

Vector Spaces. The Four Fundamental Subspaces.

Chapter 2, Linear Algebra and its Applications (G. Strang)
Chapter 2, Applied Linear Algebra (Olver and Shakiban)

A **real vector space** is a set of vectors together with rules for vector addition and multiplication by real numbers. We can add any two vectors, and we can multiply all vectors by scalars. In other words, we can take linear combinations.

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However, the vector spaces we need most are inside the standard spaces \mathbb{R}^n .

Definition. A **subspace** of a **real vector space** is a nonempty subset that satisfies the requirements for a vector space: **Linear combinations stay in the subspace.**

- (i) If we add any vectors x and y in the subspace, $x + y$ is in the subspace.
- (ii) If we multiply any vector x in the subspace by any scalar c , $c \cdot x$ is in the subspace.

Examples: Lines and planes that go through the origin.

Spanning Set of a Subspace

- Given w_1, \dots, w_l vectors in V , the set of all linear combinations of these vectors defines a subspace. In some books, it's denoted by $\langle w_1, \dots, w_l \rangle$,

$$\langle w_1, \dots, w_l \rangle = \{c_1 w_1 + \dots + c_l w_l, c_i \in \mathbb{R}\}$$

- Given w_1, \dots, w_l vectors of a subspace $W \subseteq V$, we say that these vectors **span** W if every vector $v \in W$ is linear combination of the w_i s; that is, $W = \langle w_1, \dots, w_l \rangle$.

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- Question:** The set w_1, \dots, w_l is a spanning set of $W \in \mathbb{R}^n$ **if and only if** ...

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- The set w_1, \dots, w_l is a spanning set of $W \in \mathbb{R}^n$ **if and only if** $Ax = b$ is solvable for every b in W , where A is the matrix **whose columns are** w_1, \dots, w_l .

The Column Space of a real matrix A , $C(A)$

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Important. A subspace V in \mathbb{R}^n defined by a spanning set of V can be seen as the column space of a matrix $V = C(A)$, whose columns are the spanning set.

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A line in \mathbb{R}^3 : $\{(x, y, z)/x + y + z = 0; x + 2y - z = 0\} =$

$$\{(x, y, z)/ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

That means that they are given by the nullspace of a matrix, whose rows are orthogonal to the vector space.

The Null Space of a matrix A , $N(A)$

Definition. The **nullspace of a matrix A consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$.**

It is a subspace of $\mathbb{R}^{\text{number of columns}}$, just as the column space was a subspace of $\mathbb{R}^{\text{number of rows}}$.

The nullspace of a matrix A can be seen as the solutions of a homogeneous linear system.

Obviously the vector $\mathbf{0}$ is in $N(A)$.

But, **when is $N(A)$ larger than vector 0 ?**

The Four Fundamental Subspaces

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3. The **row space** of A is the vector space spanned by the rows of A . That is, $C(A^t)$. I'll denote it by $R(A)$ (Strang, or by $\text{coimg } A$, Olver et al). **Observe** that

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4. The **left nullspace (left kernel)** of A are all vectors y such that $yA = 0$. That is, $N(A^t)$ (Strang, or by $\text{coker } A$, Olver et al).

Your turn

Describe the Four Fundamental Subspaces of the matrices

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computations: Open SAGE, <https://www.sagemath.org>

Basis for a Vector Space

Let V be a vector space.

Definition. A **basis** for V is a set of vectors having two properties at once:

1. The vectors are linearly independent (not too many vectors).
2. They span the space V (not too few vectors).

The **dimension of a Vector Space** is the number of vectors in a basis.

Definition. If $B = \{v_1, \dots, v_k\}$ is a basis for V , then any vector $v \in V$ has a **unique** representation $v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$, $c_i \in R$. The coefficients (c_1, \dots, c_k) are the **coordinates** of v with respect to the basis B

Recall: Linear independence

Definition. If $c_1 v_1 + \cdots + c_k v_k = 0$ only happens when $c_1 = \cdots = c_k = 0$, then the vectors v_1, \dots, v_k are linearly independent. If any c 's are nonzero, the v 's are linearly dependent and one vector (at least) is a combination of the others.

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Let A be the matrix whose columns are given by vectors v_1, \dots, v_k .
 v_1, \dots, v_k are L.I. $\leftrightarrow Ax=0$ iff $x=0$ $\leftrightarrow N(A) = \{\text{zero vector}\}$ \leftrightarrow
 U has k pivots $\leftrightarrow k$ rows of A are L.I. $\leftrightarrow k$ columns of A are L.I.
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Definition 2. The **rank** of a matrix is the number of independent columns, that is equal to the number of independent rows, that is, the number of pivots in the Elimination Gaussian method.

Basis for a Vector Space

Basis for $R(A)$: Given a matrix A of rank k and its echelon matrix U , the k rows of U that contain pivots are a basis for both, the row spaces of U and A . Recall that $R(A) = R(U)$.

Basis for $C(U)$: Given an echelon matrix U , the k columns that contain pivots are a basis for $C(U)$.

Basis for $C(A)$: The columns of any matrix A span its column space. If they are independent, they are a basis for the column space. However, $C(U)$ is not the same as $C(A)$, but the number of independent columns didn't change. **A basis for $C(A)$** is formed by the k columns of A that correspond, in U , to the columns containing pivots.

Basis for $N(A)$: One can obtain **a basis for $N(A)$** from the solutions of $Ax = 0$.

Dimensions of the Four Fundamental Subspaces

Given $A \in M_{m,n}(\mathbb{R})$ of rank k .

1. The dimension of **column space** of A is equal to k (the rank).
2. The dimension of **nullspace** of A is $n - k$.
3. The dimension of **row space** of A is also k .
4. The dimension of **left nullspace** of A is $m - k$.

Your turn

Basis and dimension of the Four Fundamental Subspaces of the matrices

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