NON-COMPLETE MACKEY TOPOLOGIES ON BANACH SPACES

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Abstract

Answering in the negative a question of W. Arendt and M. Kunze, we construct Banach spaces $X$ and norm closed weak*-dense subspaces $Y$ of the dual $X'$ of $X$ such that $X$ endowed with the Mackey topology $\mu(X, Y)$ of the dual pair $(X, Y)$ is not complete.

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The following problem appeared in a natural way in connection with the study of Pettis integrability with respect to norming subspaces developed by Markus Kunze in his Ph.D. thesis [5]. This question was asked to the authors by Kunze himself and his thesis advisor W. Arendt.

**Problem.** Suppose that $(X, ||\cdot||)$ is a Banach space and $Y$ is a subspace of its topological dual $X'$ which is norm closed and weak*-dense. Is there a complete topology of the dual pair $(X, Y)$ in $X$?

We use freely the notation for locally convex spaces (shortly, lcs) as in [4, 6, 7]. In particular, we denote, respectively, by $\sigma(X, Y)$ and $\mu(X, Y)$ the weak and the Mackey topology in $X$ associated to the dual pair $(X, Y)$. For a Banach space $X$ with topological dual $X'$, the weak*-topology is $\sigma(X', X)$. By the Bourbaki Robertson lemma [4, §18.4.4], there is a complete topology in $X$ of the dual pair $(X, Y)$ if and only if the space $(X, \mu(X, Y))$ is complete. Therefore, the original question is equivalent to the following

**Problem A:** Let $(X, ||\cdot||)$ be a Banach space. Is $(X, \mu(X, Y))$ complete for every norm closed weak*-dense subspace $Y$ of the dual space $X'$?

Let $(X, ||\cdot||)$ be a normed space. A subspace $Y$ of $X'$ is said to be *norming* if the function $p$ of $X$ given by $p(x) = \sup\{|x'(x)| : x' \in Y \cap B_{X'}\}$ is a norm equivalent to $||\cdot||$. We

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notice that Problem A is not affected by changing the given norm of \( X \) by any equivalent one. Thus, to study Problem A for some norming subspace \( Y \subset X' \) we can and will always assume that \( Y \) is indeed 1-norming, i.e., \( \|x\| = \sup_{x' \in Y \cap B_{X'}} |x'(x)| \).

Let us observe that under the conditions of Problem A, if \((X, \mu(X, Y))\) is quasi-complete (in particular complete), then Krein-Smulyan’s theorem, see [4, §24.5.(4)], implies that for every \( \sigma(X, Y) \)-compact subset \( H \) of \( X \) its \( \sigma(X, Y) \)-closed absolutely convex hull \( M := \overline{\text{aco}H}^{\sigma(X, Y)} \) is also \( \sigma(X, Y) \)-compact. There are several papers dealing with the validity of Krein-Smulyan theorem for topologies weaker than the weak topology; see for instance [1, 2] where it is proved that for every Banach space \( X \) not containing \( \ell^1([0, 1]) \) and every 1-norming subspace \( Y \subset X' \), if \( H \) is a norm bounded \( \sigma(X, Y) \)-compact subset of \( X \) then \( \overline{\text{aco}H}^{\sigma(X, Y)} \) is \( \sigma(X, Y) \)-compact. It was proved in [3] that the hypothesis \( \ell^1([0, 1]) \) is also necessary for the latter.

The following useful observation will be used a couple of times later.

**Proposition 1.** Let \((X, \|\cdot\|)\) be a Banach space and let \( Y \) be a 1-norming subspace of \( X' \). If \((X, \mu(X, Y))\) is quasi-complete, then every \( \sigma(X, Y) \)-compact subset of \( X \) is norm bounded.

**Proof.** Let \( H \subset X \) be \( \sigma(X, Y) \)-compact. As noted before, Krein-Smulyan’s theorem, [4, §24.5.(4)], implies that the \( \sigma(X, Y) \)-closed absolutely convex hull \( M := \overline{\text{aco}H}^{\sigma(X, Y)} \) is also \( \sigma(X, Y) \)-compact. Therefore, \( M \) is an absolutely convex, bounded and complete subset of the locally convex space \((X, \sigma(X, Y))\). Now we can apply [4, §20.11.(2)] to obtain that \( M \) is a Banach disc, i.e., \( X_M := \bigcup_{n \in \mathbb{N}} nM \) is a Banach space with the norm \( \|x\|_M := \inf \{ \lambda \geq 0 : x \in \lambda M \}, x \in X_M \).

Since \( M \) is bounded in \((X, \sigma(X, Y))\), the inclusion \( J : X_M \to (X, \sigma(X, Y)) \) is continuous, therefore \( J : X_M \to (X, \|\cdot\|) \) has closed graph, hence it is continuous by the closed graph theorem. In particular, the image of the closed unit ball \( M \) of \( X_M \) is bounded in \((X, \|\cdot\|)\), and the proof is complete. \( \square \)

As an immediate consequence of the above we have the following:

**Example 2.** Let \( X = C([0, 1]) \) be endowed with its sup norm and take

\[
Y := \text{span} \{ \delta_x : x \in [0, 1] \} \subset X'.
\]

Then \((X, \mu(X, Y))\) is not quasi-complete.

**Proof.** Notice that \( \sigma(X, Y) \) coincides with the topology \( \tau_p \) of pointwise convergence on \( C([0, 1]) \). Since there are sequences \( \tau_p \)-convergent to zero which are not norm bounded, \((X, \mu(X, Y))\) cannot be quasi-complete by Proposition 1. \( \square \)
The subspace $Y$ of $X'$ in Example 2 is weak*-dense in $X'$ but not norm closed. Another example of the same nature is the following: take $X = c_0$, $Y = \varphi$, the space of sequences with finitely many non-zero coordinates, which is norm dense in $X' = \ell_1$. In this case $\mu(X, Y) = \sigma(X, Y)$, since every absolutely convex $\sigma(X, Y)$-compact subset of $Y$ is finite dimensional by Baire category theorem. In this case $(X, \sigma(X, Y))$ is even not sequentially complete.

The following example, taken from Lemma 11 in [3], provides the negative solution to Problem A.

**Example 3.** Take $X = (\ell^1([0, 1]), \|\cdot\|_1)$ and consider the space $Y = C([0, 1])$ of continuous functions on $[0, 1]$ as a norming subspace of the dual $X' = \ell^\infty([0, 1])$. Then $(X, \mu(X, Y))$ is not quasi-complete.

**Proof.** Let $H := \{e_x : x \in [0, 1]\}$ be the canonical basis of $\ell^1([0, 1])$. The set $H$ is clearly $\sigma(X, Y)$-compact but we will prove that $\overline{\text{aco}H}^{\sigma(X, Y)}$ is not $\sigma(X, Y)$-compact, and therefore $(X, \mu(X, Y))$ cannot be quasi-complete. Indeed, proceeding by contradiction let us assume that $W := \overline{\text{aco}H}^{\sigma(X, Y)}$ is $\sigma(X, Y)$-compact. We write $M([0, 1]) = (C([0, 1]), \|\cdot\|_\infty)'$ to denote the space of Radon measures in $[0, 1]$ endowed with its variation norm. The map

$$\phi : X \to M([0, 1])$$

given by $\phi((\xi_x)_{x \in [0, 1]}) = \sum_{x \in [0, 1]} \xi_x \delta_x$ is $\sigma(X, Y)$-w*-continuous. We notice that:

1. $\phi(W) \subset \phi(\ell^1([0, 1]))$;
2. $\phi(W)$ is an absolutely convex w*-compact subset of $M([0, 1])$;
3. $\{\delta_x : x \in [0, 1]\} \subset \phi(W)$.

From the above we obtain that

$$B_{M([0, 1])} = \overline{\text{aco}\{\delta_x : x \in [0, 1]\}}^{w^*} \subset \phi(W) \subset \phi(\ell^1([0, 1])),$$

which is a contradiction because there are Radon measures on $[0, 1]$ which are not of the form $\sum_{x \in [0, 1]} \xi_x \delta_x$. The proof is complete. □

**Proposition 4.** If $X$ is a Banach space containing an isomorphic copy of $\ell^1([0, 1])$, then there is a subspace $Y \subset X'$ norm closed and norming such that $(X, \mu(X, Y))$ is not quasi-complete.

**Proof.** In the proof of [3, Proposition 3] the authors construct a norming subspace $E \subset X'$ and $H \subset X$ norm bounded $\sigma(X, E)$-compact such that $\overline{\text{aco}H}^{\sigma(X, E)}$ is not $\sigma(X, E)$-compact. If we take $Y = \overline{E} \subset X'$, norm closure, then norm bounded $\sigma(X, E)$-convergent nets in $X$ are $\sigma(X, Y)$-convergent; from here we obtain that:

(i) $H \subset X$ is $\sigma(X, Y)$-compact, and

(ii) $\overline{\text{aco}H}^{\sigma(X, E)} = \overline{\text{aco}H}^{\sigma(X, Y)}$. 

Consequently $H$ is $\sigma(X,Y)$-compact and $\overline{\operatorname{aco}H}^{\sigma(X,Y)}$ is not. Thus $(X,\mu(X,Y))$ cannot be quasi-complete and the proof is over. □

We conclude this note with a few comments about the relation of the questions considered here with Mazur property. We say that a lcs $(E,\Sigma)$ is Mazur if every sequentially $\Sigma$-continuous form defined on $E$ is $\Sigma$-continuous. We quote the following result:

**Theorem 5.** [7, Theorem 9.9.14] Let $(X,Y)$ be a dual pair. If $(X,\sigma(X,Y))$ is Mazur and $(X,\mu(X,Y))$ is complete, then $(Y,\mu(Y,X))$ is complete.

**Proposition 6.** Let $X$ be a Banach space. Let $Y$ be a proper subspace of $X'$ which is $w^*$-dense. Assume that:
1. the norm bounded $\sigma(X,Y)$-compact subsets of $X$ are weakly compact.
2. $(X,\sigma(X,Y))$ is Mazur.

Then $(X,\mu(X,Y))$ is not complete.

**Proof.** Assume that $(X,\mu(X,Y))$ is complete. Then Proposition 1 implies that every $\sigma(X,Y)$-compact subset of $X$ is norm bounded. Therefore the family of $\sigma(X,Y)$-compact subset coincide with the family of weakly compact sets. So the Mackey topology $\mu(Y,X)$ in $Y$ associated to the pair $(X,Y)$ is the topology induced in $Y$ by the Mackey topology $\mu(X',X)$ in $X'$ associated to the dual pair $(X,X')$. If we use now Theorem 5 we obtain that $Y$ is $\mu(Y,X)$ complete, that implies that $Y \subset X'$ is $\mu(X',X)$ closed. Thus

$$Y = \overline{Y}^{\mu(Y,X)} = \overline{Y}^{\mu_{w^*}} = X',$$

which is a contradiction with the fact that $Y$ is a proper subspace of $X'$. □

We observe that hypothesis (1) in the above Proposition is satisfied for Banach spaces without copies of $\ell^1([0,1])$ whenever $Y$ contains a boundary for the norm, see [1, 2].

**References**

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