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NON-COMPLETE MACKEY TOPOLOGIES ON BANACH SPACES

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Abstract

Answering in the negative a question of W. Arendt and M. Kunze, we construct Banach spaces X and norm closed weak*-dense subspaces Y of the dual X' of X such that X endowed with the Mackey topology $\mu(X, Y)$ of the dual pair $\langle X, Y\rangle$ is not complete.

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The following problem appeared in a natural way in connection with the study of Pettis integrability with respect to norming subspaces developed by Markus Kunze in his Ph.D. thesis [5]. This question was asked to the authors by Kunze himself and his thesis advisor W. Arendt.

Problem. *Suppose that $(X, \|\cdot\|)$ is a Banach space and Y is a subspace of its topological dual X' which is norm closed and weak*-dense. Is there a complete topology of the dual pair $\langle X, Y\rangle$ in X?*

We use freely the notation for locally convex spaces (shortly, lcs) as in [4, 6, 7]. In particular, we denote, respectively, by $\sigma(X, Y)$ and $\mu(X, Y)$ the weak and the Mackey topology in X associated to the dual pair $\langle X, Y\rangle$. For a Banach space X with topological dual X', the weak*-topology is $\sigma(X', X)$. By the Bourbaki Robertson lemma [4, §18.4.4], there is a complete topology in X of the dual pair $\langle X, Y\rangle$ if and only if the space $(X, \mu(X, Y))$ is complete. Therefore, the original question is equivalent to the following

Problem A: *Let $(X, \|\cdot\|)$ be a Banach space. Is $(X, \mu(X, Y))$ complete for every norm closed weak*-dense subspace Y of the dual space X'?*

Let $(X, \|\cdot\|)$ be a normed space. A subspace Y of X' is said to be *norming* if the function p of X given by $p(x) = \sup\{|x'(x)| : x' \in Y \cap B_{X'}\}$ is a norm equivalent to $\|\cdot\|$. We

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notice that Problem A is not affected by changing the given norm of X by any equivalent one. Thus, to study Problem A for some norming subspace $Y \subset X'$ we can and will always assume that Y is indeed 1-norming, *i.e.*, $\|x\| = \sup\{|x'(x)| : x' \in Y \cap B_{X'}\}$.

Let us observe that under the conditions of Problem A, if $(X, \mu(X, Y))$ is quasi-complete (in particular complete), then Krein-Smulyan's theorem, see [4, §24.5.(4)], implies that for every $\sigma(X, Y)$-compact subset H of X its $\sigma(X, Y)$-closed absolutely convex hull $M := \overline{\mathrm{aco} H}^{\sigma(X,Y)}$ is also $\sigma(X, Y)$-compact. There are several papers dealing with the validity of Krein-Smulyan theorem for topologies weaker than the weak topology; see for instance [1, 2] where it is proved that for every Banach space X not containing $\ell^1([0, 1])$ and every 1-norming subspace $Y \subset X'$, if H is a norm bounded $\sigma(X, Y)$-compact subset of X then $\overline{\mathrm{aco} H}^{\sigma(X,Y)}$ is $\sigma(X, Y)$-compact. It was proved in [3] that the hypothesis $\ell^1([0, 1]) \not\subset X$ is also necessary for the latter.

The following useful observation will be used a couple of times later.

Proposition 1. *Let $(X, \|\cdot\|)$ be a Banach space and let Y be a 1-norming subspace of X'. If $(X, \mu(X, Y))$ is quasi-complete, then every $\sigma(X, Y)$-compact subset of X is norm bounded.*

Proof. Let $H \subset X$ be $\sigma(X, Y)$-compact. As noted before, Krein-Smulyan's theorem, [4, §24.5.(4)], implies that the $\sigma(X, Y)$-closed absolutely convex hull $M := \overline{\mathrm{aco} H}^{\sigma(X,Y)}$ is $\sigma(X, Y)$-compact. Therefore, M is an absolutely convex, bounded and complete subset of the locally convex space $(X, \sigma(X, Y))$. Now we can apply [4, §20.11.(2)] to obtain that M is a Banach disc, *i.e.*, $X_M := \bigcup_{n\in\mathbb{N}} nM$ is a Banach space with the norm

$$\|x\|_M := \inf\{\lambda \geq 0 : x \in \lambda M\}, x \in X_M.$$

Since M is bounded in $(X, \sigma(X, Y))$, the inclusion $J : X_M \to (X, \sigma(X, Y))$ is continuous, therefore $J : X_M \to (X, \|\cdot\|)$ has closed graph, hence it is continuous by the closed graph theorem. In particular, the image of the closed unit ball M of X_M is bounded in $(X, \|\cdot\|)$, and the proof is complete. □

As an immediate consequence of the above we have the following:

Example 2. Let $X = C([0, 1])$ be endowed with its sup norm and take

$$Y := \mathrm{span}\,\{\delta_x : x \in [0, 1]\} \subset X'.$$

Then $(X, \mu(X, Y))$ is not quasi-complete.

Proof. Notice that $\sigma(X, Y)$ coincides with the topology τ_p of pointwise convergence on $C([0, 1])$. Since there are sequences τ_p-convergent to zero which are not norm bounded, $(X, \mu(X, Y))$ cannot be quasi-complete by Proposition 1. □

The subspace Y of X' in Example 2 is weak*-dense in X' but not norm closed. Another example of the same nature is the following: take $X = c_0$, $Y = \varphi$, the space of sequences with finitely many non-zero coordinates, which is norm dense in $X' = \ell_1$. In this case $\mu(X, Y) = \sigma(X, Y)$, since every absolutely convex $\sigma(Y, X)$-compact subset of Y is finite dimensional by Baire category theorem. In this case $(X, \sigma(X, Y))$ is even not sequentially complete.

The following example, taken from Lemma 11 in [3], provides the negative solution to Problem A.

Example 3. Take $X = (\ell^1([0, 1]), \|\cdot\|_1)$ and consider the space $Y = C([0, 1])$ of continuous functions on $[0, 1]$ as a norming subspace of the dual $X' = \ell^\infty([0, 1])$. Then $(X, \mu(X, Y))$ is not quasi-complete.

Proof. Let $H := \{e_x : x \in [0, 1]\}$ be the canonical basis of $\ell^1([0, 1])$. The set H is clearly $\sigma(X, Y)$-compact but we will prove that $\overline{\mathrm{aco}H}^{\sigma(X,Y)}$ is not $\sigma(X, Y)$-compact, and therefore $(X, \mu(X, Y))$ cannot be quasi-complete. Indeed, proceeding by contradiction let us assume that $W := \overline{\mathrm{aco}H}^{\sigma(X,Y)}$ is $\sigma(X, Y)$-compact. We write $M([0, 1]) = (C([0, 1]), \|\cdot\|_\infty)'$ to denote the space of Radon measures in $[0, 1]$ endowed with its variation norm. The map

$$\phi : X \to M([0, 1])$$

given by $\phi((\xi_x)_{x\in[0,1]}) = \sum_{x\in[0,1]} \xi_x \delta_x$ is $\sigma(X, Y)$-w*-continuous. We notice that:

1. $\phi(W) \subset \phi(\ell^1([0, 1]))$;
2. $\phi(W)$ is an absolutely convex w*-compact subset of $M([0, 1])$;
3. $\{\delta_x : x \in [0, 1]\} \subset \phi(W)$.

¿From the above we obtain that

$$B_{M\left([0,1]\right)} = \overline{\mathrm{aco}\{\delta_x : x \in [0, 1]\}}^{w^*} \subset \phi(W) \subset \phi(\ell^1([0, 1]),$$

which is a contradiction because there are Radon measures on $[0, 1]$ which are not of the form $\sum_{x\in[0,1]} \xi_x \delta_x$. The proof is complete. □

Proposition 4. *If X is a Banach space containing an isomorphic copy of $\ell^1([0, 1])$, then there is a subspace $Y \subset X'$ norm closed and norming such that $(X, \mu(X, Y))$ is not quasi-complete.*

Proof. In the proof of [3, Proposition 3] the authors construct a norming subspace $E \subset X'$ and $H \subset X$ norm bounded $\sigma(X, E)$-compact such that $\overline{\mathrm{aco}H}^{\sigma(X,E)}$ is not $\sigma(X, E)$-compact. If we take $Y = \overline{E} \subset X'$, norm closure, then norm bounded $\sigma(X, E)$-convergent nets in X are $\sigma(X, Y)$-convergent; from here we obtain that:

(i) $H \subset X$ is $\sigma(X, Y)$-compact,

and

(ii) $\overline{\mathrm{aco}H}^{\sigma(X,E)} = \overline{\mathrm{aco}H}^{\sigma(X,Y)}$.

Consequently H is $\sigma(X, Y)$-compact and $\overline{\mathrm{aco} H}^{\sigma(X,Y)}$ is not. Thus $(X, \mu(X, Y))$ cannot be quasi-complete and the proof is over. $\square$

We conclude this note with a few comments about the relation of the questions considered here with Mazur property. We say that a lcs $(E, \mathfrak{T})$ is Mazur if every sequentially $\mathfrak{T}$-continuous form defined on E is $\mathfrak{T}$-continuous. We quote the following result:

Theorem 5. *[7, Theorem 9.9.14] Let $\langle X, Y\rangle$ be a dual pair. If $(X, \sigma(X, Y))$ is Mazur and $(X, \mu(X, Y))$ is complete, then $(Y, \mu(Y, X))$ is complete.*

Proposition 6. *Let X be a Banach space. Let Y be a proper subspace of X' which is w^*-dense. Assume that:*

1. *the norm bounded $\sigma(X, Y)$-compact subsets of X are weakly compact.*
2. *$(X, \sigma(X, Y)$ is Mazur.*

Then $(X, \mu(X, Y))$ is not complete.

Proof. Assume that $(X, \mu(X, Y))$ is complete. Then Proposition 1 implies that every $\sigma(X, Y)$-compact subset of X is norm bounded. Therefore the family of $\sigma(X, Y)$-compact subset coincide with the family of weakly compact sets. So the Mackey topology $\mu(Y, X)$ in Y associated to the pair $\langle X, Y\rangle$ is the topology induced in Y by the Mackey topology $\mu(X', X)$ in X' associated to the dual pair $\langle X, X'\rangle$. If we use now Theorem 5 we obtain that Y is $\mu(Y, X)$ complete, that implies that $Y \subset X'$ is $\mu(X', X)$ closed. Thus

$$Y = \overline{Y}^{\mu(X',X)} = \overline{Y}^{w^*} = X',$$

which is a contradiction with the fact that Y is a proper subspace of X'. $\square$

We observe that hypothesis (1) in the above Proposition is satisfied for Banach spaces without copies of $\ell^1([0, 1])$ whenever Y contains a boundary for the norm, see [1, 2].

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