The Gelfand integral for multi-valued functions.

B. Cascales

Universidad de Murcia, Spain
Visiting Kent State University, Ohio. USA
http://webs.um.es/beca

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Notation

- $E$ Banach;
- $2^E$ subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;
Notation

- $E$ Banach;
- $2^E$ subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;
- $(\Omega, \Sigma, \mu)$ complete probability space;
- $\Sigma^+$ measurable sets of positive measure; for $A \in \Sigma$, $\Sigma_A^+$ measurable subsets of $A$ of positive measure.
Stay focused: kind of problems studied

Block 1.- *The setting and the single-valued case.*

Block 2.- *Measurability for multi-functions. Selectors*

Block 3.- *Integrability for multi-functions.*

Block 4.- *An open problem.*
The co-authors


[http://webs.um.es/beca/](http://webs.um.es/beca/)
Our interest: multi-functions, classical notions

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1. to take a reasonable embedding $j$ from $cwk(E)$ into the Banach space $Y(=\ell_\infty(B_E^*))$ and then study the integrability of $j \circ F$;
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4. Pettis integration for multi-functions was successfully studied in the separable case.
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NEW THINGS: The non-separable case

1. Characterization of multi-functions admitting strong selectors;
2. scalarly measurable selectors for scalarly measurable multi-functions;
3. Pettis integration;
4. existence of $w^*$-scalarly measurable selectors;
5. Gelfand integration; relationship with the previous notions.
Measurability and integrability of single-valued functions
Measurability: \( f : (\Omega, \Sigma, \mu) \to E \)

Simple function.- \( s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \), where \( \alpha_i \in E, A_i \in \Sigma \), disjoints.

Measurable function.- \( \lim_n \| s_n(w) - f(w) \| = 0 \), \( \mu \) a.e. \( w \in \Omega \).

\( w^* \)-scalarly measurable function.- when \( f : \Omega \to E^* \) and \( xf \) is measurable for \( x \in E \).

Scalarly measurable function.- \( x^*f \) is measurable for \( x^* \in E^* \).
Bochner integral.

A $\mu$-measurable $f : \Omega \to E$ is Bochner integrable, if there is a sequence of simple functions $(s_n)_n$ such that

$$\lim_{n} \int_{\Omega} \| s_n - f \| \, d\mu = 0.$$ 

The vector $\int_A f \, d\mu = \lim_n \int_A s_n \, d\mu$ is called Bochner integral of $f$.

Theorem

Let $f : \Omega \to E^*$ be a $w^*$-scalarly measurable function such that $xf \in L^1(\mu)$ such that $x \in E$. Then, the linear map

$$x_A^* : E \to \mathbb{R} \quad x \mapsto \int_A xf \, d\mu$$

lies in $E^*$, for each $A \in \Sigma$. $x_A^*$ is called the Gelfand integral of $f$ over $A$.

Gelfand integral.

$x_A^*(x) = \int_A xf \, d\mu$. 
Pettis integral

$f : \Omega \rightarrow E$ is Pettis integrable if $x^* f \in L^1(\mu)$ for every $x^* \in X^*$ and for every $A \in \Sigma$ there is $x_A \in E$ such that

$$(P) - \int_A x^* f \, d\mu := x^*(x_A), \, x^* \in X^*$$
Pettis integral

\[ f : \Omega \rightarrow E \text{ is Pettis integrable if } x^*f \in L^1(\mu) \text{ for every } x^* \in X^* \]

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Remark

Bochner Integrable \( \Rightarrow \) Pettis integrable \( \Rightarrow \) Gelfand (in \( X^{**} \))
Multi-functions: selectors
How can one define measurability for multi-function?

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- either consider the “pre-images” under $F$ of open sets;
- or consider an embedding $j : ckw(E) \rightarrow Y$ into a Banach space and then use some kind of measurability for $j \circ F$. 

Definition

$F : \Omega \rightarrow 2^E$ is said to be (Effros) measurable if

$$\{ t \in \Omega : F(t) \cap F \neq \emptyset \} \in \Sigma$$

for every open subset $F \subset E$.

Theorem (Kuratowski-Ryll Nardzewski, 1965)

Let $F : \Omega \rightarrow 2^E$ be a multi-function with closed non-empty values of $E$. If $E$ is separable and $F$ satisfies that

$$\{ t \in \Omega : F(t) \cap O \neq \emptyset \} \in \Sigma$$

for each open set $O \subset E$.

Then $F$ admits a $\mu$-measurable selector $f$. 

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\[(E)\]
Scalar measurability

**Definition**

- \( F : \Omega \rightarrow \text{cwk}(E) \) is said to be **scalarly measurable** if the real-valued map
  \[
  t \mapsto \delta^*(x^*, F(t)) := \sup \{ \langle x^*, x \rangle : x \in F(t) \}
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  is measurable for every \( x^* \in E^* \).
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- $F : \Omega \to cw^*(E^*)$ is said to be **$w^*$-scalarly measurable** if the function

  \[ t \mapsto \delta^*(x, F)(t) := \sup \{ \langle x^*, x \rangle : x^* \in F(t) \} \]

  is measurable for every $x \in E$. 
Good news & Bad news

Good news

$E$ separable, $F : \Omega \rightarrow cwk(E)$ multi-function. Then:

1. $F$ is scalarly measurable if, and only if, $F$ is Effros measurable.
2. Every scalarly measurable multi-function has a measurable selector, (Kuratowski and Ryll-Nardzewski [KRN65]).

Bad news

The above techniques do not work in the non-separable case.
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The above techniques do not work in the non-separable case.

Then...

we have a job... some other techniques are needed in the non separable case.
A new approach to find selectors

**Definition**

\( F : \Omega \to 2^E \) satisfies property (P) if for each \( \varepsilon > 0 \) and each \( A \in \Sigma^+ \) there exist \( B \in \Sigma^+_A \) and \( D \subset E \) with \( \text{diam}(D) < \varepsilon \) such that

\[ F(t) \cap D \neq \emptyset \]  
for every \( t \in B \).
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**Theorem (Kadets, Rodríguez and B. C. -2009)**

For a multi-function \( F : \Omega \rightarrow \text{wk}(E) \) TFAE:

(i) \( F \) admits a strongly measurable selector.

(ii) There exist a set of measure zero \( \Omega_0 \in \Sigma \), a separable subspace \( Y \subset E \) and a multi-function \( G : \Omega \setminus \Omega_0 \rightarrow \text{wk}(Y) \) that is Effros measurable and such that \( G(t) \subset F(t) \) for every \( t \in \Omega \setminus \Omega_0 \);

(iii) \( F \) satisfies property (P).
Scalar measurability

Theorem (Kadets, Rodriguez and B. C. - 2010)

Let $F : \Omega \to \text{cwk}(E)$ be a scalarly measurable multi-function. Then $F$ admits a scalarly measurable selector.

Proof.- Martingales and the RNP of weakly compact sets.
Scalar measurability

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**Proof.** Martingales and the RNP of weakly compact sets.

**Theorem (Kadets, Rodriguez and B. C. - 2010)**

$F : \Omega \to cwk(E)$ scalarly measurable. Then there is a collection $\{f_\alpha\}_{\alpha < \text{dens}(E^*, w^*)}$ of scalarly meas. selectors of $F$ such that

$$F(t) = \{f_\alpha(t) : \alpha < \text{dens}(E^*, w^*)\} \quad \text{for every } t \in \Omega.$$
$w^*$- scalar measurability

**Definition ($w^*$- almost selector)**

A single valued function $f : \Omega \rightarrow E^*$ is a $w^*$-almost selector of a multi-function $F : \Omega \rightarrow 2^{E^*}$ if for every $x \in E$ we have

$$\langle f, x \rangle \leq \delta^*(x, F) \mu - a.e.$$  

(the exceptional $\mu$-null set depending on $x$).
Definition ($w^*$- almost selector)

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Proposition

If $E$ be a separable Banach space, $F : \Omega \rightarrow cw^* k(E^*)$ is a multi-function and $f : \Omega \rightarrow E^*$ is a $w^*$-almost selector of $F$, then $f(t) \in F(t)$ for $\mu$-a.e. $t \in \Omega$. 
$w^*$-almost selectors

Theorem (Kadets, Rodriguez and B. C. - 2010)

Every $w^*$-scalarly measurable multi-function $F : \Omega \to cw^*k(E^*)$ admits a $w^*$-scalarly measurable $w^*$-almost selector.
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Remarks:

1. The proof uses the injectivity of $L^\infty$ and lifting techniques;
**Theorem (Kadets, Rodriguez and B. C. - 2010)**

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1. The proof uses the injectivity of $L^\infty$ and lifting techniques;
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**Remarks:**

1. The proof uses the injectivity of $L^\infty$ and lifting techniques;
2. If $E$ is separable, then $F$ admits a $w^*$-scalarly measurable selector.
3. If $E^*$ has the RNP, then $F$ admits a $w^*$-scalarly measurable selector.
Multi-functions: integrability
Debreu integrability

Let \( j : (ck(E), h) \to (\ell_\infty(B_E^*), \| \cdot \|_\infty) \) the Rådström embedding.

**Definición**

A multi-function \( F : \Omega \to ck(E) \) is said to be Debreu if the composition \( j \circ F : \Omega \to \ell_\infty(B_E^*) \) Bochner integrable.
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1. \( F \) satisfies property (P), hence \( F \) has measurable selectors;
2. For every \( A \in \Sigma \),
   \[
   \int_A Fd\mu = \{ \int_A fd\mu : f \text{ measurable selector of } F \}.
   \]
### Gelfand and Dundford Integrability: multi-functions

**Definition (Kadets, Rodríguez and B. C. -2010)**

A multi-function $F : \Omega \rightarrow cw^*k(E^*)$ is said to be Gelfand integrable if for every $x \in E$ the function $\delta^*(x, F)$ is integrable. In this case, the Gelfand integral of $F$ over $A \in \Sigma$ is defined as

$$\int_A F \, d\mu := \bigcap_{x \in E} \left\{ x^* \in E^* : \int_A \delta^*_*(x, F) \, d\mu \leq \langle x^*, x \rangle \leq \int_A \delta^*(x, F) \, d\mu \right\}.$$ 

**Theorem (Kadets, Rodríguez and B. C. -2010)**

Let $F : \Omega \rightarrow cw^*k(E^*)$ be a $w^*$-scalarly measurable multi-function.

- $F$ is Gelfand integrable iff every $w^*$-scalarly measurable $w^*$-almost selector of $F$ is Gelfand integrable.

In this case, for each $A \in \Sigma$, the set $\int_A F \, d\mu$ is non-empty and:

- $\int_A F \, d\mu = \left\{ \int_A f \, d\mu : f \text{ is a Gelfand integrable } w^*-\text{almost selector of } F \right\}$.
- $\delta^*(x, \int_A F \, d\mu) = \int_A \delta^*(x, F) \, d\mu$ for every $x \in E$. 

Properties of Gelfand integral

Theorem (Kadets, Rodríguez and B. C. -2010)

If \( F : \Omega \to cw^* k(E^*) \) is Gelfand integrable, then: for each \( A \in \Sigma \), the set \( \int_A F d\mu \) is non-empty and:

- \( \int_A F d\mu = \left\{ \int_A f d\mu : f \text{ is a Gelfand integrable } w^*\text{-almost selector of } F \right\} \).
- \( \delta^*(x, \int_A F d\mu) = \int_A \delta^*(x, F) d\mu \) for every \( x \in E \).

Taste of the proof.

1. \( S := \left\{ \int_\Omega f d\mu : f \text{ is a Gelfand int. } w^*\text{-almost sel. of } F \right\} \) is \( w^*\)-compact;
2. \( \int_\Omega F d\mu \supset S \) follows from the definitions;
3. \( \int_\Omega F d\mu \subset S \) follows from HB & \( \delta^*(x, \int_\Omega F d\mu) \leq \delta^*(x, S), \forall x \in E \);
4. Fix \( x \in E \); \( F|_x(t) := \{ x^* \in F(t) : \langle x^*, x \rangle = \delta^*(x, F)(t) \} \) is \( w^*\)-meas.;
5. let \( f : \Omega \to E^* \) be a \( w^*\)-scalarly measurable \( w^*\)-almost selector of \( F|_x \);
6. \( f \) is Gelfand integrable and \( \langle f, x \rangle = \delta^*(x, F) \mu\text{-a.e} \)
7. \( \delta^*(x, \int_\Omega F d\mu) \geq \langle \int_\Omega f d\mu, x \rangle = \int_\Omega \langle f, x \rangle d\mu = \int_\Omega \delta^*(x, F) d\mu \geq \delta^*(x, \int_\Omega F d\mu) \).
Theorem (Kadets, Rodríguez and B. C. -2009)

If $F : \Omega \rightarrow \text{cwk}(E)$ a Pettis integrable multi-function, then:

- every scalarly measurable selector is Pettis integrable;
- $F$ admits a scalarly measurable selector.

Furthermore, $F$ admits a collection $\{f_\alpha\}_{\alpha < \text{dens}(E^*, w^*)}$ of Pettis integrable selectors such that

$$F(t) = \overline{\{f_\alpha(t) : \alpha < \text{dens}(E^*, w^*)\}} \quad \text{for every } t \in \Omega.$$ 

Moreover, $\int_A F \, d\mu = \overline{\text{IS}_F(A)}$ for every $A \in \Sigma$. 
An open problem
An open question

Does any \( w^* \)-scalarly measurable multi-function \( F : \Omega \to cw^* k(E^*) \) admits a \( w^* \)-scalarly measurable selector?
Selected classical references


THANK YOU!