The Gelfand integral for multi-valued functions.

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Notation

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- 2^E subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;

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- E Banach;
- 2^E subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;
- (Ω, Σ, μ) complete probability space;
- Σ^+ measurable sets of positive measure; for $A \in \Sigma$, Σ_A^+ measurable subsets of A of positive measure.

Stay focused: kind of problems studied

Block 1.- *The setting and the single-valued case.*

Block 2.- *Measurability for multi-functions. Selectors*

Block 3.- *Integrability for multi-functions.*

Block 4.- *An open problem.*

The co-authors

- B. Cascales, **V. Kadets, and J. Rodríguez,** *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (2009), no. 3, 673–699. MR 2484932
- B. Cascales, **V. Kadets, and J. Rodríguez,** *Measurability and selections of multi-functions in Banach spaces*, J. Convex Anal. **17** (2010), no. 1.
- B. Cascales, **V. Kadets, and J. Rodríguez,** *The Gelfand integral for multi-functions*, Preprint 2010.

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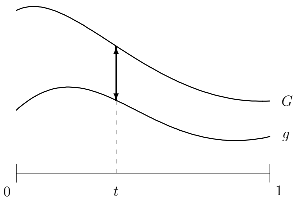
Our interest: multi-functions, classical notions

$F : \Omega \longrightarrow cwk(E)$ –convex w-compact



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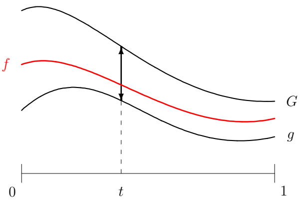

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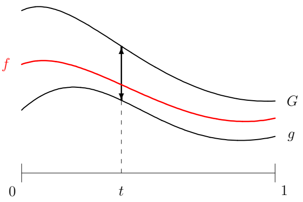

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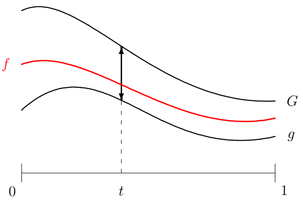

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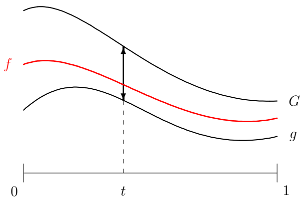

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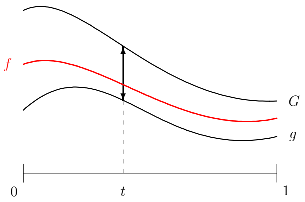

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4. Pettis integration for multi-functions was successfully studied in the separable case.

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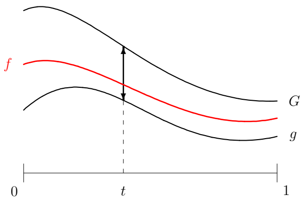

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NEW THINGS: The non-separable case

1. Characterization of multi-functions admitting strong selectors;
2. scalarly measurable selectors for scalarly measurable multi-functions;
3. Pettis integration;
4. existence of w^*-scalarly measurable selectors;
5. Gelfand integration; relationship with the previous notions.

Measurability and integrability of single-valued functions

Measurability: $f : (\Omega, \Sigma, \mu) \to E$

Simple function.- $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where $\alpha_i \in E$, $A_i \in \Sigma$, disjoints.

Measurable function.- $\lim_n \| s_n(w) - f(w) \| = 0, \quad \mu$ a.e. $w \in \Omega$.

w^*-scalarly measurable function.- when $f : \Omega \to E^*$ and xf is measurable for $x \in E$.

Scalarly measurable function.- $x^* f$ is measurable for $x^* \in E^*$.

Bochner integral

Bochner integral.- A μ-measurable $f : \Omega \longrightarrow E$ is *Bochner integrable*, if there is a sequence of simple functions $(s_n)_n$ such that

$$\lim_n \int_\Omega \| s_n - f \| \, d\mu = 0.$$

The vector $\int_A f \, d\mu = \lim_n \int_A s_n d\mu$ is called Bochner integral of f.

Theorem

Let $f : \Omega \longrightarrow E^*$ be a w^*-scalarly measurable function such that $xf \in L^1(\mu)$ such that $x \in E$. Then, the linear map

$$x_A^* : E \longrightarrow \mathbb{R} \qquad x \to \int_A xf \, d\mu$$

lies in E^*, for each $A \in \Sigma$. x_A^* is called the Gelfand integral of f over A.

Gelfand integral.- $x_A^*(x) = \int_A xf \, d\mu$.

Pettis integral

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$f : \Omega \longrightarrow E$ is Pettis integrable if $x^* f \in L^1(\mu)$ for every $x^* \in X^*$ and for every $A \in \Sigma$ there is $x_A \in E$ such that

$$(P) - \int_A x^* f \, d\mu := x^*(x_A), \; x^* \in X^*$$

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Remark

Bochner Integrable $\Rightarrow$ Pettis integrable $\Rightarrow$ Gelfand (in X^{**})

Multi-functions: selectors

How can one define measurability for multi-function?

Given $F : \Omega \to 2^E$ or $ckw(E)$

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Definition

$F : \Omega \longrightarrow 2^E$ is said to be (Effros) measurable

$$\{t \in \Omega : \; F(t) \cap F \neq \emptyset\} \in \Sigma \quad \text{for every open subset } F \subset E.$$

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Theorem (Kuratowski-Ryll Nardzewski, 1965)

Let $F : \Omega \to 2^E$ be a multi-function with closed non empty values of E. If E is separable and F satisfies that

$$\{t \in \Omega : F(t) \cap O \neq \emptyset\} \in \Sigma \text{ for each open set } O \subset E. \tag{E}$$

Then F admits a μ-measurable selector f.

Scalar measurability

Definition

- $F : \Omega \to cwk(E)$ is said to be scalarly measurable if the real-valued map

$$t \mapsto \delta^*(x^*, F(t)) := \sup\{\langle x^*, x\rangle : \ x \in F(t)\}$$

is measurable for every $x^* \in E^*$.

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is measurable for every $x^* \in E^*$.
- $F : \Omega \to cw^*(E^*)$ is said to be w^*-scalarly measurable if the function
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is measurable for every $x \in E$.

Good news & Bad news

Good news

E separable, $F : \Omega \longrightarrow cwk(E)$ multi-function. Then:

1. F is scalarly measurable if, and only if, F is Effros measurable.
2. Every scalarly measurable multi-function has a measurable selector, (Kuratowski and Ryll-Nardzewski [KRN65])

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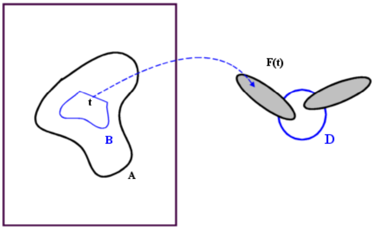
Then...

we have a job... some other techniques are needed in the non separable case.

A new approach to find selectors



Definition

$F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma^+_A$ and $D \subset E$ with $\mathrm{diam}(D) < \varepsilon$ such that

$$F(t) \cap D \neq \emptyset \text{ for every } t \in B.$$

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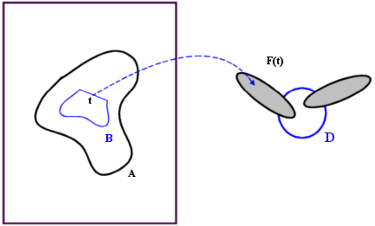

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$F:\Omega\to 2^E$ satisfies property (P) if for each $\varepsilon>0$ and each $A\in\Sigma^+$ there exist $B\in\Sigma_A^+$ and $D\subset E$ with $\operatorname{diam}(D)<\varepsilon$ such that

$$F(t)\cap D\neq\emptyset \text{ for every } t\in B.$$

Theorem (Kadets, Rodríguez and B. C. -2009)

For a multi-function $F:\Omega\to wk(E)$ TFAE:

(i) *F admits a strongly measurable selector.*

(ii) *There exist a set of measure zero $\Omega_0\in\Sigma$, a separable subspace $Y\subset E$ and a multi-function $G:\Omega\setminus\Omega_0\to wk(Y)$ that is Effros measurable and such that $G(t)\subset F(t)$ for every $t\in\Omega\setminus\Omega_0$;*

(iii) *F satisfies property (P).*

Scalar measurability

Theorem (Kadets, Rodriguez and B. C. - 2010)

Let $F : \Omega \to cwk(E)$ be a scalarly measurable multi-function. Then F admits a scalarly measurable selector.

Proof.- Martingales and the RNP of weakly compact sets.

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Theorem (Kadets, Rodriguez and B. C. - 2010)

$F : \Omega \to cwk(E)$ scalarly measurable. Then there is a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of scalarly meas. selectors of F such that

$$F(t) = \overline{\{f_\alpha(t) : \ \alpha < \mathrm{dens}(E^*, w^*)\}} \quad \text{for every } t \in \Omega.$$

w^*- scalar measurabilty

Definition (w^*- almost selector)

A single valued function $f : \Omega \to E^*$ is a w^*-almost selector of a multi-function $F : \Omega \to 2^{E^*}$ if for every $x \in E$ we have

$$\langle f, x \rangle \leq \delta^*(x, F) \ \mu - a.e.$$

(the exceptional μ-null set depending on x).

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Proposition

If E be a separable Banach space, $F : \Omega \to cw^*k(E^*)$ is a multi-function and $f : \Omega \to E^*$ is a w^*-almost selector of F, then $f(t) \in F(t)$ for μ-a.e. $t \in \Omega$.

w^*-almost selectors

Theorem (Kadets, Rodriguez and B. C. - 2010)

Every w^*-scalarly measurable multi-function $F : \Omega \to cw^*k(E^*)$ admits a w^*-scalarly measurable w^*-almost selector.

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1. The proof uses the injectivity of L^∞ and lifting techniques;
2. If E is separable, then F admits a w^*-scalarly measurable selector.
3. If E^* has the RNP, then F admits a w^*-scalarly measurable selector.

Multi-functions: integrability

Debreu integrability

Let $j : (ck(E), h) \to (\ell_\infty(B_{E^*}), \| \, \|_\infty)$ the Rådström embedding.

Definición

A multi-function $F : \Omega \longrightarrow ck(E)$ is said to be Debreu if the composition $j \circ F : \Omega \longrightarrow \ell_\infty(B_{E^*})$ Bochner integrable.

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1. F satisfies property (P), hence F has measurable selectors;
2. For every $A \in \Sigma$,

$$\int_A F d\mu = \{ \int_A f d\mu : f \text{ measurable selector of } F \}.$$

Gelfand and Dundford Integrability: multi-functions

Definition (Kadets, Rodríguez and B. C. -2010)

A multi-function $F:\Omega \to cw^*k(E^*)$ is said to be Gelfand integrable if for every $x \in E$ the function $\delta^*(x,F)$ is integrable. In this case, the Gelfand integral of F over $A \in \Sigma$ is defined as

$$\int_A F\,d\mu := \bigcap_{x\in E}\left\{x^* \in E^* : \int_A \delta_*(x,F)\,d\mu \le \langle x^*, x\rangle \le \int_A \delta^*(x,F)\,d\mu\right\}.$$

Theorem (Kadets, Rodríguez and B. C. -2010)

Let $F:\Omega \to cw^*k(E^*)$ be a w^*-scalarly measurable multi-function.

- *F is Gelfand integrable iff every w^*-scalarly measurable w^*-almost selector of F is Gelfand integrable.*

In this case, for each $A \in \Sigma$, the set $\int_A F\,d\mu$ is non-empty and:

- $\int_A F\,d\mu = \left\{\int_A f\,d\mu : \ f \text{ is a Gelfand integrable } w^*\text{-almost selector of } F\right\}$.
- $\delta^*(x, \int_A F\,d\mu) = \int_A \delta^*(x,F)\,d\mu$ *for every* $x \in E$.

Properties of Gelfand integral

Theorem (Kadets, Rodríguez and B. C. -2010)

If $F:\Omega \to cw^*k(E^*)$ is Gelfand integrable, then: for each $A \in \Sigma$, the set $\int_A F\,d\mu$ is non-empty and:

- $\int_A F\,d\mu = \left\{\int_A f\,d\mu : \ f \text{ is a Gelfand integrable } w^*\text{-almost selector of } F\right\}$.
- $\delta^*(x, \int_A F\,d\mu) = \int_A \delta^*(x,F)\,d\mu$ *for every* $x \in E$.

Taste of the proof.- $A = \Omega$

1. $S := \left\{\int_\Omega f\,d\mu : \ f \text{ is a Gelfand int. } w^*\text{-almost sel. of } F\right\}$ is w^*-compact;
2. $\int_\Omega F\,d\mu \supset S$ follows from the definitions;
3. $\int_\Omega F\,d\mu \subset S$ follows from HB & $\delta^*(x, \int_\Omega F\,d\mu) \le \delta^*(x,S)$, $\forall x \in E$;
4. Fix $x \in E$; $F|^x(t) := \{x^* \in F(t) : \ \langle x^*, x\rangle = \delta^*(x,F)(t)\}$ is w^*-meas.;
5. let $f:\Omega \to E^*$ be a w^*-scalarly measurable w^*-almost selector of $F|^x$;
6. f is Gelfand integrable and $\langle f, x\rangle = \delta^*(x,F)$ μ-a.e
7. $$\delta^*\left(x, \int_\Omega F\,d\mu\right) \ge \left\langle \int_\Omega f\,d\mu, x\right\rangle = \int_\Omega \langle f, x\rangle\,d\mu = \int_\Omega \delta^*(x,F)\,d\mu \ge \delta^*\left(x, \int_\Omega F\,d\mu\right).$$

Pettis integrability

Theorem (Kadets, Rodríguez and B. C. -2009)

If $F : \Omega \to cwk(E)$ a Pettis integrable multi-function, then:

- *every scalarly measurable selector is Pettis integrable;*
- *F admits a scalarly measurable selector.*

Furthermore, F admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of Pettis integrable selectors such that

$$F(t) = \overline{\{f_\alpha(t) : \ \alpha < \mathrm{dens}(E^*, w^*)\}} \quad \text{for every } t \in \Omega.$$

Moreover, $\int_A F \, d\mu = \overline{IS_F(A)}$ for every $A \in \Sigma$.

An open problem

An open question

Open question

Does any w^*-scalarly measurable multi-function $F : \Omega \to cw^*k(E^*)$ admits a w^*-scalarly measurable selector?

Selected classical references

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THANK YOU!