjectivity assumption in Calvert and Fitzpatrick's theorem is once again a perturbed optimization result.

Another perturbed version of James' weak compactness theorem, different from the preceding one, was established in [133, Theorem 16] as a consequence of a minimax result [133, Theorem 14]. In order to state that minimax theorem, generalizating [137, Theorem 14], the authors used the ideas of Pryce in Lemma 1.2 and a refinement of the arguments in [138]. Such a perturbed theorem reads as follows in the Banach space framework:

**Theorem 1.50.** Let $A$ be a weakly closed subset of a Banach space $E$ for which there exists $\psi \in \ell^\infty(A)$ such that

$$\text{for each } x^* \in E^*,\ x^*|_A - \psi \text{ attains its supremum.}$$

Then $A$ is weakly compact.

Here the perturbation $f$ (defined on the whole $E$) is given by

$$f(x) := \begin{cases} \psi(x), & \text{if } x \in A \\ \infty, & \text{for } x \in E \backslash A \end{cases}.$$

The second named author in this survey obtained another perturbed James type result in the class of separable Banach spaces. This result was motivated by financial applications, and once again, it was proved by applying adequately Simons' inequality. Its proof was included in the Appendix of [91]:

**Theorem 1.51.** Suppose that $E$ is a separable Banach space and that $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ is a proper function whose effective domain is bounded and such that

$$\text{for each } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Then for every $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c])$ is weakly compact.

In the preceding versions of the weak compactness theorem of James, the perturbation functions are coercive. Recently, the following characterization has been developed in [118, Theorem 5]:

**Theorem 1.52.** Let $E$ be a Banach space and suppose that $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, coercive and weakly lower semicontinuous function. Then

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E$$

if, and only if,

$$\text{for each } c \in \mathbb{R}, \text{ the sublevel set } f^{-1}((-\infty, c]) \text{ is weakly compact.}$$

The proof makes use of the perturbed technique of the undetermined function as explained in Theorem 1.14.

Let us also emphasize that there are previous topological results along the lines of Theorem 1.52, see [22, Theorems 2.1 and 2.4].

Since for any reflexive Banach space $E$ the proper, noncoercive and weakly lower semicontinuous function $f = \|\cdot\|$ satisfies that for every $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c])$ is weakly compact, although $\partial f(E) = B_{E^*}$, then the coercivity cannot be dropped in one direction of the former theorem. Nevertheless, for the converse implication Saint Raymond has just obtained the nice theorem that follows, [134, Theorem 11]:

**Theorem 1.53 (Saint Raymond).** If $E$ is a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ is a proper weakly lower semicontinuous function such that for every $x^* \in E^*$, $x^* - f$ attains its supremum, then for each $c \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, c])$ is weakly compact.

**Remark 1.54.** The fact that for a proper function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ with $\partial f(E) = E^*$ all its sublevel sets are relatively weakly compact can be straightforwardly derived from Theorem 1.53. To see it, replace $f$ with the proper weakly lower semicontinuous function $\widetilde{f} : E \longrightarrow \mathbb{R} \cup \{\infty\}$ defined for every $x \in E$ as

$$\widetilde{f}(x) := \inf\{t \in \mathbb{R} : \ (x, t) \in \overline{\operatorname{epi}(f)}^{\sigma(E \times \mathbb{R}, E^* \times \mathbb{R})}\},$$

where $\operatorname{epi}(f)$ is the *epigraph* of $f$, that is,

$$\operatorname{epi}(f) := \{(x,t) \in E \times \mathbb{R} \colon f(x) \leq t\}.$$

Furthermore, when $\operatorname{dom}(f)$ has nonempty norm-interior, we have that $E$ is reflexive as a consequence of the Baire Category theorem.

Note that Theorem 1.53 provides an answer to the problem posed in [27]: given a Banach space $E$ and a convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ whose effective domain has nonempty norm-interior, is it true that the surjectivity of its subdifferential is equivalent to the reflexivity of $E$ and the fact that for all $x^* \in E^*$, the function $x^* - f$ is bounded above?

On the other hand, Bauschke proved that each real infinite-dimensional reflexive Banach space $E$ has a proper, convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\text{for each } x^* \in E^*, \quad x^* - f \text{ is bounded above,}$$

but $f$ is not coercive, see [16, Theorem 3.6]. From here it follows that $\partial f(E) = E^*$, as seen in [118, Theorem 3]. Thus Theorem 1.53 properly extends one direction of Theorem 1.52.

Now let us show how Saint Raymond's result, Theorem 1.53, following the ideas in [118, Corollary 5], has some consequences for multivalued mappings. Let us recall that given a Banach space $E$ and a multivalued operator $\Phi : E \longrightarrow 2^{E^*}$, the *domain* of $\Phi$ is the subset of $E$

$$\mathrm{D}(\Phi) := \{x \in E : \Phi(x) \text{ is nonempty}\},$$

and its *range* is the subset of $E^*$

$$\Phi(E) := \{x^* \in E^* : \text{there exists } x \in E \text{ with } x^* \in \Phi(x)\}.$$

In addition, $\Phi$ is said to be *monotone* if

$$\inf_{\substack{x,y \in \mathrm{D}(\Phi) \\ x^* \in \Phi(x),\ y^* \in \Phi(y)}} \langle x^* - y^*, x - y\rangle \geq 0,$$

and *cyclically monotone* when the inequality

$$\sum_{j=1}^{n} \langle x_j^*, x_j - x_{j-1}\rangle \geq 0$$

holds, whenever $n \geq 2$, $x_0, x_1, \ldots, x_n \in \mathrm{D}(\Phi)$ with $x_0 = x_n$ and for $j = 1, \ldots, n$, $x_j^* \in \Phi(x_j)$.

If $\Phi$ is a cyclically monotone operator then there exists a proper and convex function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that for every $x \in E$,

$$\Phi(x) \subset \partial f(x),$$

see [128, Theorem 1], and so Theorem 1.53 leads to the following James' type result for cyclically monotone operators:

**Corollary 1.55.** Let $E$ be a Banach space and let $\Phi : E \longrightarrow 2^{E^*}$ be a cyclically monotone operator such that $\mathrm{D}(\Phi)$ has nonempty norm-interior and

$$\Phi(E) = E^*.$$

Then $E$ is reflexive.

Note that this result does not provide a satisfactory answer to the following open problem, posed in [52]: Assume that $E$ is a Banach space and $\Phi : E \longrightarrow 2^{E^*}$ is a monotone operator such that $\mathrm{D}(\Phi)$ has nonempty interior and $\Phi(E) = E^*$. Is $E$ reflexive?

To conclude this section we provide a proof of Theorem 1.53 for the wide class of Banach spaces with $w^*$-convex block compact dual unit balls, which easily follows from the unbounded Rainwater–Simons theorem, Corollary 1.7, see [119, Theorem 4]. The following lemma produces the sequence needed to apply it:

**Lemma 1.56.** Suppose that the dual unit ball of $E$ is $w^*$-convex block compact and that $A$ is a nonempty, bounded subset of $E$. Then $A$ is weakly relatively compact if, and only if, each $w^*$-null sequence in $E^*$ is also $\sigma(E^*, \overline{A}^{w^*})$-null.

*Proof.* If $A$ is weakly relatively compact, then we have $A = \overline{A}^{w^*}$ and the conclusion follows. According to Proposition 1.12, to see the reverse implication we have to check the validity of the identity

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) = 0 \tag{1.25}$$

for every bounded sequence $\{x_n^*\}_{n\geq 1}$ in $E^*$. Thus, let us fix $\{x_n^*\}_{n\geq 1}$ a bounded sequence in $B_{E^*}$. Since $B_{E^*}$ is $w^*$-convex block compact, there exists a block sequence $\{y_n^*\}_{n\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ and an $x_0^* \in B_{E^*}$ such that

$$w^*\text{-}\lim_n y_n^* = x_0^*.$$

Then, by assumption, $\{y_n^*\}_{n\geq 1}$ also converges to $x_0^*$ pointwise on $\overline{A}^{w^*} \subset E^{**}$. Mazur's theorem applied to the sequence of continuous functions $\{y_n^*\}_{n\geq 1}$ restricted to the $w^*$-compact space $\overline{A}^{w^*}$ tell us that

$$0 = \mathrm{dist}_{\|\cdot\|_{\overline{A}^{w^*}}}(x_0^*, \mathrm{co}\{y_n^* : n \geq 1\}) = \mathrm{dist}_{\|\cdot\|_A}(x_0^*, \mathrm{co}\{x_n^* : n \geq 1\}) \geq 0,$$

It is not difficult to check that $x_0^* \in L\{x_n^*\}$ and (1.25) is proved, and we have concluded the proof.

□

Following [119], we present the next proof of Theorem 1.53 for the class of Banach spaces with $w^*$-convex block compact dual unit balls:

**Theorem 1.57.** Let $E$ be a Banach space whose dual unit ball is $w^*$-convex block compact and let $f : E \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper map such that

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Then

$$\text{for every } c \in \mathbb{R}, \text{ the sublevel set } f^{-1}((-\infty, c]) \text{ is weakly relatively compact.}$$

*Proof.* We first claim that for every $(x^*, \lambda) \in E^* \times \mathbb{R}$ with $\lambda < 0$, there exists $x_0 \in E$ with $f(x_0) < +\infty$ and such that

$$\sup\{(x^*, \lambda)(x,t) : (x,t) \in \operatorname{epi}(f)\} = x^*(x_0) - \lambda f(x_0). \tag{1.26}$$

In fact, the optimization problem

$$\sup_{x\in E}\{\langle x, x^*\rangle - f(x)\} \tag{1.27}$$

may be rewritten as

$$\sup_{(x,t)\in\operatorname{epi}(f)} \{(x^*, -1), (x,t)\} \tag{1.28}$$

and the supremum in (1.27) is attained if, and only if, the supremum in (1.28) is attained.

Let us fix $c \in \mathbb{R}$ and assume that $A := f^{-1}((-\infty, c])$ is nonempty. The uniform boundedness principle and the optimization assumption on $f$ imply that $A$ is bounded. In order to obtain the relative weak compactness of $A$ we apply Lemma 1.56. Thus, let us consider a $w^*$-null sequence $\{x_n^*\}_{n\geq 1}$ in $E^*$ and let us show that it is also $\sigma(E^*, \overline{A}^{w^*})$-null.

It follows from the unbounded Rainwater–Simons theorem, Corollary 1.7, taking the Banach space $E^* \times \mathbb{R}$,

$$B := \operatorname{epi}(f) \subset C := \overline{\operatorname{epi}(f)}^{\sigma(E^{**}\times\mathbb{R}, E^*\times\mathbb{R})}$$

and the bounded sequence

$$\left\{\left(x_n^*, -\frac{1}{n}\right)\right\}_{n\geq 1},$$

that

$$\sigma(E^* \times \mathbb{R}, B)\text{-}\lim_n \left(x_n^*, -\frac{1}{n}\right) = \sigma(E^* \times \mathbb{R}, C)\text{-}\lim_n \left(x_n^*, -\frac{1}{n}\right),$$

But $w^*\text{-}\lim_{n\geq 1} x_n^* = 0$, so we have that

$$\sigma(E^* \times \mathbb{R}, C)\text{-}\lim_n \left(x_n^*, -\frac{1}{n}\right) = 0.$$

As a consequence, since $A \times \{c\} \subset B$, then $\overline{A}^{w^*} \times \{c\} \subset C$, and so

$$\sigma(E^*, \overline{A}^{w^*})\text{-}\lim_n x_n^* = 0,$$

as announced.

□

Theorem 1.57 was first presented at the meeting Analysis, Stochastics and Aplications, held at Viena in July 2010, to celebrate Walter Schachermayer's 60th Birthday, see

*www.mat.univie.ac.at/ anstap10/slides/Orihuela.pdf*,

where the conjecture of its validity for any Banach space was considered. Later on, in the Workshop on Computational and Analytical Mathematics in honor of Jonathan Borwein's 60th Birthday, held at Vancouver in May 2011, see

*http://conferences.irmacs.sfu.ca/jonfest2011/*,

Theorem 1.57 and its application Theorem 1.61 were discussed too. Both results can be found published by the second and third named authors of this survey in the paper [119]. In September 2011 we were informed by J. Saint Raymond that he had independently obtained Theorem 1.57 without any restriction on the Banach space $E$ in [134]: Saint Raymond's proof is based upon a clever and non trivial reduction to the classical James' weak compactness theorem instead of dealing with unbounded sup-limsup results as presented here, as well as in [119]. Nevertheless, our approach contains classical James' result without using it inside the proof, together with the generalizations of Simons' inequalities for unbounded sets in Section 2.

The proof of Theorem 1.57 has been obtained by means of elementary techniques for Banach spaces with a $w^*$-convex block compact dual unit ball, in particular for the separable ones. For this very reason, an easy reduction to the separable case would provide us with a basic proof of the theorem. In that direction, we suggest the following question:

**Question 4.** Let $E$ be a Banach space, $\rho : E^* \times E^* \longrightarrow [0, \infty)$ a pseudometric on $E^*$ for pointwise convergence on a countable set $A(\subset B_{E^{**}})$, where

$$A = A_0 \cup \{x_0^{**}\}, A_0 \subset E, x_0^{**} \in \overline{A_0}^{w^*}.$$

Given $\{x_n^*\}_{n\geq 1}$ a sequence in $B_{E^*}$ such that

$$\sigma(E^*, A_0)\text{-}\lim_n x_n^* = 0,$$

is it possible to find a sequence $\{y_n^*\}_{n\geq 1}$ in $E^*$ with

$$\text{w}^*\text{-}\lim_n y_n^* = 0$$

and

$$\lim_n \rho(x_n^*, y_n^*) = 0?$$

## 1.6 Applications to convex analysis and finance

Since its publication, the applicability of James' weak compactness theorem has been steady. As mentioned in the Introduction, James' weak compactness theorem implies almost straightforwardly a number of important results in Functional Analysis. In this section we focus on some consequences of Theorem 1.53, which have been recently obtained from Theorems 1.52 and 1.57 in the areas of finance and variational analysis. But before describing them, a bit of history on known applications of the theorem of weak compactness of James.

It is in 1968 when appeared the first work mentioning application: in [147] it was proved that a quasi-complete locally convex space valued measure always has a relatively weakly compact range. On the other hand, Dieudonné [47] gave an example of a Banach space for which the Peano theorem about the existence of solutions to ordinary differential equations fails. Then Cellina [37] stated, with the aid of James' reflexivity theorem, that a Banach space is reflexive provided that the Peano theorem holds true for it. Later, Godunov [62] proved that indeed the space is finite-dimensional. In [13] one can find some related results to the failure of Peano's theorem in an infinite-dimensional Banach space, as a consequence of James' reflexivity theorem. Finally, let us emphasize the well-known fact (see for instance [26, Theorem 2.2.5]) that the completeness of a metric space is equivalent to the validity of the famous Ekeland variational principle. In [143] a characterization of the reflexivity of a normed space is established, also in terms of the Ekeland variational principle, and making use once again of James' reflexivity theorem.

### *1.6.1 Nonlinear Variational Problems*

Our goal is to deal with some consequences of Theorem 1.53 for nonlinear variational problems, following the ideas in [118, §4]. For this very reason, let us first recall that variational equations are the standard setting to studying and obtaining weak solutions for large portion of differential problems. Such variational equations, in the presence of symmetry, turn into variational problems for which one has to deduce the existence of a minimum. We prove that this kind of result, always stated in the reflexive context, only make sense for this class of Banach spaces.

To be more precise, let us evoke the so-called *main theorem on convex minimum problems*, (see for instance [153, Theorem 25E, p.516]), which is a straightforward consequence of the classical theorem of Weierstrass (continuous functions defined on a compact space attain their minimum): in a reflexive Banach space $E$ the subdifferential of every proper, coercive, convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is onto, that is, for each $x^* \in E^*$, the optimization problem

$$\text{find } x_0 \in E \text{ such that } f(x_0) - x^*(x_0) = \inf_{x \in E} (f(x) - x^*(x)) \tag{1.29}$$

admits a solution. This result guarantees the solvability of nonlinear variational equations derived from the weak formulation of a wide range of boundary value problems. For instance, given $1 < p < \infty$, a positive integer $N$ and a bounded open subset $\Omega$ of $\mathbb{R}^N$, let $E$ be the reflexive Sobolev space $W_0^{1,p}(\Omega)$ and consider the coercive, convex and continuous function $f : E \longrightarrow \mathbb{R}$ defined by

$$f(x) := \frac{1}{p} \int_\Omega |\nabla x|^p d\lambda, \qquad (x \in E),$$

where $|\cdot|$ is the Euclidean norm. By the main theorem on convex minimum problems we have $\partial f(E) = E^*$. But taking into account that the $p$-laplacian operator $\triangle_p$, defined for each $x \in E$ as

$$\triangle_p(x) := \operatorname{div}\left(|\nabla x|^{p-2} \nabla x\right),$$

satisfies that for all $x \in E$

$$\partial f(x) = \{-\triangle_p x\},$$

see [98, Proposition 6.1], then given any $h^* \in E^*$, the nonlinear boundary value problem

$$\begin{cases} -\triangle_p x &= h^* \quad \text{in } \Omega \\ \quad\ x &= 0 \quad \text{on } \partial\Omega \end{cases}$$

admits a weak solution $x \in E$.

We conclude this subsection by applying Theorem 1.53 (see also Remark 1.54) to show that the adequate setting for dealing with some common variational problems, as $p$-laplacian above, is that of the reflexive spaces. To properly frame the result it is convenient to recall some usual notions. For a Banach space $E$, an operator $\Phi : E \longrightarrow E^*$ is said to be *strongly monotone* if

$$\inf_{\substack{x,y \in E \\ x \neq y}} \frac{\langle \Phi(x) - \Phi(y), x - y \rangle}{\|x - y\|^2} > 0,$$

*hemicontinuous* if for all $x, y, z \in E$, the function

$$t \in [0,1] \mapsto (\Phi(x + ty))(z) \in \mathbb{R}$$

is continuous, *bounded* when the image under $\Phi$ of a bounded set is also bounded, and *coercive* whenever the function

$$x \in E \mapsto (\Phi(x))(x) \in \mathbb{R}$$

is coercive. The result below appears in [28, Corollary 2.101] and it includes as a special case the celebrated Lax–Milgram theorem:

**Proposition 1.58.** If $E$ is a reflexive Banach space and $\Phi : E \longrightarrow E^*$ is a monotone, hemicontinuous, bounded and coercive operator, then $\Phi$ is surjective.

This result applies to several problems in nonlinear variational analysis, including one of its most popular particular cases: in a real reflexive Banach space $E$, given $x_0^* \in E^*$, the equation

$$\text{find } x \in E \text{ such that } \Phi(x) = x_0^*$$

admits a unique solution, whenever $\Phi : E \longrightarrow E^*$ is a Lipschitz continuous and strongly monotone operator. We refer to [70, Example 3.51] for usual applications.

When $\Phi$ is symmetric, that is,

$$\text{for every } x, y \in E, \quad \langle \Phi(x), y\rangle = \langle \Phi(y)), x\rangle,$$

the equation $\Phi(x) = x_0^*$ leads to the nonlinear optimization problem involving the function

$$f(x) := \frac{1}{2}(\Phi(x))(x), \qquad x \in E.$$

As a consequence of Theorem 1.53, or more specifically of Remark 1.54, the natural context for Proposition 1.58, at least with symmetry, is the reflexive one, as shown in the next corollary whose proof is completely analogous to that of [118, Corollary 3]:

**Corollary 1.59.** A Banach space $E$ is reflexive, provided there exists a monotone, symmetric and surjective operator $\Phi : E \longrightarrow E^*$.

## *1.6.2 Mathematical Finance*

We now turn our attention to some recent applications of James' weak compactness theorem in mathematical finance. Let us fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ together with $\mathscr{X}$, a linear space of functions in $\mathbb{R}^\Omega$ that contains the constant functions. We assume here that $(\Omega, \mathscr{F}, \mathbb{P})$ is atomless, although in practice this is not a restriction, since the property of being atomless is equivalent to the fact that we can define a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ that has a continuous distribution function. The space $\mathscr{X}$ will describe all possible financial positions $X : \Omega \longrightarrow \mathbb{R}$, where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. The problem of quantifying the risk of a financial position $X \in \mathscr{X}$ is modeled with functions $\rho : \mathscr{X} \longrightarrow \mathbb{R}$ that satisfy:

(i) *Monotonicity*: if $X \le Y$, then $\rho(X) \ge \rho(Y)$.
(ii)*Cash invariance*: if $m \in \mathbb{R}$ then $\rho(X + m) = \rho(X) - m$.

Such a function $\rho$ is called a *monetary measure of risk* (see Chapter 4 in [54]). When $\rho$ is also a convex function, then it is called a *convex measure of risk*. In many occasions we have $\mathscr{X} = \mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P})$, and it is important to have results for representing the risk measure as

$$\rho(X) = \sup_{Y \in \mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})} \{\mathbb{E}[Y \cdot X] - \rho^*(Y)\}. \tag{1.30}$$

Here $\rho^*$ is the Fenchel–Legendre conjugate of $\rho$, that is, for every $Y \in (\mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}))^*$,

$$\rho^*(Y) = \sup_{X \in \mathbb{L}^\infty(\Omega,\mathscr{F},\mathbb{P})} \{\langle Y, X\rangle - \rho(X)\}.$$

To have this representation is equivalent to have the so-called *Fatou property*, *i.e.,* for any bounded sequence $\{X_n\}_{n\geq 1}$ that converges pointwise almost surely (shortly, a.s) to some $X$,

$$\rho(X) \leq \liminf_n \rho(X_n)$$

(see [54, Theorem 4.31]). A natural question is whether the supremum (1.30) is attained. In general the answer is no, as it is shown by the essential supremum map on $\mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P})$, see [54, Example 4.36]. The representation formula (1.30) with a maximum instead of a supremum has been studied by Delbaen, see [41, Theorems 8 and 9] (see also [54, Corollary 4.35] in the case of coherent risk measures, that is, the convex ones that also are positively homogeneous. The fact that the order continuity of $\rho$ is equivalent to the supremum becoming a maximum, that is, for every $X \in \mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P})$:

$$\rho(X) = \max_{Y \in \mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})} \{\mathbb{E}[Y \cdot X] - \rho^*(Y)\},$$

for an arbitrary convex risk measure $\rho$, is the statement of the so-called Jouini–Schachermayer–Touzi theorem in [41, Theorem 2] (see also [91, Theorem 5.2] for the original reference). Let us remark that order sequential continuity for a map $\rho$ in $\mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P})$ is equivalent to have

$$\lim_n \rho(X_n) = \rho(X),$$

whenever $\{X_n\}_{n\geq 1}$ is a bounded sequence in $\mathbb{L}^\infty$ pointwise a.s. convergent to $X$. Indeed, it is said that a map $\rho : \mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow \mathbb{R} \cup \{+\infty\}$ verifies the *Lebesgue property* provided that it is sequentially order continuous. The precise statement is the following one:

**Theorem 1.60 (Jouini, Schachermayer and Touzi).** Let $\rho : \mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ be a convex risk measure with the Fatou property, and let $\rho^* : (\mathbb{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}))^* \longrightarrow [0, +\infty]$ be its Fenchel–Legendre conjugate. The following are equivalent:

(i) For every $c \in \mathbb{R}$, $\{Y \in \mathbb{L}^1(\Omega, \mathscr{F}, \mathbb{P}) : \rho^*(Y) \leq c\}$ is a weakly compact subset of $\mathbb{L}^1(\Omega, \mathscr{F}, \mathbb{P})$.

(ii) For every $X \in \mathbb{L}^\infty(\Omega,\mathscr{F},\mathbb{P})$, the supremum in the equality

$$\rho(X) = \sup_{Y\in\mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})} \{\mathbb{E}[XY] - \rho^*(Y)\}$$

is attained.

(iii) For every bounded sequence $\{X_n\}_{n\geq 1}$ in $\mathbb{L}^\infty(\Omega,\mathscr{F},\mathbb{P})$ tending a.s. to $X \in \mathbb{L}^\infty(\Omega,\mathscr{F},\mathbb{P})$, we have

$$\lim_n \rho(X_n) = \rho(X).$$

The proof of this result required compactness arguments of the perturbed James type and it was based on Theorem 1.51, see [91, Theorem A.1]. In [41] this result is already presented as a generalization of James' weak compactness theorem. Let us observe that we can apply Theorem 1.57 for $f = \rho^*$ to obtain the proof for the main implication (ii) $\Rightarrow$ (i) above. Indeed, $\mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})$ is weakly compactly generated and so its dual ball is $w^*$-sequentially compact.

Delbaen gave a different approach for Theorem 1.60. His proof is valid for non-separable $\mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})$ spaces, and it is based in a homogeneization trick to reduce the matter to a direct application of the classical James' weak compactness theorem, as well as the Dunford–Pettis theorem characterizing weakly compact sets in $\mathbb{L}^1(\Omega,\mathscr{F},\mathbb{P})$.

For our next application let us recall that a *Young function* $\Psi$ is an even, convex function $\Psi : E \to [0,+\infty]$ with the properties:

1. $\Psi(0) = 0$.
2. $\lim_{x\to\infty} \Psi(x) = +\infty$.
3. $\Psi < +\infty$ in a neighborhood of 0.

The Orlicz space $L^\Psi$ is defined as:

$$L^\Psi(\Omega,\mathscr{F},\mathbb{P}) := \{X \in L^0(\Omega,\mathscr{F},\mathbb{P}) : \text{there exixts } \alpha > 0 \text{ with } \mathrm{e}_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\},$$

and we consider the Luxembourg norm on it:

$$N_\Psi(X) := \inf\{c > 0 : \mathrm{e}_{\mathbb{P}}[\Psi(\frac{1}{c}X)] \leq 1\}, \qquad (X \in L^\Psi(\Omega,\mathscr{F},\mathbb{P})).$$

With the usual pointwise lattice operations, $L^\Psi(\Omega,\mathscr{F},\mathbb{P})$ is a Banach lattice and we have the inclusions:

$$L^\infty(\Omega,\mathscr{F},\mathbb{P}) \subset L^\Psi(\Omega,\mathscr{F},\mathbb{P}) \subset L^1(\Omega,\mathscr{F},\mathbb{P}).$$

Moreover, $(L^\Psi)^* = L^{\Psi^*} \oplus G$ where $G$ is the singular band and $L^{\Psi^*}$ is the order continuous band identified with the Orlicz space $L^{\Psi^*}$, where

$$\Psi^*(y) := \sup_{x\in\mathbb{R}}\{yx - \Psi(x)\}$$

is the Young function conjugate to $\Psi$, [126].

Risk measures defined on $L^{\Psi}(\Omega,\mathscr{F},\mathbb{P})$ and their robust representation are of interest in mathematical finance too. Delbaen has recently proved that a risk measure defined on $\mathbb{L}^{\infty}(\Omega,\mathscr{F},\mathbb{P})$ finitely extends to an Orlicz space if, and only if, it verifies the equivalent conditions of Theorem 1.60, see [42, Section 4.16]. Theorem 1.60 is extended to Orlicz spaces in [119, Theorem 1]:

**Theorem 1.61 (Lebesgue risk measures in Orlicz spaces).** Let $\Psi$ be a Young function with finite conjugate $\Psi^*$ and let

$$\alpha : (\mathbb{L}^{\Psi}(\Omega,\mathscr{F},\mathbb{P}))^* \to \mathbb{R}\cup\{+\infty\}$$

be a $\sigma((\mathbb{L}^{\Psi})^*,\mathbb{L}^{\Psi})$-lower semicontinuous penalty function representing a finite monetary risk measure $\rho$ as

$$\rho(X) = \sup_{Y\in\mathbb{M}^{\Psi^*}} \{-\mathbb{E}[X\cdot Y]-\alpha(Y)\}.$$

The following are equivalent:

(i) For each $c\in\mathbb{R}$, $\alpha^{-1}((-\infty,c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega,\mathscr{F},\mathbb{P})$.
(ii) For every $X\in\mathbb{L}^{\Psi}(\Omega,\mathscr{F},\mathbb{P})$, the supremum in the equality

$$\rho(X) = \sup_{Y\in\mathbb{M}^{\Psi^*}} \{-\mathbb{E}[X\cdot Y]-\alpha(Y)\}$$

is attained.
(iii) $\rho$ is order sequentially continuous.

Let us notice that order sequential continuity for a map $\rho$ in $\mathbb{L}^{\Psi}$ is equivalent to having

$$\lim_n \rho(X_n) = \rho(X)$$

whenever $(X_n)$ is a sequence in $L^{\Psi}$ a.s. convergent to $X$ and bounded by some $Z\in L^{\Psi}$, *i.e.* $|X_n|\le Z$ for all $n\in\mathbb{N}$. For that reason it is also said that a map $\rho : L^{\Psi}\to(-\infty,+\infty]$ verifies the Lebesgue property whenever it is sequentially order continuous. Orlicz spaces provide a general framework of Banach lattices for applications in mathematical finance, for a general picture see [38, 18, 19]. Non coercive growing conditions for penalty functions in the Orlicz case have been studied in [38]. More precisely, let us recall that a Young function $\Phi$ verifies the $\Delta_2$ condition if there exist $t_0>0$ and $K>0$ such that for every $t>t_0$

$$\Phi(2t)\le K\Phi(t).$$

In addition, the Orlicz heart $M^{\Psi}$ is the Morse subspace of all $X\in L^{\Psi}$ such that for every $\beta>0$

$$e_{\mathbb{P}}[\Psi(\beta X)] < +\infty.$$

In [38, Theorem 4.5] it is proved that a risk measure $\rho$, defined by a penalty function $\alpha$, is finite on the Morse subspace $\mathbb{M}^{\Psi}\subset L^{\Psi}$ if, and only if, $\alpha$ satisfies the growing condition

$$\alpha(Y) \geq a + b\|Y\|_{\Psi^*}$$

for all $Y \in \mathbb{L}^{\Psi^*}$, and fixed numbers $a,b$ with $b > 0$. Theorem 1.57 can be applied for $f = \rho^*$ because the spaces involved in the representation formulas have $w^*$-sequentially compact dual balls.

When $\Psi$ is a Young function such that either $\Psi$ or its conjugate verify the $\Delta_2$ condition we have the following result for the risk measures studied by Cheredito and Li in [38]:

**Corollary 1.62 ([119], Corollary 6 and 7).** Let $\Psi$ be a Young with finite conjugate $\Psi^*$ and such that either $\Psi$ or $\Psi^*$ verify the $\Delta_2$ condition. Let $\rho : \mathbb{L}^{\Psi}(\Omega, \mathscr{F}, \mathbb{P}) \to \mathbb{R}$ be a finite convex risk measure with the Fatou property, and

$$\rho^* : \mathbb{L}^{\Psi^*}(\Omega, \mathscr{F}, \mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$$

its Fenchel–Legendre conjugate defined on the dual space. The following are equivalent:

(i) For every $c \in \mathbb{R}$, $(\rho^*)^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathscr{F}, \mathbb{P})$.
(ii) For every $X \in \mathbb{L}^{\Psi}(\Omega, \mathscr{F}, \mathbb{P})$, the supremum in the equality

$$\rho(X) = \sup_{Y \in (\mathbb{M}^{\Psi^*})^+, \mathrm{e}(Y)=1} \{-\mathbb{E}[X \cdot Y] - \rho^*(-Y)\}$$

is attained.
(iii) $\rho$ is sequentially order continuous.
(iv) $\lim_n \rho(X_n) = \rho(X)$ whenever $X_n \nearrow X$ in $\mathbb{L}^{\Psi}$.
(v) $\mathrm{dom}(\rho^*) \subset \mathbb{M}^{\Psi^*}$.

We conclude this section with the following question:

**Question 5.** Does Corollary 1.59 remain valid in absence of symmetry?

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## References

1. M.D. Acosta, A. Aizpuru, R. Aron and F.J. García-Pacheco, *Functionals that do not attain their norm*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 407–418.
2. M. D. Acosta, J. Becerra Guerrero and M. Ruiz Galán, *Dual spaces generated by the interior of the set of norm attaining functionals*, Studia Math. **149** (2002), 175–183.
3. M. D. Acosta, J. Becerra Guerrero and M. Ruiz Galán, *Numerical-radius-attaining polynomials*, Quart. J. Math. **54** (2003), 1–10.

4. M.D. Acosta, J. Becerra Guerrero and M. Ruiz Galán, *James type results for polynomials and symmetric multilinear forms*, Ark. Mat. **42** (2004), 1–11.
5. M.D. Acosta and V. Kadets, *A characterization of reflexivity*, Math. Ann. **349** (2011), 577–588.
6. M.D. Acosta and V. Montesinos, *On a problem of Namioka on norm-attaining functionals*, Math. Z. **256** (2007), 295–300.
7. M.D. Acosta and M. Ruiz Galán, *New characterizations of the reflexivity in terms of the set of norm attaining functionals*, Canad. Math. Bull. **41** (1998), 279–289.
8. M.D. Acosta and M. Ruiz Galán, *Norm attaining operators and reflexivity*, Rend. Circ. Mat. Palermo **56** (1998), 171–177.
9. M.D. Acosta and M. Ruiz Galán, *A version of James' theorem for numerical radius*, Bull. London Math. Soc. **31** (1999), 67–74.
10. C. Angosto, *Distance to function spaces*, PhD thesis, Universidad de Murcia, 2007.
11. C. Angosto and B. Cascales, *Measures of weak noncompactness in Banach spaces*, Topology Appl. **156** (2009), 1412–1421.
12. K. Astala and H.O. Tylli, *Seminorms related to weak compactness and to Tauberian operators*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 367–375.
13. D. Azagra and T. Dobrowolski, *Smooth negligibility of compact sets in infinite-dimensional Banach spaces, with applications*, Math. Ann. **312** (1998), 445–463.
14. J. Banaś and A. Martinón, *Measures of weak noncompactness in Banach sequence spaces*, Portugal. Math. **52** (1995), 131–138.
15. P. Bandyopadhyay and G. Godefroy, *Linear structures in the set of norm-attaining functionals on a Banach space*, J. Convex Anal. **13** (2006), 489–497.
16. H.H. Bauschke, J.M. Borwein and P.L. Combettes, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3** (2001), 615–647.
17. E. Behrends, *New proofs of Rosenthal's $\ell^1$−theorem and the Josefson–Nissenzweig theorem*, Bull. Pol. Acad. Sci. Math. **43** (1996), 283–295.
18. S. Biagini and M. Fritelli, *On the Extension of the Namioka–Klee Theorem and on the Fatou Property for Risk Measures*, Optimality and Risk–Modern Trends in Mathematical Finance, The Kabanov Festschrift, Springer–Verlag, Berlin, 2009, 1–28.
19. S. Biagini and M. Fritelli, *A unified framework for utility maximization problems: an Orlicz space approach*, Ann. Appl. Prob. **18** (2008), 929–966.
20. F.S. de Blasi, *On a property of the unit sphere in a Banach space*, Colloq. Math. **65** (1992), 333–343.
21. J. Bonet and B. Cascales, *N*oncomplete Mackey topologies on Banach spaces, Bull. Aust. Math. Soc. **81** (2010), 409–413.
22. J.M. Borwein, L. Cheng, M. Fabián and J.P. Revalski, *A one perturbation variational principle and applications*, Set-Valued Anal. **12** (2004), 49–60.
23. J. Bourgain, *La propiété de Radon Nikodym*, Publications Mathématiques de l'Université Pierre et Marie Curie **36** (1979).
24. J. Bourgain and M. Talagrand, *Compacité extrêmale*, Proc. Amer. Math. Soc. **80** (1980), 68–70.
25. R.D. Bourgin, *Geometric aspects of convex sets with the Radon–Nikodym property*, Lecture Notes in Math. **993**, Springer–Verlag, Berlin, 1983
26. J.M. Borwein and Q.J. Zhu, *Techniques of variational analysis*, CMS Books in Mathematics **20**, Springer, 2005.
27. B. Calvert and S. Fitzpatrick, *Erratum: In a nonreflexive space the subdifferential is not onto*, Math. Zeitschrift **235** (2000), 627.
28. S. Carl, V.K. Le and D. Motreanu, *Nonsmooth variational problems and their inequalities. Comparison principles and applications*, Springer Monographs in Mathematics, Springer, New York, 2007.
29. B. Cascales, V.P. Fonf, J. Orihuela and S. Troyanski, *Boundaries of Asplund spaces*, J. Funct. Anal. **259** (2010), 1346–1368.

30. B. Cascales and G. Godefroy, *Angelicity and the Boundary Problem*, Mathematika **45** (1998), 105–112.
31. B. Cascales, O.F.K. Kalenda and J. Spurný, *A quantitative version of James's compactness theorem*, to appear in Proc. Edinburgh Math. Soc., 2012.
32. B. Cascales, G. Manjabacas and G. Vera, *A Krein–Šmulian type result in Banach spaces*, Quart. J. Math. Oxford **48** (1997), 161–167.
33. B. Cascales, W. Marciszesky and M. Raja, *Distance to spaces of continuous functions*, Topology Appl. **153** (2006), 2303–2319.
34. B. Cascales, M. Muñoz and J. Orihuela, *James boundaries and $\sigma$-fragmented selectors*, Studia Math. **188** (2008), 97–122.
35. B. Cascales and R. Shvydkoy, *On the Krein–Šmulian theorem for weaker topologies*, Ill. J. Math. **47** (2003), 957–976.
36. B. Cascales and G. Vera, *Topologies weaker than the weak topology of a Banach space*, J. Math. Anal. Appl. **182** (1994), 41–68.
37. A. Cellina, *On the nonexistence of solutions of differential equations in nonreflexive spaces*, Bull. Amer. Math. Soc. **78** (1972), 1069–1072.
38. P. Cheridito and T. Li, *Risks measures on Orlicz hearts*, Math. Finance **19** (2009), 189–214.
39. M. Contreras and R. Payá, *On upper semicontinuity of duality mappings*, Proc. Amer. Math. Soc. **121** (1994), 451–459.
40. H.H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796.
41. F. Delbaen, *Differentiability properties of utility functions*, F. Delbaen et al. (eds.), Optimality and risk–Modern trends in mathematical finance, Springer, 2009, 39–48.
42. F. Delbaen, *Draft: Monetary Utility Functions*, Lectures Notes in preparation.
43. R. Deville and C. Finet, *An extension of Simons' inequality and applications*, Revista Matemática Complutense, vol. XIV (2001), 95–104.
44. G. Debs, G. Godefroy and J. Saint Raymond, *Topological properties of the set of norm-attaining linear functionals*, Canad. J. Math. **47** (1995), 318–329.
45. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys Pure Appl. Math. **64**, Longman Sci. Tech., New York, 1993.
46. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Math. **92**, Springer–Verlag, New York, 1984.
47. J. Dieudonné, *Deux exemples singuliérs d'équations différentielles*, Acta Sci. Math. Szeged **12** (1950), Leopoldo Fejér et Frererico Riesz LXX annos natis dedicatus, pars B, 38–40.
48. R. Engelking, *General topology*, PWN–Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60].
49. M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics **8**, Springer–Verlag, New York, 2001.
50. M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, CMS Books in Mathematics, 2011.
51. M. Fabian, P. Hájek, V. Montesinos and V. Zizler, *A quantitative version of Krein's theorem*, Rev. Mat. Iberoamericana **21** (2005), 237–248.
52. S. Fitzpatrick and B. Calvert, *In a Nonreflexive Space the Subdifferential is Not Onto*, Math. Zeitschhrift **189** (1985), 555–560.
53. K. Floret, *Weakly Compact Sets,* Lecture Notes in Math. **801**, Springer–Verlag, Berlin, 1980.
54. H. Föllmer and A. Schied, *Stochastic Finance. An Introduction in Discrete Time*, de Gruyter Studies in Mathematics 27, 2nd Edition, Walter de Gruyter, Berlin, 2004.
55. V.P. Fonf and J. Lindenstrauss, *Boundaries and generation of convex sets*, Isr. J. Math. **136** (2003), 157–172.
56. V.P. Fonf, J. Lindenstrauss and R.R. Phelps, *Infinite-dimensional convexity*, in: Handbook of Banach Spaces I, Eds. W.B. Johnson and J.Lindenstrauss, Elsevier, 2001, 599–670.
57. F.J. García-Pacheco and D. Puglisi, Daniele, *Lineability of functionals and operators*, Studia Math. **201** (2010), 37–47.

58. I. Glicksberg, *The representation of functionals by integrals*, Duke Math. J. **19** (1952), 253–261.
59. G. Godefroy, *Boundaries of convex sets and interpolation sets*, Math. Ann. **277** (1987), 173–184.
60. G. Godefroy, *Five lectures in geometry of Banach spaces*, In: Seminar on Functional Analysis, 1987 (Murcia, 1987), volume 1 of Notas Mat., pages 9–67, Univ. Murcia, Murcia, 1988.
61. G. Godefroy, *Some applications of Simons' inequality*, Serdica Math. J. **26** (2000), 59–78.
62. A.N. Godunov, *The Peano theorem in Banach spaces*, Funkcional. Anal. i Priložen **9** (1974), 59–60.
63. M. González and J.M. Gutiérrez, *Polynomial Grothendieck properties*, Glasgow Math. J. **37** (1995), 211–219.
64. A.S. Granero, *An extension of the Krein–Šmulian theorem*, Rev. Mat. Iberoam. **22** (2006), 93–110.
65. A.S. Granero, P. Hájek and V. Montesinos Santalucía, *Convexity and $w^*$-compactness in Banach spaces*, Math. Ann. **328** (2004), 625–631.
66. A.S. Granero and J.M Hernández, *On James boundaries in dual Banach spaces*, J. Funct. Anal. **263** (2012), 429–447.
67. A.S. Granero, J.M. Hernández and H. Pfitzner, *The distance* $\operatorname{dist}(\mathscr{B},X)$ *when* $\mathscr{B}$ *is a boundary of* $B(X^{**})$, Proc. Amer. Math. Soc. **139** (2011), 1095–1098.
68. A.S. Granero and M. Sánchez, *The class of universally Krein–Šmulian Banach spaces*, Bull. Lond. Math. Soc. **39** (2007), 529–540.
69. A.S. Granero and M. Sánchez, *Convexity, compactness and distances*, in: Methods in Banach space theory, London Math. Soc. Lecture Note Ser. **337**, 215–237, Cambridge Univ. Press, Cambridge, 2006.
70. C. Grossmann, H. Roos and M. Stynes, *Numerical treatment of partial differential equations*, Universitex, Springer, Berlin, 2007.
71. A. Grothendieck, *Critères de compacité dans les espaces fonctionnels generaux*, Amer. J. Math. **74** (1952), 168–186.
72. J. Hagler and W.B. Johnson, *On Banach spaces whose dual balls are not weak$^*$ sequentially compact*, Isr. J. Math. **28** (1977), 235–330.
73. J. Hagler and E. Odell, *A Banach space not containing $\ell_1$ whose dual ball is not weak$^*$ sequentially compact*, Illinois J. Math. **22** (1978), 290–294.
74. P. Hájek, V. Montesinos, J. Vanderwerff and V. Zizler *Biorthogonal Systems in Banach Spaces* CMS Books in Mathematics **26**, Springer, 2008.
75. J. Hardtke, *Rainwater–Simons type convergence theorems for generalized convergence methods*, Acta Comm. Univ. Tartu. Math. **14** (2010), 65–74.
76. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras,* Lecture Notes in Math. **1547**, Springer–Verlag, Berlin, 1993.
77. R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford **27** (1976), 379–385.
78. R. Haydon, *Some more characterizations of Banach spaces containing $l_1$*, Math. Proc. Cambridge Philos. Soc. **80** (1976), 269–276.
79. R. Haydon, *On Banach spaces which contain $l^1(\tau)$ and types of measures on compact spaces*, Israel J. Math. **28** (1977), 313–324.
80. R. Haydon, M. Levy and E. Odell, *On sequences without weak$^*$ convergent convex block subsequences*, Proc. Amer. Mat. Soc. **100** (1987), 94–98.
81. R. Holmes, *Geometric Functional Analysis and its Applications*, Graduate Texts in Math. **24**, Springer–Verlag, New York, 1975.
82. R.C. James, *Reflexivity and the supremum of linear functionals*, Ann. of Math. **66** (1957), 159–169.
83. R.C. James, *Characterizations of reflexivity*, Studia Math. **23** (1964), 205–216.
84. R.C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. **113** (1964), 129–140.
85. R.C. James, *Weak compactness and reflexivity*, Israel. J. Math. **2** (1964), 101–119.
86. R.C. James, *A counterexample for a sup theorem in normed spaces*, Israel J. Math. **9** (1971), 511–512.

87. R.C. James, *Reflexivity and the Sup of Linear Functionals*, Isr. J. Math. **13** (1972), 289–300.
88. J.E. Jayne, J. Orihuela, A.J. Pallarés and G. Vera, *$\sigma-$fragmentability of multivalued maps and selection theorems*, J. Funct. Anal. **117** (1993), 243–273.
89. M. Jiménez-Sevilla and J.P. Moreno, *A note on norm attaining functionals*, Proc. Amer. Math. Soc. **126** (1998), 1989–1997.
90. W.B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in: Handbook of the geometry of Banach spaces, Vol. I, pages 1–84, North–Holland, Amsterdam, 2001.
91. E. Jouini, W. Schachermayer and N. Touzi, *Law invariant risk measures have the Fatou property*, Adv. Math. Econ. **9** (2006), 49–71.
92. O. Kalenda, *(I)-envelopes of unit balls and James' characterization of reflexivity*, Studia Math. **182** (2007), 29–40.
93. O. Kalenda,*(I)-envelopes of closed convex sets in Banach spaces*, Israel J. Math. **162** (2007), 157–181.
94. O.Kalenda and J.Spurny, *Boundaries of compact convex sets and fragmentability* , J. Funct. Anal. **256** (2009), 865–880.
95. J.L. Kelley, *General topology*, Graduate Texts in Mathematics **27**, Springer–Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.].
96. J.L. Kelley and I. Namioka, *Linear topological spaces*, Graduate Texts in Math., Springer–Verlag, New York, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing.
97. P.S. Kenderov, W.B. Moors and S. Sciffer, *Norm attaining functionals on $C(T)$*, Proc. Amer. Math. Soc. **126** (1998), 153–157.
98. N. Kenmochi, *Monotonicity and Compactness Methods for Nonlinear Variational Inequalities*, in: Handbook of differential equations. Stationary partial differential equations. Vol. 4, Ed. M. Chipot, Elsevier, Amsterdam, 2007.
99. K. Kivisoo and E. Oja, E. Oja, *Extension of Simons' inequality*, Proc. Amer.Math. Soc. **133** (2005), 3485–3496.
100. V.L. Klee, *Some characterizations of reflexivity*, Revista de Ciencias **52** (1950), 15–23.
101. H. König, *Theory and applications of superconvex spaces*, in: Aspects of Positivity in Functional Analysis, R . Nagel, U. Schlotterbeck and M.P.H. Wolff (editors), North–Holland, 1986.
102. G. Köthe, *Topological vector spaces. I*, Translated from the German by D. J. H. Garling, Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer–Verlag New York Inc., New York, 1969.
103. A. Kryczka, *Quantitative approach to weak noncompactness in the polygon interpolation method*, Bull. Austral. Math. Soc. **69** (2004), 49–62.
104. A. Kryczka and S. Prus, *Measure of weak noncompactness under complex interpolation*, Studia Math. **147** (2001), 89–102.
105. A. Kryczka, S. Prus and M. Szczepanik, *Measure of weak noncompactness and real interpolation of operators*, Bull. Austral. Math. Soc. **62** (2000), 389–401.
106. H.E. Lacey, *The isometric theory of classical Banach spaces*, Springer–Verlag, New York, 1974. Die Grundlehren der Mathematischen Wissenschaften, Band 208.
107. S. Mazur, *Über schwache konvergenz in den Raumen $\ell_p$*, Studia Math. **4** (1933), 128–133.
108. W.B. Moors, *An elementary proof of James' characterization of weak compactness*, Bull. Austr. Math. Soc. **84** (2011), 98–102.
109. W.B. Moors and J.R. Giles, *Generic continuity of minimal set-valued mappings*, J. Austr. Math. Soc. (Series A)**63** (1997), 238–262.
110. W.B. Moors and E.A. Reznichenko, *Separable subspaces of affine function spaces on convex compact sets*, Topology Appl. **155** (2008), 1306–1322.
111. M. Morillon, *A new proof of James' sup theorem*, Extracta Math. **20** (2005), 261–271.
112. M. Morillon, *James sequences and dependent choices*, Math. Log. Quart. **51** (2005), 171–186.
113. I. Namioka, *Radon–Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), 258–281.

114. M. Neumann, *Varianten zum Konvergenzsatz von Simons und Anwendungen in der Choquet theorie*, Arch. Math. **28** (1977), 182–192.
115. E. Oja, *A proof of the Simons's inequality*, Acta Comment. Univ. Tartuensis Math. **2** (1998), 27–28.
116. E. Oja, *Geometry of Banach spaces having shrinking approximations of the identity*, Trans. Amer. Math. Soc. **352** (2000), 2801–2823.
117. J. Orihuela, *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc., **36** (1987), 143–152.
118. J. Orihuela and M. Ruiz Galán, *A coercive James's weak compactness theorem and nonlinear variational problems*, Nonlinear Anal. **75** (2012), 598–611.
119. J. Orihuela and M. Ruiz Galán, *Lebesgue property for convex risk measures on Orlicz spaces*, Math. Finan. Econ. **6** (2012), 15–35.
120. A. Pełczyński, *On Banach spaces containing* $L_1(\mu)$, Studia Math. **30** (1968), 231–246.
121. J.I. Petunin and A.N. Plichko, *Some properties of the set of functionals that attain a supremum on the unit sphere*, Ukrain. Math. Zh. **26** (1974), 102–106.
122. H. Pfitzner, *Boundaries for Banach spaces determine weak compactness*, Invent. Math. **182** (2010), 585–604.
123. A. Plichko, *On sequential property of Banach spaces and duality*, preprint, 2003.
124. R. Pol, *On a question of H.H. Corson and some related problems*, Fund. Math. **109** (1980), 143–154.
125. J.D. Pryce, *Weak compactness in locally convex spaces*, Proc. Amer. Math. Soc. **17** (1966), 148–155.
126. M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces* Marcel Dekker, Inc. 1991
127. J. Rainwater, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963), 999.
128. R.T. Rockafellar, Characterization of the subdifferentials of convex functions., Pacific J. Math. **17** (1966), 497–510.
129. G. Rodé, *Ein Grenzwertsatz für stützende Funktionen auf superkonvexen Räumen*, Dissertation, Universität des Saarlandes, Saarbrücken, 1977.
130. G. Rodé, *Superkonvexität und schwache Kompaktheit*, Arch. Math. **36** (1981), 62–72.
131. M. Ruiz Galán, *Polynomials, symmetric multilinear forms and weak compactness*, Positivity **8** (2004), 297–304.
132. M. Ruiz Galán, *Convex numerical radius*, J. Math. Anal. Appl. **361** (2010), 481–491.
133. M. Ruiz Galán and S. Simons, *A new minimax theorem and a perturbed James's theorem*, Bull. Australian Math. Soc. **66** (2002), 43–56.
134. J. Saint Raymond, *Characterizing convex functions on a reflexive Banach space*, preprint.
135. E. Schechter, *Handbook of analysis and its foundations*, Academic Press, San Diego, 1997.
136. S. Simons, *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703–708.
137. S. Simons, *Maximinimax, minimax, and antiminimax theorems and a result of R.C. James*, Pacific J. Math. **40** (1972), 709–718.
138. S. Simons, *An eigenvector proof of Fatou's lemma for continuous functions*, Mathematical Intelligencer **17** (1995), 67–70.
139. S. Simons, *From Hahn–Banach to Monotonicity* Lecture Notes in Math. **1693**, Springer, 2008.
140. M.A. Smith M.A. and F. Sullivan, *Extremely smooth Banach spaces,* in: Banach spaces of analytic functions. Proc. Conf. Kent, Ohio, 1976. Lecture Notes in Math. **604**, pp. 125–137 (Baker, Cleaver and Diestel, Eds.), Springer–Verlag, Berlin, 1977.
141. J. Spurný, *The boundary problem for* $L^1$*-preduals*, Ill. J. Math. **52** (2008), 1183–1193.
142. S. M. Srivastava, *A Course on Borel Sets* , Graduate Text in Mathematics, vol 180, Spriger Verlag, New York, 1998.
143. T. Suzuki, *Characterizations of reflexivity and compactness via the strong Ekeland variational principle*, Nonlinear Anal. **72** (2010), 2204–2209.
144. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **307**, Providence, 1984.

145. M. Talagrand, *Sur les espaces de Banach contenant $l^1(\tau)$*, Israel J. Math. **40** (1982), 324–330.
146. S. Todorcevic, *Topics in Topology*, Lecture Notes in Math. **1652**, Springer, 1997.
147. I. Tweddle, *Weak compactness in locally convex spaces*, Glasgow Math. J. **9** (1968), 123–127.
148. M. Valdivia, *Fréchet spaces with no subspaces isomorphic to $\ell_1$*, Math. Japon. **38** (1993), 397–411.
149. M. Valdivia, *Boundaries of convex sets*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid, **87** (1993), 177–183.
150. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. **48** (1965), 161–228.
151. D. Werner, *New classes of Banach spaces which are M-ideals in their biduals,* Math. Proc. Cambridge Philos. Soc., **111** (1992), 337–354.
152. M. De Wilde, *Pointwise compactness in spaces of functions and R. C. James theorem*, Math. Ann. **208** (1974), 33–47.
153. E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B Nonlinear Monotone Operators*, Springer-Verlag 1990.