THE BISHOP-PHELPS-BOLLOBÁS THEOREM AND OPERATORS ON BANACH SPACES

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TABLE OF CONTENTS

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INTRODUCTION

Let X be a Banach space. A functional $x^* \in X^*$ is called norm-attaining if there exists $x_0 \in S_X$ so that $|x^*(x_0)| = 1$. A bounded linear operator $T \in L(X,Y)$ is norm-attaining, if there exists $x_0 \in S_X$ so that $\|T(x_0)\| = 1$. A fundamental theorem by R. C. James [32] gives a characterization of reflexivity through norm-attaining functionals: every bounded linear functional on X is norm-attaining if and only if the space X is reflexive. R. Phelps studied the spaces whose set of norm-attaining functionals is dense, the so-called "sub-reflexive spaces". Together with E. Bishop, R. Phelps proved that every Banach space is subreflexive.

The main interest of this thesis is in the following quantitative extension of the Bishop-Phelps theorem, the Bishop-Phelps-Bollobás theorem.

Theorem 0.1 ([15]). *Suppose $x_0 \in S_X$, $x^* \in S_{X^*}$ and $|x^*(x_0) - 1| \leq \varepsilon^2/2$, where $0 < \varepsilon < \frac{1}{2}$. Then there exist $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(u_0) = 1$, $\|x^* - y^*\| \leq \varepsilon$ and $\|x_0 - u_0\| < \varepsilon + \varepsilon^2$.*

We consider this theorem from three different aspects: norm-attaining functionals, norm-attaining operators, and numerical-radius attaining operators.

Chapter 1 combines three preliminary discussions on the Bishop-Phelps-Bollobás theorem in the classical sense – for norm-attaining functionals on Banach spaces. First, we consider two variational principles, the Ekeland's variational principle and the Brønsted-Rockafellar principle. We show how to prove the Bishop-Phelps-Bollobás theorem from the Brønsted-Rockafellar principle (Theorem 1.2). This version will be used in Chapter

2. Then we discuss the density theorems in the case of complex Banach spaces, a problem originally stated during a conference in Kent State in 1985 [45]. We finish the chapter with a survey on how "sharp" Theorem 0.1 is, which includes "the Bishop-Phelps-Bollobás modulus" of a Banach space Φ_X (Definition 1.2).

In Chapter 2, we study Theorem 0.1 in the setting of operators on Banach spaces. We recall in section 2.1 the notions of Asplund space and Asplund operator, with a comment on complex Banach spaces. In this section, we also prove a central technical result (Lemma 2.3) that will be used to prove our main result Theorem 2.4.

Theorem 2.4 establishes that if $T : X \to C_0(L)$ is an Asplund operator and $\|T(x_0)\| \approxeq \|T\|$ for some $\|x_0\| = 1$, then there is a norm attaining Asplund operator $S : X \to C_0(L)$ and $\|u_0\| = 1$ with $\|S(u_0)\| = \|S\| = \|T\|$ such that $u_0 \approxeq x_0$ and $S \approxeq T$.

Three consequences follow:

(A) If T is weakly compact, then S can also be taken being weakly compact (see Corollary 2.5).

(B) If X is Asplund, then the pair $(X, C_0(L))$ has the BPBP for all L (see Corollary 2.6).

(C) If L is scattered, then the pair $(X, C_0(L))$ has the BPBP for all X (see Corollary 2.7).

We note that in Corollary 2.5 even the part of the density of norm attaining weakly compact operators from X to $C_0(L)$ in the family of weakly compact operators $\mathcal{W}(X, C_0(L))$ seems to be new. Corollary 2.6 strengthens a result in [33] and Corollary 2.7 can be alternatively proved using a result in [5].

In Chapter 3, we investigate an analogue of the Bishop-Phelps-Bollobás property for operators 2.1 but in relation with numerical radius attaining operators. In Definition 3.1, we call it the *Bishop-Phelps-Bollobás property for numerical radius*, BPBp-ν for short. In section 3.1 we prove two constructive versions of Theorem 0.1 for $\ell_1(\mathbb{C})$–Theorems 3.4 and Theorem 3.6. Both versions are used later in the chapter to prove the main result in vein of numerical radius attaining operators: $\ell_1(\mathbb{C})$ has BPBp-ν, Theorem 3.7. Then using adjoint operators, a natural consequence of this result follows in Theorem 3.10: $c_0(\mathbb{C})$ has BPBp-ν. In particular, these provide quantitative versions and strengthen the results in [20].

Finally, we finish the chapter by showing that the aforesaid results hold true for $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$, where Γ is a general non-empty set and K could be either $\mathbb{R}$ or $\mathbb{C}$.

CHAPTER 1

THE BISHOP-PHELPS-BOLLOBÁS THEOREM AND VARIATIONAL PRINCIPLES

1.1 The Brønsted-Rockafellar Principle and Ekeland's Variational Principle

The Bishop-Phelps theorem [14] states that the set of bounded linear functionals attaining their maximum on a closed convex set of a real Banach space is dense in the topological dual. The essence of the proof relies on introducing a convex hull, getting a point on the boundary through a certain partial ordering and Zorn's lemma, and then employing the separation theorem to show that a hyperplane tangent at this point exists.

Similar arguments were used to prove both the Brønsted-Rockafellar principle [18] and Ekeland's variational principle [28]. On the other hand, either one of them implies the Bishop-Phelps theorem. Though I. Ekeland proved his famous variational principle shortly afterwards, the Brønsted-Rockafellar principle is often presented as a consequence of Ekeland's variational principle. We will follow this fashion. Our particular interest comes from the fact that the Bishop-Phelps-Bollobás theorem follows easily from the Brønsted-Rockafellar principle, with a better estimate than the original version.

Though the Ekeland's variational principle holds in every complete metric space, we will mostly work with Banach spaces. Let X be a Banach space, and let f be an extended real-valued function on X. The *effective domain* of f is the set $dom(f) = \{x \in X : f(x) < \infty\}$. The function is called *proper* if f is not identically $+\infty$, i.e. $dom(f) \neq \emptyset$. We say that f is *lower semicontinuous* provided $\{x \in X : f(x) \leq r\}$ is closed in X for every $r \in R$.

Theorem 1.1 (Ekeland's Variational Principle, [47])**.** *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous function that is bounded below. Let $\varepsilon > 0$ and suppose that at a given point x_0,*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon.$$

Then for any $\lambda > 0$, there exists $z \in dom(f)$ so that:

(i) $\lambda \|z - x_0\| \leq f(x_0) - f(z)$,

(ii) $\|z - x_0\| \leq \varepsilon / \lambda$,

(iii) $\lambda \|x - z\| + f(x) > f(z)$, *whenever* $x \neq z$.

Roughly, the theorem states that if a convex lower semicontinuous function f is close to its lower bound at some x_0, then a small Lipschitz continuous perturbation of f attains a minimum at a point z close to x_0.

Now, the Brønsted-Rockafellar principle studies the differentiability properties of convex functions. We will follow the presentation of ([47], pp.47–48) and [18].

Definition 1.1. *Let f be a proper convex lower semicontinuous function on X, $x \in dom(f)$, and $\varepsilon > 0$. For each ε, define the ε-subdifferential $\partial_\varepsilon f(x)$ by*

$$\partial_\varepsilon f(x) = \{x^* \in X^* : x^*(y) - x^*(x) \leq f(y) - f(x) + \varepsilon \textit{ for all } y \in X\}.$$

For each $\varepsilon > 0$ and $x \in dom(f)$, $\partial_\varepsilon f(x)$ is non-empty, and $\partial_\varepsilon f(x)$ is a w^*-closed set in X^*. As ε decreases, so does $\partial_\varepsilon f(x)$. The intersection over ε of the nest $\partial_\varepsilon f(x)$ is the *subdifferential*

$$\partial f(x) = \{x^* \in X^* : x^*(y) - x^*(x) \leq f(y) - f(x) \text{ for all } y \in X\}.$$

The subdifferential may be empty in arbitrary locally convex spaces. If at least one subgradient $x^* \in \partial f(x)$ exists, then f is *subdifferentiable* at x. As was noted by the authors themselves in [18], the theorem below estimates how well $\partial_\varepsilon f$ approximates ∂f.

For the rest of this section assume that X is a real Banach space.

Theorem 1.2 (The Brønsted-Rockafellar Principle)**.** *Assume that f is a convex proper lower semicontinuous function on X. Given $x_0 \in dom(f)$, $\varepsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\varepsilon f(x_0)$, there exist vectors $x \in dom(f)$ and $x^* \in \partial f(x)$ such that $\|x - x_0\| \leq \varepsilon/\lambda$ and $\|x^* - x_0^*\| \leq \lambda$.*

Proof. Since $x_0^* \in \partial_\varepsilon f(x_0)$, then

$$x_0^*(x) - x_0^*(x_0) \leq f(x) - f(x_0) + \varepsilon \text{ for any } x \in X.$$

Define a function $g(x) = f(x) - x_0^*(x)$, $x \in X$. Then g is proper, lower semicontinuous, and $dom(g) = dom(f)$. Moreover,

$$g(x_0) \leq \inf_{x \in X} g(x) + \varepsilon, \quad x \in X.$$

From Ekeland's variational principle 1.1, there exists $z \in dom(f)$ such that

$$\lambda \|z - x_0\| \leq \varepsilon \text{ and } \lambda \|x - z\| + g(x) \geq g(z)$$

for all $x \in X$. Call $h(x) = \lambda \|x - z\|$, $x \in X$. Then $h(x) + g(x) \geq g(z)$, or $h(x) + g(x) - h(z) - g(z) \geq 0$ for all $x \in X$. In particular,

$$h(x+z) - h(z) \geq -[g(x+z) - g(z)] \text{ for all } x \in X. \tag{1.1}$$

Define two sets in $X \times \mathbb{R}$:

$$C_1 = \{(x,r) : r \geq g(x+z) - g(z)\}, \tag{1.2}$$

$$C_2 = \{(x,r) : r < -[h(x+z) - h(z)]\}. \tag{1.3}$$

Since C_1 is an epigraph of a lower semicontinuous function $g(x+z)-g(z)$, C_1 is closed [47]. On the other hand, C_2 is an open convex cone, and from 1.1, $C_1 \cap C_2 = \emptyset$.

Hence, from the Hahn-Banach theorem, there is a hyperplane $(x^*, r^*) \in X^* \times \mathbb{R}$, with $(x^*, r^*) \neq (0,0)$, separating the sets 1.2 and 1.3:

$$-[h(x+z) - h(z)] \leq z^*(x) \leq g(x+z) - g(z),$$

$$-\lambda \|x\| \leq z^*(x) \leq f(x+z) - f(z) - x_0^*(x) \quad \text{for all } x \in X.$$

Set $x^* = z^* + x_0^*$ and $x = z$. From the right side of this inequality, it follows that for all $y \in X$, $z^*(y) + x_0^*(y) \leq f(y+x) - f(x)$, or, equivalently: $x^*(y) \leq f(y+x) - f(x)$. Hence, $x^* \in \partial f(x)$.

Lastly, observe that from the left side of the inequality: $z^*(\frac{x}{\|x\|}) \geq -\lambda$. Hence,

$$\|x^* - x_0^*\| = \sup_{x \in X} |z^*(\frac{x}{\|x\|})| \leq \lambda.$$

□

Now if C is a non-empty convex subset of X, then let $f = \delta_C$ be the indicator function of C: $f(x) = 0$, if $x \in C$, and $f(x) = \infty$, if $X \setminus C$. Note that δ_C is a proper convex function, which is lower semicontinuous if and only if C is closed. Application of the variational principle 1.2 to $f = \delta_{B_X}$ with ε^2 instead of ε and $\lambda = \varepsilon$ gives the following result.

Theorem 1.3. *Let X be a Banach space and $\varepsilon > 0$. Suppose $x_0 \in X$ and $x_0^* \in X^*$, $\|x_0^*\| = 1 = \|x_0\|$, and*

$$x_0^*(x_0) \geq 1 - \varepsilon^2.$$

Then there exist $u_0 \in B_X$ and $y^* \in X^*$ such that y^* attains its norm at u_0,

$$y^*(u_0) = \sup_{x \in B_X} |y^*(x)|,$$

with $\|x_0 - u_0\| < \varepsilon$ *and* $\|x_0^* - y^*\| < \varepsilon$.

Proof. Let $x \in B_X$. Then

$$x_0^*(x_0) \geq \sup_{c \in B_X} |x_0^*(c)| - \varepsilon^2 \geq x_0^*(x) - \varepsilon^2.$$

Hence, trivially:

$$(x_0^*, x - x_0) \leq \varepsilon^2 = \delta_{B_X}(x) - \delta_{B_X}(x_0) + \varepsilon^2, \quad \text{for all } x \in B_X.$$

Therefore, x_0^* is in the ε^2-subdifferential of δ_{B_X} at x_0. But then from the Brønsted-Rockafellar principle 1.2, there exist $u_0 \in B_X$ and $y^* \in X^*$ with $y^* \in \partial\delta_{B_X}(u_0)$, satisfying:

$$\|x_0 - u_0\| \leq \varepsilon^2/\varepsilon = \varepsilon \quad \text{and} \qquad \|x^* - y_0^*\| \leq \varepsilon.$$

In fact, $y^* \in \partial\delta_{B_X}(u_0)$ means that y^* is a subgradient of δ_{B_X} at u_0 and

$$y^*(y) - y^*(u_0) \leq \delta_{B_X}(y) - \delta_{B_X}(u_0) \text{ for all } y \in B_X.$$

Or, using the supremum:

$$\sup_{y \in B_X} y^*(y) \leq y^*(u_0).$$

Hence, y^* attains its maximum on B_X at u_0. □

Note that Theorem 1.3 differs from the Bishop-Phelps-Bollobás theorem 0.1, since the approximating functional y^* does not necessarily have norm 1. This is fixed in the following corollary, but with a sacrifice to the estimate. B. Bollobás in his original paper [15] constructed an example illustrating that the approximation below is the best possible.

Theorem 1.4. *Let* X *be a Banach space and* $\varepsilon > 0$. *Suppose* $x_0 \in X$ *and* $x_0^* \in X^*$, $\|x_0^*\| = 1 = \|x_0\|$, *and*

$$x_0^*(x_0) > 1 - \frac{\varepsilon^2}{2}.$$

Then there exist $u_0 \in X$ *and* $y^* \in X^*$, $\|y^*\| = 1 = \|u_0\|$, *such that*

$$y^*(u_0) = \sup_{x \in B_X} |y^*(x)|, \quad \|x_0 - u_0\| \leq \varepsilon, \quad \textit{and} \quad \|x^* - y^*\| \leq \varepsilon.$$

Proof. Following the reasoning in the proof of Theorem 1.3,

$$x_0^* \in \partial_{\varepsilon^2/2} \delta_{B_X}(x_0).$$

Apply the Brønsted-Rockafellar principle 1.2 and set $\lambda = \varepsilon/2$, which produce $u_0 \in dom(f)$ and a functional $z^* \in \partial \delta_{B_X}(u_0)$, with the following approximation

$$\|x_0 - u_0\| \leq \frac{\varepsilon^2/2}{\varepsilon/2} = \varepsilon \ \text{ and } \ \|x_0^* - z^*\| \leq \frac{\varepsilon}{2}.$$

Let $y^* = \dfrac{z^*}{\|z^*\|}$. Then $\|y^*\| = 1$ and

$$y^*(u_0) = \frac{z^*(u_0)}{\|z^*\|} = \sup_{x \in B_X} \frac{z^*(x)}{\|z^*\|} = \sup_{x \in B_X} y^*(x)$$

Moreover, y^* is sufficiently close to x_0^*:

$$\|x_0^* - y^*\| = \left\| x_0^* - \frac{z^*}{\|z^*\|} \right\| \leq \|x_0^* - z^*\| + \left\| z^* - \frac{z^*}{\|z^*\|} \right\| \leq 2 \cdot \varepsilon/2 = \varepsilon$$

This follows from the approximation above, and employing the following trick:

$$\|z^*\| \left\| 1 - \frac{1}{\|z^*\|} \right\| = \|1 - \|z^*\|\| = \|\|x_0^*\| - \|z^*\|\| \leq \|x_0^* - z^*\| < \varepsilon/2.$$

$\square$

Remark 1.4.1. *Even though the statements of Theorem 1.3 and Theorem 1.4 involve norm-attaining functionals, the results are also true for support functionals on a closed bounded convex set* C.

1.2 The Bishop-Phelps-Bollobás Theorem in the Complex Case

For a complex Banach space, it is important to establish first whether the Bishop-Phelps-Bollobás theorem is stated for the unit ball or a closed convex set.

In the case of the unit ball B_X, it was shown in [22] that the proof of the complex version of Theorem 1.4 can be reduced to the real case by the means of a canonical mapping, associating the duals of a complex Banach space X and its subjacent real space $X_{\mathbb{R}}$. Namely, define the map $\mathcal{R} : X^* \to (X_{\mathbb{R}})^*$ by

$$\mathcal{R}(x^*)(x) = \operatorname{Re} x^*(x) \quad \text{for } x \in X. \tag{1.4}$$

It is norm-preserving and $\mathbb{R}$-linear. To show that $\mathcal{R}$ is onto, let $y^* \in (X_{\mathbb{R}})^*$. Define the map $\mathcal{G} : (X_{\mathbb{R}})^* \to X^*$ by

$$\mathcal{G}(y^*)(x) = y^*(x) - iy^*(ix). \tag{1.5}$$

Then $\mathcal{G}$ is $\mathbb{R}$-linear and, moreover, $\mathbb{C}$-linear:

$$\mathcal{G}(y^*)(ix) = y^*(ix) - iy^*(-x) = y^*(ix) + iy^*(x) = i(y^*(x) - iy^*(ix)) = i\mathcal{G}(y^*)(x).$$

Thus, $\mathcal{G} \in X^*$, and $\mathcal{R}(\mathcal{G}) = y^*$. Hence, $\mathcal{R}$ is an $\mathbb{R}$-linear isomorphism and isometry. It is a standard material, see [29]. Moreover, in Chapter 2 it will be shown that $\mathcal{R}$ is a homeomorphism from (X^*, w^*) onto $((X_{\mathbb{R}})^*, w^*)$.

Here, Theorem 1.4 is stated in a more general form, regardless of X being a real or a complex Banach space.

Theorem 1.5 ([22]). *Let X be a Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $|x_0^*(x_0)| \geq 1 - \varepsilon^2/2$, where $0 < \varepsilon < \sqrt{2}$. Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that*

$$\|x_0^* - x^*\| \leq \varepsilon \quad \text{and} \quad \|x_0 - x\| \leq \varepsilon.$$

The idea of the proof is to apply the Theorem 1.4 to a real functional $\mathcal{R}(x_0^*)$ and a point $\lambda x_0 \in X$, where λ is a complex number such that $\lambda x_0^*(x_0) = |x_0^*(x_0)|$. This gives an approximating real-valued functional, attaining the norm at a point from $X_{\mathbb{R}}$. Thus, the Bishop-Phelps-Bollobás theorem can be used to get the approximating norm-attaining functional and a point at which this functional attains the norm. Then convert back to X, using the isometric properties of $\mathcal{R}$.

Now let $C \subseteq X$ be a balanced closed convex bounded set. A set C is called *balanced*, if $\alpha C \subseteq C$ for all $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$. First, for a balanced set C, $|x^*|$ attains its supremum on C if and only if $\mathrm{Re}\, x^*$ does. Moreover, $\sup_{x\in C} |x^*(x)| = \sup_{x\in C} |\mathrm{Re}\, x^*(x)|$. The proof is based on a standard argument using linearity. Indeed, for all $z \in \mathbb{C}$:

$$|\mathrm{Re}\, x^*(e^{-i\ \arg\ x^*(z)} \cdot z)| = |\mathrm{Re}\, (e^{-i\ \arg\ x^*(z)} \cdot x^*(z))| = |\mathrm{Re}\, |x^*(z)|| = |x^*(z)|,$$

Now suppose that $\mathrm{Re}\, x^*$ attains its maximum at some $z_0 \in C$. Then

$$\begin{aligned} \sup_{z\in C} |x^*(z)| &= \sup_{z\in C} |\mathrm{Re}\, x^*(e^{-i\ \arg\ x^*(z)} \cdot z)| \\ &\leq \sup_{z\in C} |\mathrm{Re}\, x^*(z)| = |\mathrm{Re}\, x^*(z_0)| \\ &= |x^*(e^{i\ \arg\ x^*(z_0)} \cdot z_0)| \leq \sup_{z\in C} |x^*(z)|. \end{aligned}$$

Since C is balanced and $|e^{-i\ \arg\ x^*(z_0)}| = 1$, then $e^{-i\ \arg\ x^*(z_0)} \cdot z_0 \in C$. The other direction follows similarly.

If C is non-balanced, in the same survey [3], as well as in [46], it was noted that two different approaches are possible. The first approach is that x^* attains the norm if and only if $\mathrm{Re}\, x^*$ attains the norm. The second approach is that x^* attains the norm if and only if $|x^*|$ attains the norm.

In the first case, the structure of the underlying real Banach space determines whether the set of support functionals is dense in X^*, allowing to apply the real version of the Bishop-Phelps-Bollobás theorem. This is the approach used in 1.5.

The second approach is more complicated, but intrinsic. J. Bourgain showed that if X is a complex Banach space with the Radon-Nikodým property [16], then the unit ball could be replaced by an arbitrary closed convex bounded set. From one of the equivalent definitions, X is said to have the *Radon-Nikodým property* (RNP), if every bounded subset of X is dentable, i.e. it has slices of an arbitrary small diameter. The question whether the Bishop-Phelps theorem would be true in general remained open for almost twenty years. It was answered negatively by V. Lomonosov in [39].

Let C be a subset of a Banach space. Then a point x is called a support point of C if there is a norm-attaining functional x^* which attains its supremum on C at x. Then x^* is called a support functional. V. Klee in [37] asked if each closed bounded convex subset of a Banach space has a support point. The Bishop-Phelps theorem could be restated in terms of support points and support functionals, and it implies that the set of support functionals is norm dense.

V. Lomonosov [39] constructed a very "pathological" (but in a natural setting) counterexample: a complex Banach space X and a closed convex bounded set $C \subset X$ with no support points whatsoever. Hence, there are no functionals in X^* that attain their norms on C at all. In contrast, in [46], R. R. Phelps proved that the set of "modulus support functionals" of a closed convex bounded subset of a real Banach space is dense in the dual space:

Theorem 1.6. *[46] Suppose X is a real Banach space and that $C \subset X$ is a nonempty closed convex bounded set. Given $x^* \in X^*$, $\|x^*\| = 1$, and $\varepsilon > 0$, there exists $y^* \in X^*$, $\|y^*\| = 1$, such that $\sup_{x \in X} |y^*(x)| = |y^*(u)|$ for some $u \in C$, and $\|x^* - y^*\| < \varepsilon$.*

Consider the counterexample constructed by V. Lomonosov [39]. Let $\mathcal{H}^\infty$ be the algebra of bounded analytic functions on a unit disc D, with the norm $\|f\| = \sup_{z\in D}|f(z)|$. Then $\mathcal{H}^\infty \cong X^*$ for some Banach space X.

For every $z \in D$, consider the point evaluations φ_z, i.e. $\varphi_z(f) = f(z)$. Let L be a line generated by the point evaluation at 0. Define the quotient map $\pi : X \to X/L$. By the isomorphism theorem, $(X/L)^* \cong L^\perp$, where

$$\begin{aligned} L^\perp &= \{g \in \mathcal{H}^\infty : g(c\cdot\varphi_0) = 0 \text{ for all } c \in \mathbb{C}\} \\ &= \{g \in \mathcal{H}^\infty : g(0) = 0\}. \end{aligned}$$

Let $S_1 = \pi(\overline{\text{conv}}\{\varphi_z\}) \subset X/L$. Note that $\overline{\text{conv}}\{\varphi_z\}$ is a closed convex bounded set. It was shown in [39, Theorem 2], that no functional from $L^\perp$ attains its supremum on S_1. Note also that S_1 is a closed convex bounded set, which is *non-balanced*.

1.3 How Sharp is the Bishop-Phelps-Bollobás Theorem?

The Bishop-Phelps-Bollobás theorem has appeared in several different forms, with varying approximation of functionals and points([15], [5], [22], [10]). Consider the following three versions. First, the original statement by B. Bollobás:

Theorem 1.7 ([15]). *Suppose $x_0 \in S_X$, $x^* \in S_{X^*}$ and $|x^*(x_0) - 1| \le \varepsilon^2/2$, where $0 < \varepsilon < \frac{1}{2}$. Then there exist $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(u_0) = 1$, $\|x^* - y^*\| \le \varepsilon$, and $\|x_0 - u_0\| < \varepsilon + \varepsilon^2$.*

In addition, B. Bollobás constructed an example using $\mathbb{R}^2$, showing that in order for two specific functionals to satisfy $\|x^* - y^*\| < \varepsilon$, the points would have to be at least ε apart. Thus, he stated the "best possible version" of this result as the remark below.

Remark 1.7.1. *[15] For any $0 < \varepsilon < 1$, there exist a Banach space X, point $x \in S_X$ and functional $x^* \in S_{X^*}$ such that $x^*(x) = 1 - \varepsilon^2/2$ but if $u \in S_X$, $y^* \in S_{X^*}$ and $y^*(u) = 1$, then either $\|x^* - y^*\| \geq \varepsilon$ or $\|x - u\| \geq \varepsilon$.*

The bottom line is that for a closer approximation of the functionals, the approximating points will have to be further apart.

In the next version of Theorem 1.7, the point x_0 lies inside the unit ball rather than on the boundary. This comes at a price: in order to get the same ε approximation, $x^*(x_0)$ has to be closer to 1.

Theorem 1.8. *[5] Let $\varepsilon > 0$, $x_0 \in B_X$, $x^* \in S_{X^*}$ such that $|x^*(x_0) - 1| < \varepsilon^2/4$. Then there are $u_0 \in S_X$, $y^* \in S_{X^*}$ such that $y^*(u_0) = 1 = \|y^*\|$,*

$$\|x^* - y^*\| < \varepsilon, \quad \textit{and} \quad \|x_0 - u_0\| < \varepsilon.$$

Finally, the Brønsted-Rockafellar principle 1.2 gives a sharper form of the Bishop-Phelps-Bollobás theorem:

Theorem 1.9. *[22] Let $\varepsilon > 0$, and $x_0 \in S_X$, $x^* \in S_{X^*}$ such that $|x^*(x_0)| > 1 - \varepsilon^2/2$. Then there exists $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(u_0) = 1$,*

$$\|y^* - x^*\| \leq \varepsilon, \quad \textit{and} \quad \|u_0 - x_0\| \leq \varepsilon.$$

Hence, a natural question:

What is the best estimate in the Bishop-Phelps-Bollobás theorem?

The meaning of this question is the following: what is the largest $\varepsilon(\delta) > 0$ such that whenever $x^* \in S_X^*$ and $x_0 \in S_X$ are such that $|x^*(x_0)| > 1 - \varepsilon(\delta)$, then there is $y^* \in S_{X^*}$ and $u_0 \in S_X$ such that $|y^*(u_0)| = 1$,

$$\|y^* - x^*\| \leq \varepsilon, \quad \text{and} \quad \|u_0 - x_0\| \leq \varepsilon?$$

This question was first properly studied in [23]. The authors introduced the infimum of all such $\varepsilon(\delta)$, called the modulus of a Banach space $\Phi_X(\delta)$. Let X be a real or a complex Banach space and $\delta > 0$. To define $\Phi_X(\delta)$, first consider the set of pairs $(x, x^*) \in B_X \times B_{X^*}$ such that $x^*(x)$ is almost 1:

$$A_X(\delta) = \{(x, x^*) \in B_X \times B_{X^*} : \; Re(x^*(x)) > 1 - \delta\}.$$

Define a set of functionals that attain their norms:

$$\Pi(X) = \{(x, x^*) \in X \times X^* : \; x \in S_X, \; x \in S^*, \; x^*(x) = 1\}.$$

Definition 1.2. *[23] The Bishop-Phelps-Bollobás modulus of a Banach space X is the function $\Phi_X : (0, 2) \to \mathbb{R}^+$ such that given $\delta \in (0, 2)$, $\Phi_X(\delta)$ is the infimum of all $\varepsilon > 0$ such that for every $(x, x^*) \in B_X \times B_{X^*}$ with $Re(x^*(x)) > 1 - \delta$, there is a pair $(u, y) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.*

It is clear that a smaller $\Phi_X(\delta)$ gives a better approximation on X, and according to [23, Theorem 2.1], the upper bound for $\Phi_X(\delta)$ is $\sqrt{2\delta}$. Note when $\delta = \varepsilon^2/2$, $\Phi_X(\delta) \leq \varepsilon$. Hence, Theorem 1.9, the version from the Brønsted-Rockafellar principle 1.2, gives the sharpest form of the Bishop-Phelps-Bollobás theorem.

Theorem 1.10. *[23] Let $0 < \varepsilon < 2$. Suppose $x \in B_X$ and $f \in B_X^*$ are such that*

$$Ref(x) > 1 - \frac{\varepsilon^2}{2}.$$

Then there exists a pair $(y, g) \in \Pi(X)$ with

$$\|x - y\| < \varepsilon \;\; \textit{and} \;\; \|f - g\| < \varepsilon.$$

A counterpart of $\Phi_X(\delta)$ for $x_0 \in S_X$ is called *the spherical modulus of the Banach space*.

Definition 1.3. *[23] The spherical Bishop-Phelps-Bollobás modulus of a Banach space X is the function $\Phi_X^S : (0,2) \to \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X^S(\delta)$ is the infimum of all $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $Rex^*(x) > 1-\delta$, there is a pair $(u,y) \in \Pi(X)$ satisfying $\|x-u\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.*

The best possible upper bound is

$$\Phi_X^S(\delta) \leq \Phi_X(\delta) \leq \sqrt{2\delta}, \text{ where } 0 < \delta < 2.$$

(Recall Theorem 1.8.) And in the worst approximation, both moduli coincide:

$$\Phi_X(\delta) = \sqrt{2\delta} \text{ if and only if } \Phi_X^S(\delta) = \sqrt{2\delta}.$$

CHAPTER 2

ASPLUND SPACES AND THE BISHOP-PHELPS-BOLLOBÁS PROPERTY

In this chapter we are concerned with the study of simultaneously approximating both operators and the points at which they almost attain their norms by norm attaining operators and the points at which they attain their norms. Namely, we study the Bishop-Phelps-Bollobás property:

Definition 2.1 (M. Acosta, R. Aron, D. García, and M. Maestre, [5])**.** *A pair of Banach spaces (X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBp for short) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t \to 0} \beta(t) = 0$, such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then there are $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying*

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \beta(\varepsilon), \textit{ and } \|T - S\| < \varepsilon.$$

Thus, a pair of Banach spaces (X, Y) has the BPBp if a "Bishop-Phelps-Bollobás" type theorem can be proved for the set of operators from X to Y. This property implies, in particular, that the norm attaining operators from X to Y are dense in the whole space of continuous linear operators $L(X, Y)$. However, as shown in [5], the converse is not true. Consequently, the BPB property is more than a quantitative tool for studying the density of norm attaining operators.

In general, the BPB property fails for a pair of Banach spaces X and Y. This is a simple consequence of the fact that the set of norm-attaining operators on arbitrary Banach spaces X and Y might not be dense in $L(X, Y)$, [38]. To add structure to the research of this very general question, J. Lindentrauss introduced properties A and B. The domain space X is said to have *property A*, if for every Banach space Y, the set of norm-attaining operators $T : X \to Y$ is dense in $L(X,Y)$. Similarly, the range space Y is said to have *property B*, if the set of norm-attaining operators $T : X \to Y$ is norm dense in $L(X,Y)$ for every Banach space X.

In [5], the authors described a number of cases of pairs (X,Y) with BPBp. For instance, if Y has property (β), see [51, Definition 1.2], then (X,Y) has BPBp for every Banach space X. Also, (ℓ_1, Y) has BPBp for Y in a large class of Banach spaces that includes the finite dimensional Banach spaces, uniformly convex Banach spaces, spaces $L_1(\mu)$ for a σ-finite measure μ and spaces $C(K)$, where K is a compact Hausdorff space. Although some particular results can be found in [5, Section 5] for pairs of the form (ℓ_n^∞, Y) (for instance, Y uniformly convex), the authors of [5] comment that their methods do not work for pairs of the form (c_0, Y). Recently, S. K. Kim showed that (c_0, Y) has BPBp for Y uniformly convex, [34]. In addition, if Y is a strictly convex (real) space, then (c_0, Y) and (l_1, Y) are examples of pairs of spaces such that the set of norm-attaining operators is dense, but the BPBp fails, [34, 5].

The results in this chapter are based on joint work with R. Aron and B. Cascales [10]. Here, we devise a method to study the Bishop-Phelps-Bollobás property that in particular addresses this question when $Y = C_0(L)$, L a locally compact Hausdorff space.

2.1 Asplund Operators on Real and Complex Spaces

Let X be a complex Banach space. Recall the map 1.4, associating the dual of X and its subjacent real space $X_{\mathbb{R}}$, $\mathcal{R} : X^* \to (X_{\mathbb{R}})^*$ defined by $\mathcal{R}(x^*)(x) = \operatorname{Re} x^*(x)$, for $x \in X$.

Note that $\mathcal{R}$ is a homeomorphism from (X^*, w^*) onto $((X_{\mathbb{R}})^*, w^*)$. Indeed, let us show that $\mathcal{R}$ is w^*-w^* continuous. Suppose $\{x_\alpha^*\}_{\alpha \in I} \subseteq X^*$ and $x_\alpha^* \xrightarrow{w^*} x^*$. Hence for all $x \in X$,

$x^*_\alpha(x) \to x^*(x)$ pointwise. Since it is a net of complex numbers, $\text{Re } x^*_\alpha(x) \to \text{Re } x^*(x)$ and $\text{Im } x^*_\alpha(x) \to \text{Im } x^*(x)$. Hence, $(\mathcal{R}x^*_\alpha)(x) = \text{Re } x^*_\alpha(x) \to \text{Re } x^*(x) = (\mathcal{R}x^*)(x)$, yielding $\mathcal{R}x^*_\alpha \xrightarrow{w^*} \mathcal{R}x^*$.

Now we prove that the inverse map $\mathcal{G}$ is w^*-w^* continuous. Let $y^* \in X_{\mathbb{R}}$ and define the inverse map 1.5:

$$\mathcal{G} : (X_{\mathbb{R}})^* \to X^*,$$

$$\mathcal{G}(y^*)(x) = y^*(x) - iy^*(ix).$$

Suppose $\{y^*_\alpha\}_{\alpha \in I} \subseteq (X_{\mathbb{R}})^*$ and $y^*_\alpha \xrightarrow{w^*} y^*$. Fix $x \in X$. Since $y^*_\alpha(x) \to y^*(x)$, then for all $\varepsilon > 0$, there exists $\alpha_1 \in I$, such that whenever $\beta_1 > \alpha_1$, then $\|y^*_{\beta_1}(x) - y^*(x)\| < \varepsilon$. Similarly, there is some $\alpha_2 \in I$, such that for all $\beta_2 > \alpha_2$, $\|y^*_{\beta_2}(ix) - y^*(ix)\| < \varepsilon$. Let $\alpha > max\{\alpha_1, \alpha_2\}$. Then

$$\begin{aligned} |\mathcal{G}(y^*_\alpha)(x) - \mathcal{G}(y^*)(x)| &= \|y^*_\alpha(x) - iy^*_\alpha(ix) - (y^*(x) - iy^*(ix))\| \\ &< \|y^*_\alpha(x) - y^*(x)\| + \|y^*_\alpha(ix) - y^*(ix)\| \\ &< \varepsilon. \end{aligned}$$

Therefore, $\mathcal{G}(y^*_\alpha) \xrightarrow{w^*} \mathcal{G}(y^*)$. Hence, $\mathcal{R}$ is a homeomorphism.

The Banach space X is called an *Asplund space* if, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [11] under the name *strong differentiability space*. Asplund spaces have been used profusely since they were introduced. The versatility of this concept is in part explained by its multiple characterizations via topology or measure theory, as presented for instance in the theorem below.

Recall that a subset C of (X^*, w^*) is said to be *fragmented by the norm* if for each non-empty subset A of C and for each $\varepsilon > 0$ there exists a non-empty w^*-open subset U of X^* such that $U \cap A \neq \emptyset$ and $\|\cdot\|\text{-}\mathrm{diam}(U \cap A) \leq \varepsilon$, [40].

Theorem 2.1. *[41] Let X be a Banach space. Then the following conditions are equivalent:*

(i) X is an Asplund space;

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has a separable dual;

(iv) X^* has the Radon-Nikodým property.

The proof in [41] is given for a real Banach space. However, it could be adjusted for a complex Banach space using the homeomorphism $\mathcal{R}$. Note that a complex Banach space X is Asplund if the subjacent real space $X_{\mathbb{R}}$ is Asplund.

Proof. Let X be a real Banach space. The Radon-Nikodým property was defined in Chapter 1. For the equivalent notions of this property we refer to [17, 26]. The equivalence (iii) $\Leftrightarrow$ (iv) is due to Stegall [54], (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) can be found in the paper by Namioka and Phelps [42] and (iii) $\Rightarrow$ (ii) is due again to Stegall [55]. We note that if C is w^*-compact convex, then C is fragmented by the norm if, and only if, C has the Radon-Nikodým property, see [17, Theorem 4.2.13].

Now let X be a complex Asplund space, $C \subset X^*$ be a w^*-compact set, $A \subset C$ a non-empty subset, and $\varepsilon > 0$. Since $\mathcal{R}$ is w^*-continuous, then $\mathcal{R}(C) \subset X^*_{\mathbb{R}}$ is w^*-compact, $\mathcal{R}(A) \subset \mathcal{R}(C)$, $\mathcal{R}(A) \neq \emptyset$. The underlying real space $X_{\mathbb{R}}$ is Asplund, hence from (ii) there exists a w^*-open $U \subset X^*_{\mathbb{R}}$ such that $U \cap \mathcal{R}(A) \neq \emptyset$, and $\|\cdot\|\text{-}\mathrm{diam}(U \cap \mathcal{R}(A)) \leq \varepsilon$.

Note that $\mathcal{R}$ is a w^*-w^* homeomorphism, thus $\mathcal{R}^{-1}(U)$ is w^*-open. Since $\mathcal{R}$ is an isometry, $\|\cdot\|$-$\mathrm{diam}(\mathcal{R}^{-1}(U) \cap A) \leq \varepsilon$, and $\mathcal{R}^{-1}(U) \cap A \neq \emptyset$. Therefore, C is w^*-fragmented by the norm, and (i) $\Rightarrow$ (ii). The other side of the equivalence, (ii) $\Rightarrow$ (i), follows similarly. $\square$

An operator $T \in L(X,Y)$ is said to be an *Asplund operator* if it factors through an Asplund space, *i.e.*, there are an Asplund space Z and operators $T_1 \in L(X,Z)$, $T_2 \in L(Z,Y)$ such that $T = T_2 \circ T_1$, see [27, 56]. Note that every weakly compact operator $T \in \mathcal{W}(X,Y)$ factors through a reflexive Banach space, see [24], and hence T is an Asplund operator (it is easy to see that the equivalence (i)-(iii) in Theorem 2.1 implies that every reflexive space is Asplund).

An operator $T \in L(X,Y)$ is called a *Radon-Nikodým operator* if for every probability space (Ω, Σ, μ) and every μ-continuous vector measure $\mu : \Sigma \to X$ with finite variation, there exists a Bochner integrable function $g : \Omega \to Y$ such that $(T \circ \nu)(E) = \int_E g d\mu$ for all $E \in \Sigma$, i.e. the measure $T \circ \nu$ has a Radon-Nikodým derivative in $L_1(\Omega, \Sigma, \mu)$, [27, 49]. However, we are interested more in the definition of the Radon-Nikodým operator that comes from the duality aspect (i)$\Leftrightarrow$(iv) of Theorem 2.1. An operator $T \in L(X,Y)$ is Asplund if and only if $T^* : Y^* \to X^*$ is a Radon-Nikodým operator. This equivalence is due to C. Stegall:

Theorem 2.2 ([56], Theorem 2.11)**.** *Let $T : X \to Y$ be a bounded linear operator. Then the following are equivalent:*

(i) T^* factors through a space W with W having RNP;

(ii) T factors through a space Z with Z being Asplund.

2.2 The Bishop-Phelps-Bollobás Theorem and Asplund Operators on $C(K)$

In this section we present our main result related to the operators on spaces of continuous functions, Theorem 2.4. We begin with Lemma 2.3, which isolates the technicalities that we need to prove our main theorem. Originally, it was proved in [10]. But Lemma 2.3 was improved quantitatively in [22], and that shall be the version we present. In the proof of the lemma, we use the Bishop-Phelps-Bollobás Theorem 1.5.

Recall that a subset $B \subset B_{Y^*}$ is said to be *1-norming* if

$$\|y\| = \sup_{b^* \in B} |b^*(y)|,$$

for every $y \in Y$. Recall that if $T \in L(X, Y)$, then its adjoint $T^* \in L(Y^*, X^*)$ is also w^*-w^* continuous.

Lemma 2.3. *[22, Lemma 3.5] Let $T : X \to Y$ be an Asplund operator with $\|T\| = 1$, $0 < \varepsilon < \sqrt{2}$, and $x_0 \in S_X$ such that $\|Tx_0\| > 1 - \varepsilon^2/2$. For any given 1-norming set $B \subset B_{Y^*}$ if we write $M = T^*(B)$ then, for every $r > 0$ there exist:*

(i) a $weak^*$-open set $U_r \subset X^*$ with $U_r \cap M \neq \emptyset$, and

(ii) points $y_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|y_r^*(u_r)| = 1$ such that

$$\|x_0 - u_r\| \leq \varepsilon \quad \textit{and} \quad \|z^* - y_r^*\| \leq r + \frac{\varepsilon^2}{2} + \varepsilon \ \textit{for every } z^* \in U_r \cap M. \tag{2.1}$$

Proof. Observe first that if T is an Asplund operator, then its adjoint T^* sends the unit ball of Y^* into a w^*-compact subset of (X^*, w^*) that is norm fragmented. Indeed, if $T = T_2 \circ T_1$ is a factorization of T through the Asplund space Z, then its adjoint T^* factors through Z^*

$$\begin{tikzcd} Y^* \arrow[rr, "T^*"] \arrow[dr, "T_2^*"'] & & X^* \\ & Z^* \arrow[ur, "T_1^*"'] & \end{tikzcd}$$

Since T_2^* is w^*-w^* continuous, $T_2^*(B_{Y^*})$ is a w^*-compact subset of Z^*, and we can now appeal to Theorem 2.1 to conclude that $T_2^*(B_{Y^*}) \subset (Z^*, w^*)$ is fragmented by the norm of Z^*. On the other hand, $T_1^* : Z^* \to X^*$ is norm-to-norm and w^*-w^* continuous and, therefore it sends the fragmented w^*-compact set $T_2^*(B_{Y^*}) \subset (Z^*, w^*)$ onto the w^*-compact set $T^*(B_{Y^*}) \subset (X^*, w^*)$ that is fragmented by the norm of X^*, see [40, Lemma 2.1], and our observation is proved. (Alternatively, the observation can be proved using [56, Theorem 2.11] and [17, Theorem 4.2.13].)

Now we really start the proof of the lemma. Use that $B \subset B_{Y^*}$ is 1-norming and pick $b_0^* \in B$ such that

$$|T^*(b_0^*)(x_0)| = |b_0^*(T(x_0))| > 1 - \frac{\varepsilon^2}{2}.$$

Defining $U_1 = \{x^* \in X^* : \ |x^*(x_0)| > 1 - \varepsilon^2/2\}$, we have that

$$T^*(b_0^*) \in U_1 \cap M \subset T^*(B_{Y^*}) \subset B_{X^*}.$$

Fix $r > 0$. Since $T^*(B_{Y^*})$ is fragmented and $U_1 \cap M$ is non-empty, there exists a w^*-open set $U_2 \subset X^*$ such that $(U_1 \cap M) \cap U_2 \neq \emptyset$ and

$$\|\cdot\|\text{- diam}\big((U_1 \cap M) \cap U_2\big) \leq r. \tag{2.2}$$

Let $U := U_1 \cap U_2$ and fix $x_0^* \in U \cap M$. We have

$$1 \geq \|x_0^*\| \geq |x_0^*(x_0)| > 1 - \frac{\varepsilon^2}{2}. \tag{2.3}$$

If we normalize we still have

$$1 \geq \frac{|x_0^*(x_0)|}{\|x_0^*\|} \geq |x_0^*(x_0)| \geq 1 - \frac{\varepsilon^2}{2}. \tag{2.4}$$

Then by applying the Bishop-Phelps-Bollobás Theorem 1.5 to $\dfrac{x_0^*}{\|x_0^*\|}$ and x_0, we obtain $y_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|y_r^*(u_r)| = 1$ such that

$$\|x_0 - u_r\| < \varepsilon \text{ and } \left\| \frac{x_0^*}{\|x_0^*\|} - y_r^* \right\| < \varepsilon. \tag{2.5}$$

Let $z^* \in U \cap M$ be an arbitrary element. Then,

$$\begin{aligned}
\|z^* - y_r^*\| &\leq \|z^* - x_0^*\| + \left\| x_0^* - \frac{x_0^*}{\|x_0^*\|} \right\| + \left\| \frac{x_0^*}{\|x_0^*\|} - y_r^* \right\| \\
&\overset{(2.2),(2.5)}{\leq} r + \|x_0^*\| \left| 1 - \frac{1}{\|x_0^*\|} \right| + \varepsilon \\
&\overset{(2.3)}{\leq} r + \frac{\varepsilon^2}{2} + \varepsilon,
\end{aligned}$$

and the proof is over. □

Theorem 2.4. *Let $T : X \to C_0(L)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that*

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{2}.$$

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X, C_0(L))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| \leq \varepsilon \text{ and } \|T - S\| < 2\varepsilon.$$

Proof. The natural embedding $\xi : L \to C_0(L)^*$ given by $\xi(s) := \delta_s$, for $s \in L$, is continuous for the topology of L and the w^*-topology in $C_0(L)^*$. Hence the composition $\phi := T^* \circ \xi : L \to X^*$ is continuous for the w^* topology in X^*.

Apply now Lemma 2.3 for $Y := C_0(L)$, $B := \{\delta_s : s \in L\} \subset B_{C_0(L)^*}$, our given operator T, ε, and $0 < r < \varepsilon - \varepsilon^2/2$. Note that the set B is 1-norming, since

$$\|f\| = \sup_{s \in L} |f(s)| = \sup_{\{\delta_s : \, \delta_s \in B\}} |\delta_s(f)|.$$

Thus, we produce the w^*-open set U and the functional $y^* \in S_{X^*}$ satisfying properties (a) and (b) in the aforesaid lemma. Note that with our new notation we have $\phi(L) = M$. Since $U \cap M \neq \emptyset$ we can pick $s_0 \in L$ such that $\phi(s_0) \in U$. The w^*-continuity of ϕ ensures that the set $W = \{s \in L : \ \phi(s) \in U\}$ is an open neighborhood of s_0. By Urysohn's lemma, [50, Lemma 2.12], we can find a continuous function $f : L \to [0,1]$ with compact support, satisfying:

$$f(s_0) = 1 \quad \text{and} \quad \operatorname{supp}(f) \subset W. \tag{2.6}$$

Define now the linear operator $S : X \to C_0(L)$ by the formula

$$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s). \tag{2.7}$$

It is easily checked that S is well-defined and that $\|S\| \leq 1$. On the other hand, $1 = |y^*(u_0)| = |S(u_0)(s_0)| \leq \|S(u_0)\| \leq 1$ and therefore S attains the norm at the point $u_0 \in S_X$ for which we had $\|u_0 - x_0\| \leq \varepsilon$.

Now, bearing in mind (2.6), (2.7), Lemma 2.3 and the definition of W we conclude that

$$\begin{aligned} \|T - S\| &= \sup_{x \in B_X} \|Tx - Sx\| = \sup_{x \in B_X} \sup_{s \in L} f(s)|T(x)(s) - y^*(x)| \\ &= \sup_{x \in B_X} \sup_{s \in W} f(s)|\phi(s)(x) - y^*(x)| \leq \sup_{s \in W} \sup_{x \in B_X} |\phi(s)(x) - y^*(x)| \\ &= \sup_{s \in W} \|\phi(s) - y^*\| \leq r + \varepsilon^2/2 + \varepsilon < 2\varepsilon. \end{aligned} \tag{2.8}$$

To finish we prove that S is also an Asplund operator. This is based on the fact that the family of Asplund operators between Banach spaces is an operator ideal, see [56, Theorem 2.12]. Observe that S appears as the sum of a rank one operator and the operator $x \mapsto (1 - f)T(x)$; the latter is the composition of a bounded operator from $C_0(L)$ into itself with T. Therefore S is an Asplund operator and the proof is over. $\square$

Remark 2.4.1. *It is possible to get an estimate better than 2ε in Theorem 2.4. From 2.8, it follows that $\|T - S\| < \delta + \varepsilon$, where $\varepsilon^2/2 < \delta < \varepsilon$.*

2.3 Operator Ideals and Other Corollaries

Recall that an *operator ideal* $\mathcal{I}$ is a way of assigning to each pair of Banach spaces (X, Y) a linear subspace $\mathcal{I}(X, Y) \subset L(X, Y)$ that contains all finite rank operators from X to Y and satisfies the following property: $T_2 \circ T \circ T_1 \in \mathcal{I}(Z, V)$ whenever $T \in \mathcal{I}(X, Y)$, $T_1 \in L(Z, X)$, and $T_2 \in L(Y, V)$, see [25, 48].

If we denote by $\mathcal{A}$ the ideal of Asplund operators between Banach spaces, the above theorem applies as well to any sub-ideal $\mathcal{I} \subset \mathcal{A}$.

Corollary 2.5. *Let $\mathcal{I} \subset \mathcal{A}$ be an operator ideal. Let $T \in \mathcal{I}(X, C_0(L))$ with $\|T\| = 1$, $0 < \varepsilon < \sqrt{2}$, and $x_0 \in S_X$ be such that*

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{2}.$$

Then there are $u_0 \in S_X$ and $S \in \mathcal{I}(X, C_0(L))$ with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| \leq \varepsilon \text{ and } \|T - S\| < 2\varepsilon.$$

We should stress that because $\mathcal{W} \subset \mathcal{A}$, see [24], the above corollary applies in particular to the ideals of finite rank operators $\mathcal{F}$, compact operators $\mathcal{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathcal{W}$ themselves. Results in this vein can be found in the literature for weakly compact operators but, with spaces of continuous functions as domain spaces and only for the so-called *Bishop-Phelps property*: Schachermayer proved, see [52, Theorem B], that any $T \in \mathcal{W}(C(K), X)$ can be approximated by norm attaining operators. This result was generalized later for operators $T \in \mathcal{W}(C_0(L), X)$, see [9]). With spaces of continuous functions in the range, Johnson and Wolfe, see [33, Theorem 3], proved that any $T \in \mathcal{K}(X, C(K))$ can be approximated by finite rank norm attaining operators. Note then that Corollary 2.5 adds several new versions of the vector-valued

Bishop-Phelps theorem. Moreover, these cases provide the Bollobás part of approximation of points at which the norm is attained.

Standard $\varepsilon - \delta$ tricks suffice to prove that *for a pair of Banach spaces* (X, Y) *the following are equivalent:*

(i) (X, Y) *has the BPB property according to Definition 2.1;*

(ii) there are functions $\eta : (0, +\infty) \to (0, 1)$ *and* $\beta, \gamma : (0, +\infty) \to (0, +\infty)$ *with*

$$\lim_{t \to 0} \beta(t) = \lim_{t \to 0} \gamma(t) = 0,$$

such that given $\varepsilon > 0$*, for all* $T \in S_{L(X,Y)}$*, if* $x_0 \in S_X$ *and* $\|T(x_0)\| > 1 - \eta(\varepsilon)$*, then there exist a point* $u_0 \in S_X$ *and* $S \in S_{L(X,Y)}$ *satisfying*

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \beta(\varepsilon), \text{ and } \|T - S\| < \gamma(\varepsilon).$$

Once again, in (ii) above we can take $\beta(t) = \gamma(t) = t$, but of course changing η if needed!. Consequently we arrive to the following straightforward consequence of Theorem 2.4:

Corollary 2.6. *For any Asplund space* X *and any locally compact Hausdorff topological space* L *the pair* $(X, C_0(L))$ *has the BPBp.*

This corollary extends and strengthens Theorem 2 in [33]. Also, we can take as X any $c_0(\Gamma)$ (Γ arbitrary set), or more generally any $C_0(S)$ where S is a scattered locally compact Hausdorff space (see, for instance, [44] for scattered or dispersed spaces). Indeed for a locally compact space S, the space $C_0(S)$ is Asplund if, and only if, S is scattered. This can be proved in the following way:

(1) It is known that for K compact, $C(K)$ is Asplund if, and only if, K is scattered,

combine [44, Main Theorem] with Theorem 2.1 or alternatively see [42, Theorem 18].

(2) It is easy to check that if S is locally compact, then S is scattered if, and only if, its Alexandroff compactification $S \cup \{\infty\}$ is scattered,

(3) Now use that Asplundness is a *three space property*, see [42, Theorems 11,12 and 14], and conclude that $C_0(S)$ is Asplund if, and only, if $C(S \cup \{\infty\})$ is Asplund.

(4) Summarizing, $C_0(S)$ is Asplund if, and only if, S is scattered.

Note that whereas the hypothesis of X being Asplund in the above corollary is an isomorphic property, for the range space we have to use the sup norm in $C_0(L)$. Indeed, Lindenstrauss [38, Proposition 4] established that if $(c_0, \|\cdot\|)$ is a strictly convex renorming of c_0 then $id : c_0 \to (c_0, \|\cdot\|)$ cannot be approximated by norm attaining operators. Notice also, that Corollary 2.6 may fail when X is not Asplund: Schachermayer [52] gave an example of an operator $T \in L(L^1[0,1], C[0,1])$ that cannot be approximated by norm attaining operators.

With our comments above together with Theorem 2.4 we have:

Corollary 2.7. *For any Banach space X and any scattered locally compact Hausdorff topological space L the pair $(X, C_0(L))$ has the BPBp.*

An alternative proof for this corollary can be obtained using the fact that for such an L the space $Y = C_0(L)$ has property (β), see [51, Definition 1.2], and for spaces Y with property (β), every pair (X, Y) has BPBp, see [5, Theorem 2.2].

In a different line of ideas, we point out that Lindenstrauss proved in [38, Theorem 1] that every operator $T \in L(X, Y)$ can be approximated by operators $S \in L(X, Y)$ such that $S^{**} \in L(X^{**}, Y^{**})$ attains the norm on $B_{X^{**}}$. In [5, Example 6.3] it is established that the counterpart of the above Lindenstrauss' result is no longer valid for the corresponding natural Bishop-Phelps-Bollobás theorem with bi-adjoints operators. The example again uses c_0 as a domain space. Replacing Y^{**} by $C(B_{Y^*}, w^*)$, we state the last result in this Chapter.

Corollary 2.8. *Let $T : X \to Y$ be an Asplund operator with $\|T\| = 1,\ \ 0 < \varepsilon < \sqrt{2}$, and $x_0 \in S_X$ be such that*

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{2}.$$

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X,C(B_{Y^*}))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| \leq \varepsilon \text{ and } \|i \circ T - S\| < 2\varepsilon,$$

where $i : Y \hookrightarrow C(B_{Y^*})$ is the natural embedding.

Finally, we would like to comment on some extensions of this work. There is a string of results related to uniformly convex spaces and the spaces of continuous functions. In [6], it was shown that if Y is uniformly convex, then the Bishop-Phelps-Bollobás theorem holds for compact bounded linear operators $T \in L(C_0(L), Y)$. If the condition T being compact is removed, then $(C(K), C(S))$ has the BPBp, where $C(K)$ and $C(S)$ are real-valued continuous functions on compact Hausdorff spaces K and S, [6]. Further it was extended to $(C_0(S), C_0(L))$, where S is a locally compact metrizable space and L is a locally compact Hausdorff space, [35].

CHAPTER 3

THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR NUMERICAL RADIUS

> The infinite we do immediately, the finite takes a little longer.
>
> S. Ulam and P. Erdös.

B. Bollobás extended the Bishop-Phelps theorem in a quantitative way in order to work on problems related to the numerical range of an operator [15]. The known proofs of this fact have an existence nature: they are based on the Hahn-Banach extension theorem, Ekeland's variational principle or the Brøndsted-Rockafellar principle. We begin this chapter by constructing explicit expressions of the approximating pair (x_0, x_0^*) when $X = \ell_1(\mathbb{C})$ and $X = c_0(\mathbb{C})$, a necessary tool for our main results related to numerical radius. The results in this Chapter are based on joint work with A. J. Guirao [31].

Paralleling the research of norm attaining operators initiated by Lindenstrauss in [38], B. Sims raised the question of the norm denseness of the set of numerical radius attaining operators [53]. Partial positive results are known. M. Acosta in her Ph.D. thesis [1] initiated a systematic study of the problem. Other important works in this direction are the renorming result by M. Acosta in [2] and the joint findings of this author with R. Payá [7, 8]. Prior to them, I. Berg and B. Sims in [13] gave a positive answer for uniformly convex spaces and C. S. Cardassi obtained positive answers for ℓ_1, c_0, $C(K)$, $L_1(\mu)$, and uniformly smooth spaces [19, 20, 21].

Using a renorming of c_0, R. Payá provided an example of a Banach space X such that the set of numerical radius attaining operators on X is not norm dense, answering in the negative Sims' question, [43]. In the same year, M. Acosta, F. Aguirre, and R. Payá in [4] gave another counterexample: $X = \ell_2 \oplus_\infty G$, where G is the Gowers space.

We investigate here an analogue of the Bishop-Phelps-Bollobás property for operators 2.1 but in relation with numerical radius attaining operators, called the *Bishop-Phelps-Bollobás property for numerical radius*, or BPBp-ν for short. The relation between norm attaining and numerical radius attaining operators is far from clear, although the existence of an interconnection is evident. Amongst the first examples, we show that $\ell_1(\mathbb{C})$ and $c_0(\mathbb{C})$ satisfy BPBp-ν (Theorems 3.7 and 3.10). This brings an extension as well as a quantitative version of C. S. Cardassi's results in [20]. Other recent results include BPBp-ν for $C(K)$, when K is metrizable, [12]; a construction of BPBp-ν for L_1 by J. Falcó, [30]; and a series of results in [36].

Observe that the counterexamples provided in [4] and [43] imply, in particular, that there exist Banach spaces failing the Bishop-Phelps-Bollobás property for numerical radius.

Recall the set $\Pi(X)$ (1.3) from Chapter 1:

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : \ x^*(x) = 1\}.$$

Given $x \in S_X$ and $x^* \in S_{X^*}$, we set

$$\pi_1(x^*) := \{x \in S_X : \ x^*(x) = 1\}.$$

For a given $T \in L(X)$, its *numerical radius* $\nu(T)$ is defined by

$$\nu(T) = \sup\{|x^*(Tx)| : \ (x, x^*) \in \Pi(X)\}.$$

It is well known that the numerical radius of a Banach space X is a continuous seminorm on X which is, in fact, an equivalent norm when X is complex. In general, there exists a constant $n(X)$, called the *numerical index* of X, such that

$$n(X)\,\|T\| \leq \nu(T) \leq \|T\|\,, \text{ for all } T \in L(X).$$

In this work we consider the spaces of numerical index 1, $n(X) = 1$, where the norm and the numerical radius coincide.

We say that $T \in L(X)$ attains its numerical radius if there exists $(x, x^*) \in \Pi(X)$ such that $|x^*(Tx)| = \nu(T)$. The set of numerical radius attaining operators will be denoted by $\mathrm{NRA(X)} \subset \mathrm{L(X)}$.

Definition 3.1 (BPBp-ν)**.** *A Banach space X is said to have the* Bishop-Phelps-Bollobás property for numerical radius *if for every $0 < \varepsilon < 1$, there exists $\delta > 0$ such that for a given $T \in L(X)$ with $\nu(T) = 1$ and a pair $(x, x^*) \in \Pi(X)$ satisfying $|x^*(Tx)| \geq 1 - \delta$, there exist $S \in L(X)$ with $\nu(S) = 1$, and a pair $(y, y^*) \in \Pi(X)$ such that*

$$\nu(T - S) \leq \varepsilon,\ \|x - y\| \leq \varepsilon,\ \|x^* - y^*\| \leq \varepsilon\ \textit{ and }\ |y^*(Sy)| = 1. \tag{3.1}$$

Observe that if X is a Banach space with $n(X) = 1$, then the seminorm $\nu(\cdot)$ can be replaced by $\|\cdot\|$ in the definition above, which is precisely the case for all the spaces studied below.

Let $\arg(\cdot)$ stand for the function which sends a non zero complex number z to the unique $\arg(z) \in [0, 2\pi)$ such that $z = |z|e^{\arg(z)i}$. For convenience we extend the function to $\mathbb{C}$ by writing $\arg(0) = 0$.

Throughout 3.1 to 3.3, the spaces ℓ_1, ℓ_∞, and c_0 stand respectively for $\ell_1(\mathbb{C})$, $\ell_\infty(\mathbb{C})$, and $c_0(\mathbb{C})$. The standard basis of ℓ_1 is denoted by $\{e_n\}_{n\in\mathbb{N}}$, and its biorthogonal functionals by $\{e_n^*\}_{n\in\mathbb{N}}$. Given a sequence $\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and a complex function $f \colon \mathbb{C} \to \mathbb{C}$ we write $f(\xi)$ to mean the sequence $(f(\xi_j))_{j\in\mathbb{N}}$.

Before stating the results, we need to introduce the following sets.

Given $x = (x_j)_{j\in\mathbb{N}} \in \ell_1$, $\varphi = (\varphi_j)_{j\in\mathbb{N}} \in \ell_\infty$ we define

$$\mathcal{N}_{(x,\varphi)} = \{j \in \mathbb{N}\colon\ \varphi_j\, x_j = |x_j|\}, \tag{3.2}$$

$$\operatorname{supp}(x) = \{j \in \mathbb{N}\colon\ |x_j| \neq 0\}.$$

For $r > 0$ we consider

$$\mathcal{A}_\varphi(r) = \{j \in \mathbb{N}\colon\ |\varphi_j| \geq 1 - r\}, \tag{3.3}$$

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \operatorname{supp}(x)\colon\ \operatorname{Re}(\varphi_j\, x_j) \geq (1 - r)|x_j|\}. \tag{3.4}$$

Observe that $\mathcal{P}_{(x,\varphi)}(r) \subset \mathcal{A}_\varphi(r)$. If x is *positive*, i.e. $x_j \geq 0$ for all $j \in \mathbb{N}$, then

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \operatorname{supp}(x)\colon\ \operatorname{Re}(\varphi_j) \geq (1 - r)\}.$$

For a given set Γ, a subset $A \subset \Gamma$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathbb{1}_A$ the characteristic function of A, that is, the element in $\mathbb{K}^\Gamma$ such that $(\mathbb{1}_A)_\gamma = 1$ if $\gamma \in A$ and $(\mathbb{1}_A)_\gamma = 0$ otherwise.

3.1 The Bishop-Phelps-Bollobás Theorem in $\ell_1(\mathbb{C})$

In this section we present two constructive versions of Theorem 0.1, which are the main tool in the proof of Theorems 3.7 and 3.13.

Lemma 3.1. *Let $(x, \varphi) \in S_{\ell_1} \times S_{\ell_\infty}$. Then $x \in \pi_1(\varphi)$ if and only if $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$.*

Proof. Given a pair $(x, \varphi) \in S_{\ell_1} \times S_{\ell_\infty}$ satisfying $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$, one can compute $\varphi(x) = \sum_{j\in\mathbb{N}} \varphi_j\, x_j \overset{(3.2)}{=} \sum_{j\in\mathbb{N}} |x_j| = \|x\| = 1$, which implies that $(x, \varphi) \in \Pi(\ell_1)$.

Conversely, let us assume that $(x, \varphi) \in \Pi(\ell_1)$ then,

$$1 = \operatorname{Re}(\varphi(x)) = \sum_{j\in\mathbb{N}} \operatorname{Re}(\varphi_j\, x_j) \leq \sum_{j\in\mathbb{N}} |\varphi_j\, x_j| \leq \sum_{j\in\mathbb{N}} |x_j| = 1,$$

which implies that $\mathrm{Re}(\varphi_j\, x_j) = |\varphi_j\, x_j| = |x_j|$ for $j \in \mathbb{N}$. Therefore, $\varphi_j\, x_j = |x_j|$ for every $j \in \mathbb{N}$, which finishes the proof. $\square$

Lemma 3.1 provides the essential insight into the properties of $\Pi(\ell_1)$ needed for the proof of Theorems 3.4 and 3.6. In particular, Lemma 3.1 gives an intuitive characterization of the norm attaining functionals on ℓ_1, $\mathrm{NA}(\ell_1)$.

Corollary 3.2. $\mathrm{NA}(\ell_1) = \{\varphi \in \ell_\infty \colon\ \exists \mathrm{n} \in \mathbb{N}$ *with* $|\varphi_{\mathrm{n}}| = \|\varphi\|\}$.

The following lemma is an adaptation of [5, Lemma 3.3].

Lemma 3.3. *Let* $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ *and* $0 < \delta < 1$ *such that* $\varphi(x) \geq 1 - \delta$. *Then, for every* $\delta < r < 1$ *we have* $\left\|\mathrm{Re}\big(e^{\arg(\varphi) i}\, x\big) \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1 - (\delta/r)$.

Proof. By assumption, we have that

$$\begin{aligned} 1 - \delta \leq \mathrm{Re}(\varphi(x)) &= \sum_{j \in \mathbb{N}} \mathrm{Re}(\varphi_j\, x_j) = \sum_{j \in \mathbb{N}} |\varphi_j|\, \mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big) \\ &\leq \sum_{\mathcal{P}_{(x,\varphi)}(r)} \mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big) + (1 - r) \sum_{\mathbb{N} \setminus \mathcal{P}_{(x,\varphi)}(r)} |x_j| \\ &\leq r \sum_{\mathcal{P}_{(x,\varphi)}(r)} \left|\mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big)\right| + (1 - r), \end{aligned}$$

which implies that

$$\left\|\mathrm{Re}\big(e^{\arg(\varphi) i}\, x\big)\, \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| = \sum_{j \in \mathcal{P}_{(x,\varphi)}(r)} \left|\mathrm{Re}\Big(e^{\arg(\varphi_j) i}\, x_j\Big)\right| \geq 1 - (\delta/r),$$

as we wanted to show. $\square$

Observe that the previous lemma implies, in particular, that

$$\left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1 - (\delta/r).$$

We present next the two constructive versions of the Bishop-Phelps-Bollobás theorem.

Theorem 3.4 (First constructive version.). *Given* $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ *and* $0 < \varepsilon < 1$ *such that* $\varphi(x) \geq 1 - \frac{\varepsilon^3}{4}$. *Then, there exists* $(x_0, \varphi_0) \in \Pi(\ell_1)$ *such that* $\|x - x_0\| \leq \varepsilon$, $\|\varphi - \varphi_0\| \leq \varepsilon$. *Moreover, we can take*

$$x_0 := \left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}\right\|^{-1} \cdot x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}. \tag{3.5}$$

Proof. Set $P := \mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)$, as defined in (3.4). Applying Lemma 3.3 with $\delta = \varepsilon^2/2$ and $r = \varepsilon$ gives that

$$M := \|x \cdot \mathbb{1}_P\| \geq 1 - (\varepsilon/2). \tag{3.6}$$

Then set

$$\varphi_0 := \varphi \cdot \mathbb{1}_{\mathbb{N} \setminus P} + e^{-\arg(x)i} \cdot \mathbb{1}_P \in S_{\ell_\infty} \tag{3.7}$$

and

$$x_0 := M^{-1} x \cdot \mathbb{1}_P \in S_{\ell_1}. \tag{3.8}$$

On one hand, we can compute

$$\begin{aligned} \|x - x_0\| &\overset{(3.8)}{=} \left\|x - M^{-1} x \cdot \mathbb{1}_P\right\| = (M^{-1} - 1)\,\|x \cdot \mathbb{1}_P\| + \left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \\ &\overset{(3.6)}{=} (1 - M) + \left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \overset{\|x\| \leq 1}{\leq} 2 - 2M \overset{(3.6)}{\leq} \varepsilon, \end{aligned}$$

and use the fact that the support of x_0 is contained in P (from (3.8)), to deduce that

$$\varphi_0(x_0) = \sum_{j \in P} (\varphi_0)_j\,(x_0)_j \overset{(3.7)}{=} \sum_{j \in P} e^{-\arg(x_j)i}\,(x_0)_j \overset{(3.8)}{=} \sum_{j \in P} |(x_0)_j| = \|x_0\| = 1,$$

which is equivalently expressed as $(x_0, \varphi_0) \in \Pi(\ell_1)$.

On the other hand, using that

$$|z - 1| \leq \sqrt{2(1 - \mathrm{Re}(z))} \text{ for every } z \in \mathbb{C} \text{ such that } |z| \leq 1, \tag{3.9}$$

we conclude

$$\begin{aligned}\|\varphi-\varphi_0\| &\overset{(3.7)}{=} \sup_{j\in P}\{|\varphi_j-(\varphi_0)_j|\} \overset{(3.7)}{=} \sup_{j\in P}\left\{\left|\varphi_j-e^{-\arg(x_j)i}\right|\right\}\\ &= \sup_{j\in P}\left\{\left|e^{\arg(x_j)i}\,\varphi_j-1\right|\right\} \overset{(3.9)}{\leq} \sup_{j\in P}\left\{\sqrt{2-2\operatorname{Re}\!\left(e^{\arg(x_j)i}\,\varphi_j\right)}\right\}\\ &\leq \sqrt{2-2(1-\varepsilon^2/2)}=\varepsilon,\end{aligned}$$

which finishes the proof. □

An immediate consequence of Theorem 3.4 is the following version of the Bishop-Phelps-Bollobás theorem for $\ell_1(\mathbb{C})$.

Corollary 3.5. *Let* $0<\varepsilon<1$ *and* $(x,\,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ *such that* $|\varphi(x)|\geq 1-\frac{\varepsilon^3}{4}$. *Then, there exists* $(x_0,\varphi_0)\in S_{\ell_1}\times S_{\ell_\infty}$ *such that* $\|x-x_0\|\leq\varepsilon$, $\|\varphi-\varphi_0\|\leq\varepsilon$, *and* $|\varphi_0(x_0)|=1$.

Proof. Apply Theorem 3.4 to the pair $\left(e^{-\arg(\varphi(x))i}\,x,\varphi\right)$ obtaining (z_0,φ_0) belonging to $\Pi(\ell_1)$ such that $\left\|e^{-\arg(\varphi(x))i}\,x-z_0\right\|\leq\varepsilon$ and $\|\varphi-\varphi_0\|\leq\varepsilon$. Therefore, if we set

$$x_0{:=}e^{\arg(\varphi(x))i}\,z_0,$$

the pair (x_0,φ_0) satisfies the conclusions of the corollary. □

Given a pair (x,φ) and $0<\varepsilon<1$, Theorem 3.4 ensures the existence of a pair (x_0,φ_0), defined by (3.8)-(3.7), and satisfying the conclusions of the Bishop-Phelps-Bollobás theorem. However, φ_0 depends on x, in fact, on $\arg(x)$. In order to prove Theorem 3.7 we will need a functional φ_0 depending only on the given ε and φ. So, we present another constructive version of the Bishop-Phelps-Bollobás theorem for ℓ_1.

Theorem 3.6 (Second constructive version.)**.** *Let* $(x,\,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ *and* $0<\varepsilon<1$ *be such that* $\varphi(x)\geq 1-\frac{\varepsilon^3}{60}$. *Then there exists* $(x_0,\varphi_0)\in\Pi(\ell_1)$ *such that* $\|x-x_0\|\leq\varepsilon$,

$\|\varphi - \varphi_0\| \leq \varepsilon$. *Moreover, the functional* φ_0 *can be defined as*

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/20)} + e^{\arg(\varphi)i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/20)}. \tag{3.10}$$

Proof. To simplify the proof, we will first apply the isometry $S\colon \ell_1 \to \ell_1$ defined by

$$\langle e_j^*, Sy\rangle = e^{\arg(\varphi_j)i}\, y_j, \text{ for } y \in \ell_1 \text{ and } j \in \mathbb{N}. \tag{3.11}$$

Set $\widetilde{x} = Sx$ and $\widetilde{\varphi} = \varphi \circ S^{-1}$. Then, it is clear that the pair $(\widetilde{x}, \widetilde{\varphi})$ is in $B_{\ell_1} \times B_{\ell_\infty}$, that $\widetilde{\varphi}(\widetilde{x}) \geq 1 - \frac{\varepsilon^3}{60}$ and that $\widetilde{\varphi} = (|\varphi_j|)_{j\in\mathbb{N}}$ is positive. Fix $r{:=}\frac{\varepsilon^2}{20}$. Then denote by A and P respectively the sets $\mathcal{A}_{\widetilde{\varphi}}(r)$ and $\mathcal{P}_{(\widetilde{x},\widetilde{\varphi})}(r)$, as defined in (3.3) and (3.4). Let

$$\widehat{\varphi}{:=}\widetilde{\varphi} \cdot \mathbb{1}_{\mathbb{N}\setminus A} + \mathbb{1}_A \in S_{\ell_\infty} \tag{3.12}$$

and

$$\widehat{x}{:=}M^{-1}\mathrm{Re}(\widetilde{x}) \cdot \mathbb{1}_P \in S_{\ell_1}, \tag{3.13}$$

where $M{:=}\|\mathrm{Re}(\widetilde{x}) \cdot \mathbb{1}_P\|$. Applying Lemma 3.3 with $\delta = \varepsilon^3/60$ and r, gives that $M \geq 1 - \frac{\varepsilon}{3}$. In particular, this means that P, and thus A, are non-empty.

We can compute that

$$\begin{aligned}\|\widetilde{\varphi} - \widehat{\varphi}\| &\overset{(3.12)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j - \widehat{\varphi}_j|\} \overset{(3.12)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j - 1|\} \\ &= \sup_{j\in A}\{(1 - \widetilde{\varphi}_j)\} \overset{(3.3)}{\leq} r \leq \varepsilon,\end{aligned} \tag{3.14}$$

and, since by (3.4) and (3.13) the support of $\widehat{x}$ is $P \subset A$ –which, in particular, implies that $\widehat{x}_j > 0$ for $j \in P$, we deduce that

$$\widehat{\varphi}(\widehat{x}) = \sum_{j\in P} \widehat{\varphi}_j \widehat{x}_j \overset{(3.12)}{=} \sum_{j\in P} \widehat{x}_j = \sum_{j\in P} |\widehat{x}_j| = 1. \tag{3.15}$$

Or, equivalently: $(\widehat{x}, \widehat{\varphi}) \in \Pi(\ell_1)$.

In order to show that $\|\widetilde{x}-\widehat{x}\| \leq \varepsilon$, let us observe first that

$$\|\widetilde{x} \cdot \mathbb{1}_P\| = \sum_{j \in P} |\widetilde{x}_j| \geq \sum_{j \in P} |\mathrm{Re}(\widetilde{x}_j)| = M \geq 1 - \frac{\varepsilon}{3}, \tag{3.16}$$

from which

$$\begin{aligned} \|\widetilde{x}-\widehat{x}\| &\overset{(3.13)}{=} \left\|\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x}) \cdot \mathbb{1}_P\right\| = \left\|\widetilde{x} \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| + \left\|(\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\right\| \\ &\overset{(3.16)}{\leq} \frac{\varepsilon}{3} + \left\|(\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\right\|. \end{aligned} \tag{3.17}$$

We need a bit more care to estimate the last term in (3.17). From the very definition of P, we know that for every $j \in P$ it holds

$$|\widetilde{x}_j| \leq (1-r)^{-1} \widetilde{\varphi}_j \, \mathrm{Re}(\widetilde{x}_j). \tag{3.18}$$

Therefore,

$$\begin{aligned} \|(\widetilde{x} - \mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\| &= \sum_{j \in P} |\widetilde{x}_j - \mathrm{Re}(\widetilde{x}_j)| = \sum_{j \in P} |\mathrm{Im}(\widetilde{x}_j)| \\ &= \sum_{j \in P} \sqrt{|\widetilde{x}_j|^2 - \mathrm{Re}(\widetilde{x}_j)^2} \\ &\overset{(3.18)}{\leq} \sum_{j \in P} |\mathrm{Re}(\widetilde{x}_j)| \sqrt{(1-r)^{-2} - 1} \\ &\leq \|\widetilde{x}\| \sqrt{(1-r)^{-2} - 1} \overset{r = \frac{\varepsilon^2}{20}}{\leq} \frac{\varepsilon}{3}, \end{aligned} \tag{3.19}$$

which implies that

$$\begin{aligned} \left\|(\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\right\| &\leq \|(\widetilde{x} - \mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\| + \left\|(1 - M^{-1})\mathrm{Re}(\widetilde{x}) \cdot \mathbb{1}_P\right\| \\ &\overset{(3.19)}{\leq} \frac{\varepsilon}{3} + (M^{-1} - 1) \|\mathrm{Re}(\widetilde{x}) \cdot \mathbb{1}_P\| \\ &= \frac{\varepsilon}{3} + (1 - M) \leq \frac{2\varepsilon}{3}. \end{aligned} \tag{3.20}$$

Putting together (3.17) and (3.20), we finish the core part of the proof:

$$\|\widetilde{x}-\widehat{x}\| \leq \frac{\varepsilon}{3} + \left\|(\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x})) \cdot \mathbb{1}_P\right\| \leq \varepsilon. \tag{3.21}$$

Now, we define

$$x_0{:=}S^{-1}\widehat{x} \quad \text{and} \quad \varphi_0 = S^*(\widehat{\varphi}) = \widehat{\varphi} \circ S, \tag{3.22}$$

which by (3.15) gives that $\varphi_0(x_0) = \widehat{\varphi}(\widehat{x}) = 1$. Since S and S^* are isometries, we deduce from (3.14), (3.21), (3.22) and the definition of $\widetilde{x}$ and $\widetilde{\varphi}$ that

$$\|x - x_0\| \leq \varepsilon, \ \ \|\varphi - \varphi_0\| \leq \varepsilon.$$

Therefore, (x_0, φ_0) is the pair in $\Pi(\ell_1)$ we were looking for.

Bearing in mind (3.22), one computes

$$(\varphi_0)_j = \varphi_0(e_j) \overset{(3.22)}{=} \widehat{\varphi}(Se_j) \overset{(3.11)}{=} \widehat{\varphi}\big(e^{\arg(\varphi_j)i}\, e_j\big) = e^{\arg(\varphi_j)i}\, \widehat{\varphi}_j,$$

which together with (3.12) implies that $\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus A} + e^{\arg(\varphi)i} \cdot \mathbb{1}_A$. Finally, noting that $A = \mathcal{A}_{\widetilde{\varphi}}(r) = \mathcal{A}_{\varphi}(r)$, the validity of (3.10) has been shown. □

Remark 3.6.1. *Observe that the function φ_0 provided by Theorem 3.6 and defined by* (3.10) *only depends on ε and φ itself and also satisfies $\pi_1(\varphi) \subset \pi_1(\varphi_0)$.*

3.2 BPB Property for Numerical Radius in $\ell_1(\mathbb{C})$

As a consequence of Theorems 3.4 and 3.6 we show that ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.

Theorem 3.7. *Let $T \in S_{L(\ell_1)}$, $0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(\ell_1)$ such that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $T_0 \in S_{L(\ell_1)}$ and $(x_0, \varphi_0) \in \Pi(\ell_1)$ such that*

$$\|T - T_0\| \leq \varepsilon, \ \ \|x - x_0\| \leq \varepsilon, \ \ \|\varphi - \varphi_0\| \leq \varepsilon \textit{ and } \varphi_0(T_0x_0) = 1. \tag{3.23}$$

Proof. First of all, fix $\mu := \sqrt{\varepsilon^3/240}$. Using a suitable isometry, we can assume that x is positive. In particular, by Lemma 3.1 and the definition of $\mathcal{N}_{x,\varphi}$ in (3.2), we can assume that $\varphi_j = 1$ for $j \in \operatorname{supp}(x)$. Since $\mu^3/4 \geq (\varepsilon/9)^{9/2}$, Theorem 3.4 can be applied to the pair $(x, T^*\varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and μ instead of ε giving $x_0 \in \pi_1(\varphi)$ such that $\|x - x_0\| \leq \mu \leq \varepsilon$. Moreover, by (3.5) we know that

$$x_0 = \|x \cdot \mathbb{1}_P\|^{-1} \cdot x \cdot \mathbb{1}_P, \tag{3.24}$$

where the non-empty set P is defined by

$$P := \mathcal{P}_{(x, T^*\varphi)}(\mu^2/2) = \{j \in \operatorname{supp}(x) \colon \operatorname{Re}(T^*\varphi(e_j)) \geq 1 - \mu^2/2\}. \tag{3.25}$$

In particular, x_0 is positive.

Since $\mu^2/2 = \frac{(\varepsilon/2)^3}{60}$, for each $j \in P$ we can apply Theorem 3.6 to the pair $(e^{-\arg(\varphi(Te_j))i}\,Te_j, \varphi)$ and $\varepsilon/2$ to find $(z_j, \varphi_0) \in \Pi(\ell_1)$ such that

$$\|Te_j - a_j z_j\| \leq \varepsilon/2, \quad \|\varphi - \varphi_0\| \leq \varepsilon/2$$

and $\Pi_1(\varphi) \subset \Pi_1(\varphi_0)$ –see Remark 3.6.1, where $a_j = e^{\arg(\varphi(Te_j))i}$. Observe that φ_0 can be chosen independently on $j \in P$ and by (3.10) explicitly written as

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/80)} + e^{\arg(\varphi)i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/80)}. \tag{3.26}$$

Let us define T_0 as the unique operator in $L(\ell_1)$ such that $T_0 e_i = Te_i$ for $i \notin P$ and $T_0 e_j = z_j$ for $j \in P$. Equivalently,

$$T_0 x = \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j\in P} e_j^*(x) z_j, \text{ for } x \in \ell_1. \tag{3.27}$$

It is clear from (3.27) that

$$\|T_0\| = \sup_{n\in\mathbb{N}}\{\|T_0 e_n\|\} = \max\left\{\sup_{j\notin P}\{\|Te_j\|\}, \sup_{j\in P}\{\|z_j\|\}\right\} = 1.$$

Given $j \in P$, the identity (3.25) ensures that $\mathrm{Re}(\varphi(Te_j)) \geq 1 - \mu^2/2$. Using again the general fact (3.9), we deduce that $|a_j - 1| \leq \mu \leq \varepsilon/2$.

Therefore,

$$\begin{aligned}\|T - T_0\| &= \sup_{n\in\mathbb{N}}\{\|Te_n - T_0e_n\|\} = \sup_{j\in P}\{\|Te_j - z_j\|\} \\ &\leq \sup_{j\in P}\{\|Te_j - a_jz_j\|\} + \sup_{j\in P}\{\|a_jz_j - z_j\|\} \\ &\leq \frac{\varepsilon}{2} + \sup_{j\in P}\{|a_j - 1|\} \leq \varepsilon.\end{aligned}$$

Since $x_0 \in \pi_1(\varphi)$ and $\pi_1(\varphi) \subset \pi_1(\varphi_0)$, we deduce that (x_0, φ_0) belongs to $\Pi(\ell_1)$. It remains to show that $\varphi_0(T_0x_0) = 1$ to prove the validity of (3.23). But, since x_0 is positive, we obtain that

$$\begin{aligned}\varphi_0(T_0x_0) &\overset{(3.27)}{=} \sum_{j\in P}(x_0)_j\varphi_0(z_j) + \sum_{j\notin P}(x_0)_j\varphi_0(Te_j) \\ &\overset{(3.24)}{=} \sum_{j\in P}(x_0)_j = \sum_{j\in P}|(x_0)_j| = \|x_0\| = 1,\end{aligned}$$

and the proof is over. □

Remark 3.7.1. *We cannot replace the condition $(x, \varphi) \in \Pi(\ell_1)$ in Theorem 3.7 by the more general $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$. Indeed, let us consider the operator $T: \ell_1 \to \ell_1$ defined by $Te_j = e_j$ for $j \geq 2$ and $Te_1 = e_2$. Take $(e_1, e_2^*) \in B_{\ell_1} \times B_{\ell_\infty}$, $T_0 \in L(\ell_1)$, and $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ such that $\|T - T_0\| \leq \varepsilon$, $\|e_1 - x\| \leq \varepsilon$, and $\|e_2^* - \varphi\| \leq \varepsilon$. Then*

$$|\varphi(x)| \leq |\varphi(x) - e_2^*(x)| + |e_2^*(x) - e_2^*(e_1)| + |e_2^*(e_1)| \leq 2\varepsilon,$$

which implies that (x, φ) cannot be in $\Pi(\ell_1)$.

Corollary 3.8. *The Banach space ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.*

Proof. Let us consider $T \in L(\ell_1)$ with $\nu(T) = 1$ and $0 < \varepsilon < 1$. Let us take a pair $(x, \varphi) \in \Pi(\ell_1)$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$. In fact, we can assume that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$; otherwise, we proceed with $\widetilde{T} = e^{-\arg(\varphi(Tx))i}\, T$. Then Theorem 3.7 gives the existence of an operator $T_0 \in S_{L(\ell_1)}$ and a pair $(x_0, \varphi_0) \in \Pi(\ell_1)$ that satisfy conditions in (3.23), which are precisely the requirements (3.1) in Definition 3.1. $\square$

Corollary 3.9 ([20]). *The set* $\mathrm{NRA}(\ell_1)$ *is dense in* $L(\ell_1)$.

3.3 BPB Property for Numerical Radius in $c_0(\mathbb{C})$

Theorem 3.7 allows us to show that c_0 has the Bishop-Phelps-Bollobás property for numerical radius as well. Indeed, we rely on the fact that our constructions in ℓ_1 can be dualized.

Theorem 3.10. *Let* $T \in S_{L(c_0)}$, $0 < \varepsilon < 1$ *and* $(x, \varphi) \in \Pi(c_0)$ *such that* $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}$. *Then there exist* $S \in S_{L(c_0)}$ *and* $(x_0, \varphi_0) \in \Pi(c_0)$, *such that*

$$\|T - S\| \leq \varepsilon,\ \ \|x - x_0\| \leq \varepsilon,\ \ \|\varphi - \varphi_0\| \leq \varepsilon,\ \ \textit{and}\ \varphi_0(Sx_0) = 1.$$

Proof. Throughout this proof we identify the elements in c_0 with their image in ℓ_∞ through the natural embedding of c_0 into ℓ_∞. The adjoint operator of T, $T^* \colon \ell_1 \to \ell_1$ satisfies

$$|x(T^*\varphi)| = |T^*(\varphi)(x)| = |\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}.$$

Without loss of generality, we can assume that $x(T^*\varphi) \geq 1 - (\varepsilon/9)^{9/2}$. Otherwise, employing techniques from the proof of Corollary 3.8, define the operator $\widetilde{T} = e^{-\arg(x(T^*\varphi))i}\, T^*$ and proceed with the proof for $x(\widetilde{T}\varphi) = |x(T^*\varphi)|$.

By Theorem 3.7, there exists $T_0 \in L(\ell_1)$, $\|T_0\| = 1$, and $(\varphi_0, x_0) \in \Pi(\ell_1)$ such that

$$\|T^* - T_0\| \leq \varepsilon, \quad \|\varphi - \varphi_0\| \leq \varepsilon, \quad \|x - x_0\| \leq \varepsilon$$

and $x_0(T_0\varphi_0) = 1$.

We assume that (x_0, φ_0) is the needed pair. To show this, we will reexamine the proof of Theorem 3.7 to establish how x_0, φ_0 and T_0 are defined. Indeed, from (3.25), (3.24), (3.26) and (3.27) we have respectively

$$\begin{aligned} P &= \mathcal{P}_{(\varphi, T^{**}x)}(\varepsilon^3/480), \\ \varphi_0 &= \|\varphi \cdot \mathbb{1}_P\|^{-1} \cdot \varphi \cdot \mathbb{1}_P, \\ x_0 &= x \cdot \mathbb{1}_{\mathbb{N}\setminus A_x(\varepsilon^2/80)} + e^{\arg(x)i} \cdot \mathbb{1}_{A_x(\varepsilon^2/80)}, \\ T_0 x &= \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j \in P} e_j^*(x) z_j, \text{ for } x \in \ell_1, \end{aligned} \tag{3.28}$$

where $\{z_j\}_{j\in P} \subset \pi_1(\varphi_0)$.

Note that $\mathcal{A}_x(\varepsilon^2/80) = \{j \in N \colon |x_j| \geq 1 - \varepsilon^2/80\}$ and that $x \in c_0$. Thus, $A_x(\varepsilon^2/80)$ is finite which, by (3.28), implies that $x_0 \in c_0$.

We shall show that T_0 is an adjoint operator and thus that there exists $S \in L(c_0)$ such that $S^* = T_0$. It will be enough to show that $T_0^*|_{c_0} \subset c_0$. Set $t_{ij} = \langle e_i, T(e_j)\rangle$ for $i, j \in \mathbb{N}$. Fix $i \in \mathbb{N}$, then for $j \in \mathbb{N}$

$$\langle e_j, T_0^*(e_i)\rangle = \begin{cases} t_{ji} & \text{if } j \notin P, \\ (z_j)_i & \text{if } j \in P. \end{cases}$$

Since $x \in c_0$, $T^{**}x$ belongs to c_0, which implies that P is finite. Accordingly, only finitely many terms of the form $\langle e_j, T_0^*(e_i)\rangle$ differ from the corresponding t_{ji}. On the other hand, since T belongs to $L(c_0)$, it holds that $\lim_j |t_{ji}| = 0$. Therefore, we deduce that $|\langle e_j, T_0^*(e_i)\rangle| \to 0$ when $j \to \infty$. This implies that $T_0^* e_i \in c_0$ and, since $i \in \mathbb{N}$ is arbitrarily chosen, we deduce that $T_0^*|_{c_0} \subset c_0$.

Hence we obtain the operator $S = T_0^*|_{c_0} \in L(c_0)$ and the pair $(x_0, \varphi_0) \in \Pi(c_0)$ satisfying:

$$\varphi_0(Sx_0) = S^*\varphi_0(x_0) = x_0(S^*\varphi_0) = x_0(T_0\varphi_0) = 1,$$

and

$$\|S - T\| = \|(S - T)^*\| = \|S^* - T^*\| = \|T_0 - T^*\| \leq \varepsilon,$$

which finishes the proof. □

Theorem 3.10 implies the following two corollaries.

Corollary 3.11. *The Banach space c_0 has the Bishop-Phelps-Bollobás property for numerical radius.*

Corollary 3.12 ([20])**.** *The set* $\mathrm{NRA}(c_0)$ *is dense in* $L(c_0)$.

3.4 Generalizations and Remarks

All the results that have been presented in sections 3.1, 3.2 and 3.3 were stated and proved for the Banach spaces $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. However, the arguments could be easily adjusted for $\ell_1(\mathbb{R})$ and $c_0(\mathbb{R})$, yielding shorter proofs and better estimates. More generally, given a non-empty set Γ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, these results are still valid for $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$. The spaces $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$ are, respectively, the ℓ_1-sum and the c_0-sum of Γ copies of the field $\mathbb{K}$. Note that in particular $\ell_1(\mathbb{N}, \mathbb{K}) = \ell_1(\mathbb{K})$.

The Banach space $c_0(\Gamma, \mathbb{K})$ is a predual of $\ell_1(\Gamma, \mathbb{K})$. Observe that both $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$ have numerical index 1. Previous considerations imply that both of them also have the BPB property for numerical radius. The ω^* topology of $\ell_1(\Gamma, \mathbb{K})$ stands here for the topology induced on $\ell_1(\Gamma, \mathbb{K})$ by pointwise convergence on elements of $c_0(\Gamma, \mathbb{K})$.

On the other hand, the proof of Theorem 3.10 shows that in Theorem 3.7 we proved more than was stated. Indeed, by putting together Theorem 3.7, the ideas on duality in the proof of Theorem 3.10 and considerations above, one easily proves the following theorem.

Theorem 3.13. *Let $T \in S_{L(\ell_1(\Gamma,\mathbb{K}))}$, $0 < \varepsilon < 1$ and $(x,\varphi) \in \Pi(\ell_1(\Gamma,\mathbb{K}))$ such that $\varphi(Tx) \geq 1-(\varepsilon/9)^{9/2}$. Then there exist $T_0 \in S_{L(\ell_1(\Gamma,\mathbb{K}))}$ and $(x_0,\varphi_0) \in \Pi(\ell_1(\Gamma,\mathbb{K}))$ such that*

$$\|T-T_0\| \leq \varepsilon,\ \|x-x_0\| \leq \varepsilon,\ \|\varphi-\varphi_0\| \leq \varepsilon \ \textit{ and } \ \varphi_0(T_0x_0)=1.$$

Moreover, if T is ω^*-ω^*-continuous and φ is ω^*-continuous, then T_0 and φ_0 will be ω^*-ω^*-continuous and ω^*-continuous, respectively.

Below are two consequences of Theorem 3.13.

Theorem 3.14. *The Banach space $\ell_1(\Gamma,\mathbb{K})$ has the BPB property for numerical radius.*

Theorem 3.15. *The Banach space $c_0(\Gamma,\mathbb{K})$ has the BPB property for numerical radius.*

Proof. Fix $0<\varepsilon<1$, $\delta \leq (\varepsilon/9)^{9/2}$, $T \in S_{L(c_0(\Gamma,\mathbb{K}))}$ and $(x,x^*) \in \Pi(c_0(\Gamma,\mathbb{K}))$ such that $x^*(Tx)\geq 1-\delta$. Applying Theorem 3.13 to the ω^*-ω^*-continuous operator $T^* \in S_{L(\ell_1(\Gamma,\mathbb{K}))}$, the pair (x^*,x) and ε, gives a new $T_0 \in S_{L(c_0(\Gamma,\mathbb{K}))}$ and a new pair $(x_0^*,x_0^{**}) \in \Pi(\ell_1(\Gamma,\mathbb{K}))$ satisfying

$$\left\|T^*-T_0^*\right\| \leq \varepsilon,\ \|x-x_0^{**}\| \leq \varepsilon,\ \|x^*-x_0^*\| \leq \varepsilon \text{ and } x_0^{**}\left(T_0^*x_0^*\right)=1. \tag{3.29}$$

Moreover, x_0^{**} is ω^*-continuous, so we can identify it with some $x_0 \in S_{c_0(\Gamma,\mathbb{K})}$. Therefore, the conditions in (3.29) become

$$\left\|T-T_0\right\| \leq \varepsilon,\ \|x-x_0\| \leq \varepsilon,\ \|x^*-x_0^*\| \leq \varepsilon \text{ and } x_0^*\left(T_0x_0\right)=1.$$

which are the requirements (3.1) in Definition 3.1. Consequently, $c_0(\Gamma,\mathbb{K})$ has the Bishop-Phelps-Bollobás property for numerical radius. □

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