EDPs y series de Fourier, 3º Matemáticas/4º PES — Test 2, 27/4/2021

Nombre: SOLUCIÓN

1. Considera la función signo

$$\text{sgn}(x) = \begin{cases} -1, & -\frac{1}{2} < x < 0 \\ 1, & 0 < x < \frac{1}{2} \end{cases}$$

a) Encuentra la serie de Fourier **real** de $\text{sgn}(x)$

b) Justifica si la serie converge y a qué valor, en cada punto $x \in \mathbb{R}$

c) Utiliza lo anterior para calcular $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

d) Determina a qué función corresponde la serie $\sum_{n=1}^{\infty} \frac{\cos(2\pi(2n-1)x)}{(2n-1)^2}$.

a) $$\hat{f}(n) = \int_0^{\frac{1}{2}} e^{-2\pi i n x}\,dx - \int_{-\frac{1}{2}}^{0} e^{-2\pi i n x}\,dx = \int_0^{\frac{1}{2}} \left(e^{-2\pi i n t} - e^{2\pi i n t}\right) dt$$

$x = -t$

$$= -2i \int_0^{\frac{1}{2}} \text{sen}(2\pi n t)\,dt \longrightarrow \hat{f}(0) = 0$$

$$\underset{(n \neq 0)}{=} 2i \left.\frac{\cos(2\pi n t)}{2\pi n}\right]_{t=0}^{t=\frac{1}{2}} = \cancel{2}i\,\frac{(-1)^n - 1}{\cancel{2}\pi n} = \begin{cases} 0 & n = \text{par} \\ -\frac{2i}{\pi n} & n = \text{impar.} \end{cases}$$

$$\Rightarrow f(x) \sim -\frac{2i}{\pi} \sum_{m \in \mathbb{Z}} \frac{e^{2\pi i (2m-1)x}}{2m-1} = -\frac{2i}{\pi} \sum_{m=1}^{\infty} \underbrace{\left(\frac{e^{2\pi i(2m-1)x}}{2m-1} - \frac{e^{-2\pi i(2m-1)x}}{2m-1} \right)}_{\frac{2i\,\text{sen}(2\pi(2m-1)x)}{2m-1}}$$

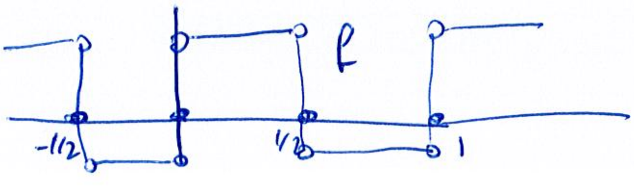
$$\Rightarrow \boxed{f(x) \sim \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\text{sen}(2\pi(2m-1)x)}{2m-1}}$$

b) Llamo f = extensión 1-periódica de sign

Como f es derivable en $\mathbb{R} \setminus (\frac{1}{2}\mathbb{Z}) \xrightarrow{\text{Dini}}$ la SF converge en esos ptos;

Como en cada $a \in \frac{1}{2}\mathbb{Z}$ existen $f(a)^{\pm}$ $(= \pm 1)$ y $f'(a)^{\pm}$ $(= 0)$

$\Rightarrow$ la SF converge en esos ptos a $\frac{f(a)^+ + f(a)^-}{2} = 0$.



c) En particular, si $x = \frac{1}{4}$

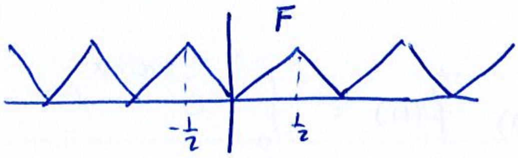
$$\operatorname{sen}\left((2m-1)\tfrac{\pi}{2}\right) = (-1)^{m+1}$$

$$\Rightarrow \quad 1 = f\left(\tfrac{1}{4}\right) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\operatorname{sen}\left(2\pi(2m-1)\frac{1}{4}\right)}{2m-1}$$

$$= \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} = \frac{4}{\pi}\Big(\underbrace{\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}_{\gamma}\Big)$$

$$\Rightarrow \quad \boxed{\gamma = \frac{\pi}{4}}.$$

d) Sea $F(x) = |x|$, $|x| \le \frac{1}{2}$ (y extendida 1-periódica) una primitiva de $f(x)$



Por el Tma de Integr de SF (como $\hat{f}(0) = 0$)

$$\Rightarrow \quad F(x) \sim c_0 + \sum_{n \neq 0} \frac{\hat{f}(n)}{2\pi i n} e^{2\pi i n x}$$

$$\sim c_0 - \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{\cos(2\pi(2m-1)x)}{(2m-1)^2}$$

donde $c_0 = \hat{F}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x|\, dx = 2\int_0^{\frac{1}{2}} x\, dx = x^2\Big]_0^{\frac{1}{2}} = \frac{1}{4}$.

Por la conv unif en el Tma anterior

$$|x| = \frac{1}{4} - \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{\cos(2\pi(2m-1)x)}{(2m-1)^2}, \quad \forall\, |x| \le 1/2$$

$$\Rightarrow \quad \boxed{\sum_{m=1}^{\infty} \frac{\cos(2\pi(2m-1)x)}{(2m-1)^2} = \frac{\pi^2}{2}\left(\frac{1}{4} - |x|\right), \quad |x| \le 1/2}$$