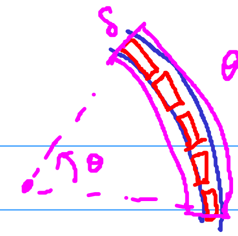
Scheme of proof



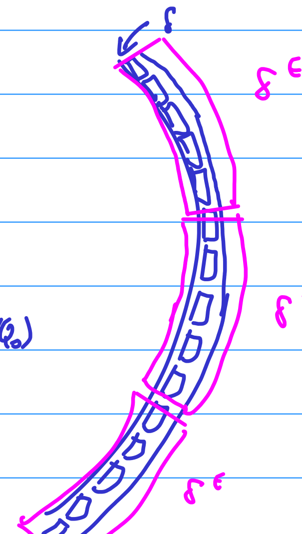
Let $f = \sum f_j$, $g = \sum g_j$.

<u>Notations</u> • $f_\theta = \sum_{\pi_j \subseteq \theta} f_j$ if θ is a sector of $S^1 + O(\delta)$

• $\sum_{\theta \sim \delta_1} f_\theta$ means sum over all δ_1-sectors θ

Step 1

Pick a fixed (very small) $\epsilon > 0$



$$\| f \cdot g \|_{L^{p/2}_\#(Q_0)} \overset{CS}{\le} R^\epsilon \left\| \Big(\sum_{\theta \sim \delta^\epsilon} |f_\theta|^2 \Big)^{1/2} \Big(\sum_{\theta \sim \delta^\epsilon} |g_\theta|^2 \Big)^{\frac{1}{2}} \right\|_{L^{p/2}_\#(Q_0)}$$

minimal loss

$|f_\theta|$ kills possible cancellation between different f_θ's ...

... but $|f_\theta| \approx$ const in R^ϵ-cubes $\longrightarrow$ so don't loose too much...

More notation:

Now we start from

size of freq arcs

$$A_p(f, g, \delta^\epsilon, Q_0) = \left\| \Big(\sum_{\theta \sim \delta^\epsilon} |f_\theta|^2 \Big)^{1/2} \Big(\sum_{\theta \sim \delta^\epsilon} |g_\theta|^2 \Big)^{\frac{1}{2}} \right\|_{L^{p/2}_\#(Q_0)}$$

and want to finish with

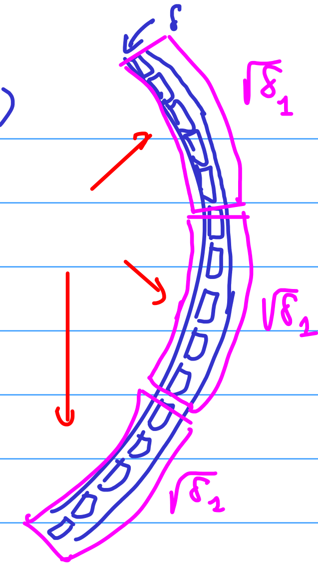
$$D_p(f, g, \delta_1, Q_0) = \Big(\sum_{\theta \sim \delta_1} \| f_\theta \|^2_{L^p_\#(Q_0)} \Big)^{\frac{1}{2}} \Big(\sum_{\theta \sim \delta_1} \| g_\theta \|^2_{L^p_\#(Q_0)} \Big)^{1/2}$$

size freq arcs $\hookrightarrow$ In the last step want $\delta_1 = \sqrt{\delta}$

Notes:

① $A_p(f, g, \delta_1, Q_0) \le D_p(f, g, \delta_1, Q_0)$ by CS + Minkowski

$\hookrightarrow$ but this is too inefficient ...



group in big arcs

② $$D_p(f, g, \sqrt{\delta_1}, Q_0) \leq (R/R_1)^{2\alpha} \cdot D_p(f, g, \sqrt{\delta}, Q_0)$$

parabolic rescaling

I. H. in each block

Step 2 Induction on scales

We successively group the long blocks

$$\delta^{\epsilon} \longrightarrow \delta^{2\epsilon} \longrightarrow \delta^{4\epsilon} \longrightarrow \dots \longrightarrow \delta^{2^{s-1}\epsilon} = \sqrt{\delta} \qquad \epsilon = \frac{1}{2^s}$$

and apply "parabolic rescaling" + H.I in each step

... but we need to improve the inefficient passage

$$A_p(f, g, \delta_1, Q_0) \leq D_p(f, g, \delta_1, Q_0) \quad \dots$$

This is achieved by exploiting transversality, via Kakeya bilinear estimate. The crucial lemma is

Lemma Let $p \geq 4$. Then for all $\eta > 0$,

δ_1^2- long blocks

target

$$A_p(f, g, \delta_1, Q_0) \lesssim_\eta (1/\delta_1)^{\epsilon_1} \cdot A_p(f, g, \delta_1^2, Q_0)^{1-\kappa_p} \cdot D_p(f, g, \delta_1, Q_0)^{\kappa_p}$$

δ_1- long blocks

gain!

where $\kappa_p \in (0,1)$ comes from interpol $\|\cdot\|_{p/2} \leq \|\cdot\|_2^{1-\kappa_p} \cdot \|\cdot\|_p^{\kappa_p}$

check $p = 6 \rightarrow \kappa_6 = 1/2$

Step 3: Iteration of lemma

Iterating the lemma and collecting the estimates

$$A_p(f,g,\delta^{\epsilon},Q_0) \lessapprox A_p(f,g,\delta^{2\epsilon},Q_0)^{1-\kappa} \cdot D_p(f,g,\delta^{\epsilon},Q_0)^{\kappa}$$

$$\lessapprox A_p(f,g,\delta^{2^2\epsilon},Q_0)^{(1-\kappa)^2} \cdot D_p(f,g,\delta^{2\epsilon},Q_0)^{(1-\kappa)\kappa} \cdot D_p(f,g,\delta^{\epsilon},Q_0)^{\kappa}$$

$$\lessapprox A_p(f,g,\underbrace{\delta^{2^{s-1}\epsilon}}_{\sqrt{\delta}},Q_0)^{(1-\kappa)^{s-1}} \cdot \prod_{j=1}^{s-1} D_p(f,g,\delta^{2^{j}\epsilon},Q_0)^{\kappa(1-\kappa)^{j-1}}$$

By CS + Mink we have

$$A_p(f,g,\sqrt{\delta},Q_0) = \Big\| \Big(\sum_{\theta\sim\sqrt{\delta}} |f_\theta|^2\Big)^{\frac12} \Big(\sum_{\theta\sim\sqrt{\delta}} |g_\theta|^2\Big)^{\frac12} \Big\|_{L^{p/2}_{\#}(Q_0)}$$

$$\le \Big\| \Big(\sum_{\theta\sim\sqrt{\delta}} |f_\theta|^2\Big)^{1/2} \Big\|_{L^{p}_{\#}(Q_0)} \cdot \Big\| \Big(\sum_{\theta\sim\sqrt{\delta}} |g_\theta|^2\Big)^{1/2} \Big\|_{L^{p}_{\#}(Q_0)}$$

$$\overset{\text{Mink}}{\le} \Big(\sum_{\theta\sim\sqrt{\delta}} \|f_\theta\|^2_{L^{p}_{\#}(Q_0)}\Big)^{1/2} \Big(\sum_{\theta\sim\sqrt{\delta}} \|g_\theta\|^2_{L^{p}_{\#}(Q_0)}\Big)^{1/2} = D_p(f,g,\sqrt{\delta},Q_0)$$

target

So, linking with step 1 and collecting all estimates

$$\|f\cdot g\|_{L^{p/2}_{\#}(Q_0)} \lessapprox R^{\epsilon} \cdot D_p(f,g,\sqrt{\delta},Q_0)^{(1-\kappa)^{s-1}} \cdot \prod_{j=1}^{s-1} D_p(f,g,\sqrt{\delta}^{2^{j}\epsilon},Q_0)^{\kappa(1-\kappa)^{j-1}}$$

target

$$\xrightarrow{\text{parab rescaling}} \quad \lessapprox \underbrace{R^{\epsilon} \cdot \prod_{j=1}^{s-1} \left(\delta^{2^{j}\epsilon}/\delta\right)^{2\alpha\cdot\kappa(1-\kappa)^{j-1}}}_{\text{loss}} \cdot D_p(f,g,\sqrt{\delta},Q_0)$$

Doing the numerology, the loss is

$$R^{?} \cdot R^{\epsilon + 2\alpha\cdot\sum_{j=1}^{s-1}(1-2^{j-s})\kappa(1-\kappa)^{j-1}} = R^{?}\cdot R^{\epsilon+2\alpha\cdot c_s}$$

$p=6 \Rightarrow$ $\kappa = 1/2$ where $c_s = \sum_{j=1}^{s-1}(1-2^{j-s})\cdot\frac{1}{2^j} = 1 - \frac{2s-1}{2^{s+1}} < 1$!!

gain

A careful analysis gives a new I.H. with

$$\| f \cdot g \|_{L^{p/2}} \le R^{2\alpha(1-\epsilon_0)} \cdot \left(\sum \|f_j\|_p^2 \right)^{1/2} \left(\sum \|g_j\|_p^2 \right)^{1/2}$$

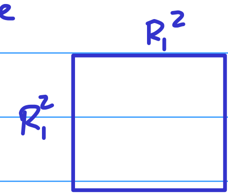
¡¡gain!! → can iterate to get a const as small as desired.

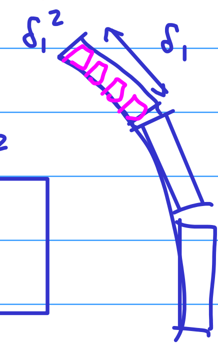
Step 4 Proof of Lemma → tools:
- Kakeya integral estimate
- Interpolation $\|\cdot\|_{p/2} \le \|\cdot\|_2^{1-u} \cdot \|\cdot\|_p^{u}$
- OG in L^2-norms.

To show

$$A_p(f, g, \delta_1, Q_0) \lesssim_n (1/\delta_1)^n \cdot A_p(f, g, \delta_1^2, Q_0)^{1-u_p} \cdot D_p(f, g, \delta_1, Q_0)^{u_p}.$$

- By localization, WMA $Q_0 = R_1^2$-cube





- Recall

$$A_p^{p/2} = \int_{Q_0} \left| \left(\sum_{\theta\sim\delta_1} |f_\theta|^2 \right)^{1/2} \left(\sum |g_\theta|^2 \right)^{1/2} \right|^{p/2}$$

Group freq arcs by size of $\|f_\theta\|_{p/2}$

$$I_\ell = \{ \theta \sim \delta_1 \,/\, \|f_\theta\|_{L^{p/2}_\#(Q_0)} \simeq 2^\ell \} \quad \text{and} \quad J_m = \{ \theta \sim \delta_1 \,/\, \|g_\theta\|_{L^{p/2}_\#(Q_0)} \simeq 2^m \}$$

A pigeon-hole argument allows to assume $\ell = m = 0$.
(at the expense of $(\log R)^c$ consts).

Note that $\displaystyle \left(\sum_{\theta\in I} |f_\theta|^2 \right)^{1/2} \overset{\text{Holder } p/4}{\le} |I|^{\frac12 - \frac2p} \cdot \left(\sum_\theta |f_\theta|^{p/2} \right)^{\frac2p}$

Thus

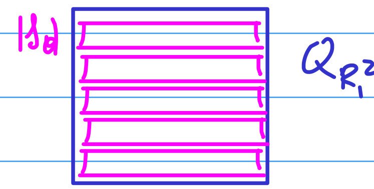
$$A_p^{p/2} \lesssim \left(|\mathcal{I}|\cdot|\mathcal{Y}|\right)^{\left(\frac{1}{2}-\frac{2}{p}\right)\frac{p}{2}} \cdot \int_{Q_0} \Big(\sum_{\mathcal{I}} |f_\theta|^{\frac{p}{2}}\Big)\cdot\Big(\sum_{\mathcal{Y}} |g_\theta|^{\frac{p}{2}}\Big)$$

and we are in the position of applying <u>bilinear Kakeya</u>

Note that each $|f_\theta| \simeq$ const in tubes

and f-tubes are transversal to g-tubes

bilinear assumption



Hence

$$\int_{Q_0} \overbrace{\Big(\sum_{\mathcal{I}} |f_\theta|^{\frac{p}{2}}\Big)}^{h_1}\cdot\overbrace{\Big(\sum_{\mathcal{Y}} |g_\theta|^{\frac{p}{2}}\Big)}^{h_2} \overset{\text{Kakeya}}{\lesssim} \Big(\int_{Q_0} \sum |f_\theta|^{p/2}\Big)\cdot\Big(\int_{Q_0} \sum |g_\theta|^{p/2}\Big)$$

$$\simeq \Big(\sum_{\mathcal{I}} \|f_\theta\|^{p/2}_{L^{p/2}_{\#}(Q_0)}\Big)\Big(\sum_{\mathcal{Y}} \|g_\theta\|^{p/2}_{L^{p/2}_{\#}(Q_0)}\Big)$$

So

$$A_p^{p/2} \lesssim \left(|\mathcal{I}|\cdot|\mathcal{Y}|\right)^{\left(\frac{1}{2}-\frac{2}{p}\right)\frac{p}{2}} \Big(\sum_{\mathcal{I}} \|f_\theta\|^{p/2}_{L^{p/2}_{\#}(Q_0)}\Big)\Big(\sum_{\mathcal{Y}} \|g_\theta\|^{p/2}_{L^{p/2}_{\#}(Q_0)}\Big)$$

by def $\mathcal{I}, \mathcal{Y}$ $\rightarrow$

$$\simeq \Big(\sum_{\mathcal{I}} \|f_\theta\|^{2}_{L^{p/2}_{\#}(Q_0)}\Big)^{p/4}\Big(\sum_{\mathcal{Y}} \|g_\theta\|^{2}_{L^{p/2}_{\#}(Q_0)}\Big)^{p/4}$$

i can reverse Hölder $p/4$!

We are not done yet... Need to pass from $\|\cdot\|_{L^{p/2}} \longrightarrow \|\cdot\|_{L^p}$.

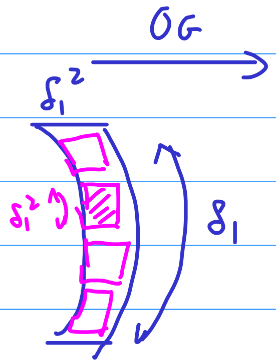
Since $p \geq 4$, we interpolate

$$\|h\|_{p/2} \leq \|h\|_2^{1-\alpha_p}\cdot\|h\|_p^{\alpha_p}.$$

$$\Big(\sum_{\theta\sim\delta_1} \|f_\theta\|^2_{L^{p/2}_{\#}(Q_0)}\Big)^{1/2} \lesssim \Big(\sum_{\theta} \|f_\theta\|^{2(1-\alpha)}_{L^2_{\#}(Q_0)}\cdot\|f_\theta\|^{2\alpha}_{L^p_{\#}(Q_0)}\Big)^{\frac{1}{2}}$$

$$\text{Hölder} \leq \Big(\sum_{\theta\sim\delta_1} \|f_\theta\|^2_{L^2_{\#}(Q_0)}\Big)^{\frac{1-\alpha}{2}}\cdot\Big(\sum_{\theta\sim\delta_1} \|f_\theta\|^2_{L^p_{\#}(Q_0)}\Big)^{\frac{\alpha}{2}}$$

The last term is our target, but we have paid with a new term with L^2-norms. We also want to scale δ_1^2...



$$\Big(\sum_{\theta\sim\delta_1} \|f_\theta\|^2_{L^2_\#(Q_0)}\Big)^{1/2} \approx \Big(\sum_{\theta'\sim\delta_1^2} \|f_{\theta'}\|^2_{L^2_\#(Q_0)}\Big)^{1/2}$$

$Q_0 = (1/\delta_1)^2$- cube

$\Rightarrow$ incompatible with OG!

target arcs!!

- Finally, since $|f_{\theta'}| \approx$ const in $Q_0 = (1/\delta_1)^2$- cube

$\operatorname{supp} \widehat{f_{\theta'}} \subseteq \boxed{\ }\ \delta_1^2 \times \delta_1^2$

$$\Rightarrow \Big(\sum_{\theta\sim\delta_1} \|f_\theta\|^2_{L^2_\#(Q_0)}\Big)^{\frac{1-u}{2}} \Big(\sum_{\theta\sim\delta_1} \|g_\theta\|^2_{L^2_\#(Q_0)}\Big)^{\frac{1-u}{2}}$$

$$\approx \Big[\Big(\sum_{\theta'\sim\delta_1^2} \|f_{\theta'}\|^2_{L^2_\#(Q_0)}\Big)^{\frac12} \Big(\sum_{\theta'\sim\delta_1^2} \|g_{\theta'}\|^2_{L^2_\#(Q_0)}\Big)^{\frac12}\Big]^{1-u}$$

$|f_{\theta'}| \approx$ const in Q_0

$$\approx \Big[\fint_{Q_0} \Big|\Big(\sum_{\theta'\sim\delta_1^2} |f_{\theta'}|^2\Big)^{1/2} \cdot \Big(\sum_{\theta'\sim\delta_1^2} |g_{\theta'}|^2\Big)^{1/2}\Big|^{p/2} dx\Big]^{\frac{1-u}{p/2}}$$

$$= A_p(f, g, \delta_1^2, Q_0)^{1-u}.$$

This proves the Key lemma ∎

Notes:

① This notes are informal, and many technical steps have been omitted (like Schwartz tails, etc...).

↳ See Demeter's book for more detailed proof.