A survey on Bergman projections in tubes over cones

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1. Project started by Békollé, Bonami in 1995...
2. ... further developed in various papers by Békollé, Bonami, Peloso, Ricci, GG, Debertol, Gonessa, Nana, Sehba,....
3. ... settled (for light-cones) using the decoupling ineq of Bourgain-Demeter

Goal: survey these results + some open questions

Domains $D \subset \mathbb{C}^n$ of interest:

$$D = T_\Omega = \mathbb{R}^n + i\Omega\,, \quad \text{where } \Omega = \text{symmetric cone in } \mathbb{R}^n.$$

If $n = 1$ then $\Omega = (0, \infty)$, and $D =$ upper half-plane in $\mathbb{C}$

Some motivations

1. Together with Siegel domains

$$D(\Omega, \Phi) = \big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m \;:\; \operatorname{Im} z \in \Omega + \Phi(w, w)\big\}$$

   provide unbded realizations of **all** *BSD* in $\mathbb{C}^N$ (via Cayley transf)

2. D has "global" geometric structure

$$D = \big\{g \cdot (i\mathbf{e}) \;:\; g \in G(D)\big\}$$

   $\longrightarrow$ explicit Bergman kernels

3. Bergman projections related with interesting operators in HA (Bochner-Riesz type multipliers)
4. Further extension occurs if $\Omega =$ homogeneous cone...

Def: $\Omega \subset \mathbb{R}^n$ is a symmetric cone if

open convex + homogeneous + self-dual

1. Light-cone (rank $r = 2$)

$$\Lambda_n = \Big\{ y \in \mathbb{R}^n \;:\; \Delta(y) = y_1^2 - |y'|^2 > 0,\; y_1 > 0 \Big\}$$

2. Cones of matrices: if $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (or $\mathbb{O}$ if $r \leq 3$)

$$\mathrm{Her}_+(r, \mathbb{K}) = \Big\{ y \;:\; \Delta_j(y) > 0,\; j = 1, \ldots, r \Big\}$$

Some properties of Ω

- Ω identifies with a group $H = NA$ $\longrightarrow$ lower triang matrices
- Explicit characters of H $\longrightarrow$ $\Delta^{\boldsymbol{\nu}} = \prod_{j=1}^r (\Delta_j / \Delta_{j-1})^{\nu_j}$, $\boldsymbol{\nu} \in \mathbb{R}^r$
- Gamma integral in Ω: for all $y \in \Omega$

$$\int_\Omega e^{-\langle y | \xi \rangle} \, \Delta^{\boldsymbol{\nu}}(\xi) \, \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)} = \Gamma(\boldsymbol{\nu}) \Delta_*^{-\boldsymbol{\nu}}(y), \quad \text{if } \boldsymbol{\nu} > \mathbf{g}_0$$

Consider, for each weight $d_\nu z = \Delta^{\nu - \frac{n}{r}}(y)\,dx\,dy$, $\nu > \mathbf{g}_0$

- Bergman space:

$$A^p_\nu = \Big\{ f \in \mathcal{H}(T_\Omega) \;:\; \|f\|_{L^p(T_\Omega; d_\nu z)} < \infty \Big\}$$

- Bergman projection $P_\nu f(z) = \iint_{T_\Omega} B_\nu(z,w)\,f(w)\,d_\nu w$
- Bergman kernel (explicit!)

$$B_\nu(z,w) = \int_\Omega \Delta^\nu_*(\xi)\,e^{i\langle z - \bar{w} | \xi\rangle}\,d\xi \;=\; \frac{c_\nu}{\Delta(z-\bar{w})^{\nu + \frac{n}{r}}}$$

Question:

When is $P_\nu : L^{p,q}_\nu(T_\Omega) \to L^{p,q}_\nu(T_\Omega)$, for $p \neq 2$?

- More generally, for *mixed normed spaces* $L^{p,q}_\nu$

$$\|f\|_{L^{p,q}_\nu} = \Big[\int_\Omega \|f(\cdot + iy)\|^q_{L^p(\mathbb{R}^n)}\,d_\nu y \Big]^{\frac{1}{q}}$$

- **Key fact:** expect **proper** range since $B_\nu(z,\cdot) \notin L^{p'}_\nu(T_\Omega)$ if $p \geq \tilde{p}_\nu$...

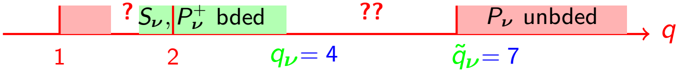
A first (naive) approach

$$\begin{aligned} \big\|P_\nu^+ f(\cdot+iy)\big\|_{L^p(dx)} &\leq \int_\Omega\int_{\mathbb{R}^n} \big\|f(\cdot+iv)\big\|_{L^p(dx)} |B_\nu(iy,u+iv)|\,du\,d_\nu v \\ &= c_\nu \int_\Omega \frac{g(v)}{\Delta^\nu(y+v)}\,d_\nu v \;=:\; S_\nu g(y) \end{aligned}$$

This can be reversed, so that P_ν^+ bded in $L^{p,q}_\nu(T_\Omega)$ iff

$$S_\nu : L^q_\nu(\Omega) \to L^q_\nu(\Omega) \qquad (\star)$$

But... $(\star)$ can only hold in a proper range $q'_\nu < q < q_\nu$...



- first proved by [BB'95] in Λ_n
- **Note:** in $(\star)$, optimal q_ν only known if $r=2$, or if $\boldsymbol{\nu}=(\nu,\ldots,\nu)$
- **Q1 ([GNa'20]):** charact bdedness S_ν in $L^q_\nu(\Omega)$ if $\boldsymbol{\nu}\in\mathbb{R}^r$ (and $r\geq 3$)

Geometric approach

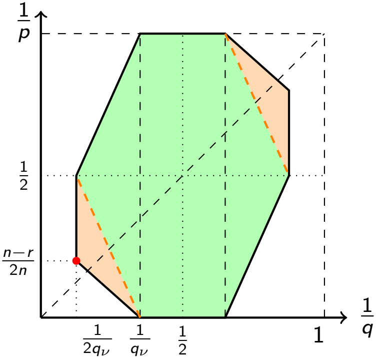
Assume $\boldsymbol{\nu} = (\nu, \dots, \nu)$. In BBPR98, BBG00, BBGR04, a new approach gives

$$P_\nu \text{ bounded in } L^{2,q}_\nu(T_\Omega) \text{ iff } \quad (2q_\nu)' < q < 2q_\nu \; = 8$$

By interpolation this gives...

- New key tools are
  1. Mixed normed spaces $L^{p,q}_\nu$
  2. Spectral decomposition of Ω
  3. New counterexamples
- Ideas extend to homog cones Ω

  $\longrightarrow$ BeNa'07, NaTr'11...



The spectral decomposition of Ω

From $G(\Omega) = \{g \in \mathrm{Gl}(n, \mathbb{R}) \ : \ g(\Omega) = \Omega\}$ one constructs

1. $d(\xi, \xi')$, a G-invariant distance in Ω
2. $\{\xi_\lambda\}$, a 1-separated set in Ω
3. $\{B_\lambda = B_1(\xi_\lambda)\}$, a covering of Ω with FIP
4. $\widehat{\psi}_\lambda \in C_c^\infty(B_\lambda)$ such that $\sum_\lambda \widehat{\psi}_\lambda = \mathbf{1}_\Omega$

- A key feature is that, when $\xi \in B_\lambda$,

$$\Delta_j(\xi) \approx \Delta_j(\xi_\lambda), \qquad j = 1, \dots, r$$

and also

$$\frac{(y|\xi)}{(y|\xi_\lambda)} \approx 1, \quad \text{unif in } y \in \Omega.$$

Sketch of the proof

$$\begin{aligned} P_\nu f(x+iy) &= \int_\Omega \int_{\mathbb{R}^n} B_\nu(x-u+i(y+v))\, f(u+iv)\, du\, d_\nu v \\ &= \mathcal{F}^{-1}\Big[\Delta_+^\nu(\xi)\, e^{-(y|\xi)} \int_\Omega e^{-(v|\xi)} f(\hat{\xi}, v) d_\nu v\Big](x) \\ &= \mathcal{F}^{-1}\Big[\Delta_+^\nu(\xi)\, e^{-(y|\xi)}\ \widehat{G}(\xi)\Big](x) \end{aligned}$$

▸ kernel

- Bdedness of P_ν in $L^{p,q}_\nu$ then follows from 3 steps

$$\big\|P_\nu f(\cdot + iy)\big\|_{L^p(dx)} \lesssim \Big(\sum_\lambda \big[\Delta(\xi_\lambda)^\nu\, e^{-(y|\xi_\lambda)}\, \|G * \psi_\lambda\|_p\big]^s\Big)^{\frac{1}{s}} \qquad (1)$$

$\rightarrow$ consequence of spectral decomposition, since

$$\|\sum_\lambda g * \psi_\lambda\|_p \lesssim \Big[\sum_\lambda \|g * \psi_\lambda\|_p^s\Big]^{\frac{1}{s}}, \quad \text{if } s = p \wedge p' \qquad (*)$$

Given

$$\big\|P_\nu f(\cdot+iy)\big\|_{L^p(dx)} \lesssim \Big(\sum_\lambda \big[\Delta(\xi_\lambda)^\nu\, e^{-(y|\xi_\lambda)}\, \|G*\psi_\lambda\|_p\big]^s\Big)^{\frac1s}, \quad \forall\, y\in\Omega \qquad (1)$$

- Next, step 2 consists in showing

$$\int_\Omega |\mathrm{RHS}|^q\, d_\nu y \;=\; \int_\Omega \Big[\sum_\lambda \Delta(\xi_\lambda)^{\nu s}\, e^{-\gamma(y|\xi_\lambda)}\, \|G*\psi_\lambda\|_p^s\Big]^{\frac qs} d_\nu y$$

$$(**) \;\lesssim\; \sum_\lambda \Delta(\xi_\lambda)^{\nu q}\, \|G*\psi_\lambda\|_p^q \int_\Omega e^{-\gamma'(y|\xi_\lambda)}\, d_\nu y\; \Delta^{-\nu}(\xi_\lambda) \qquad (2)$$

- $(**)$ requires a Schur-test argument, ie Hölder ineq using as test functions

$$\prod_{j=1}^r \Delta_j^{\alpha_j}(\xi), \quad \text{for suitable } (\alpha_1,\ldots,\alpha_r)$$

but... this produces a restriction $2\le q<Q_\nu(s)$...

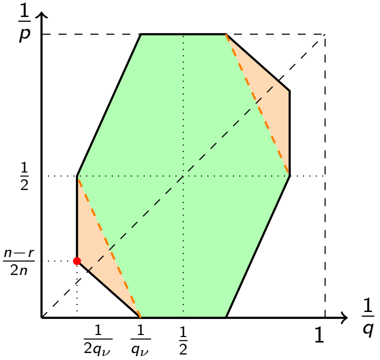
which for $s=p\wedge p'$ matches the green hexagon! ▸ fig

- Step 3: one can "dualize" the argument (2) to obtain

$$\mathrm{RHS}_{(2)} \lesssim \ldots \lesssim \|f\|^q_{L^{p,q}_\nu} \qquad (3)$$

- **Key observation:** $\|\sum_\lambda g * \psi_\lambda\|_p \lesssim \left[\sum_\lambda \|g * \psi_\lambda\|_p^s\right]^{\frac{1}{s}}$ with $s = 2$, would imply (2) in the conjectured range

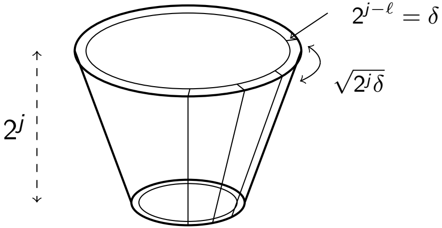
$$2 \leq q < Q_\nu(2) = 2q_\nu \quad !!$$



- **Q:** Is (*) true in the range $2 \leq p \leq \frac{2n}{n-r}$?
- Need more about the geometry of the balls B_λ...

The case of light-cones $\Omega = \Lambda_n$

In Λ_n the invariant balls B_λ take the explicit form



Key fact: Suppose $j = 0$, $\hat{g}$ supported in δ-slice S_δ, and

$$\Big\| \sum_{\lambda \in S_\delta} g * \psi_\lambda \Big\|_p \lesssim \delta^{-\varepsilon} \Big(\sum_\lambda \| g * \psi_\lambda \|_p^s \Big)^{\frac{1}{s}}, \quad 2 \le p \le \tfrac{2n}{n-2} \qquad (*)$$

then, one still can prove

$$\big\| P_\nu f(\cdot + iy) \big\|_p \lesssim \Big(\sum_\lambda \big[\Delta(\xi_\lambda)^{\nu-\varepsilon} e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|_p \big]^s \Big)^{\frac{1}{s}}, \quad \forall\, y \in \Lambda_n \qquad (1)$$

- These implications are discussed in BBGR'04; the partial results at that time were based on Wolff's ineq from 2000 (and small improvements in G-Seeger'2009).
- In 2015 Bourgain and Demeter showed that: for $2 \leq p \leq \frac{2n}{n-2}$ and all $\varepsilon > 0$

$$\| \sum_{\lambda \in S_\delta} g * \psi_\lambda \|_p \lesssim \delta^{-\varepsilon} \Big(\sum_\lambda \|g * \psi_\lambda\|_p^2 \Big)^{\frac{1}{2}} \qquad (*)$$

- So, as a consequence of $(*)$ and the previous reasoning one concludes

Theorem (BBGR04 + BouDem'15)

P_ν is bded in $L^{p,q}_\nu(T_{\Lambda_n})$ in the full conjectured region

Bergman projection $P_{\boldsymbol{\nu}}$ with $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$

Let $\boldsymbol{\nu} > \mathbf{g}_0 := (0, \frac{d}{2}, \dots, (r-1)\frac{d}{2})$, $\quad d = \dim \mathbb{K}$

when is $P_{\boldsymbol{\nu}}$ bded in $L^{p,q}_{\boldsymbol{\nu}}(T_\Omega)$?

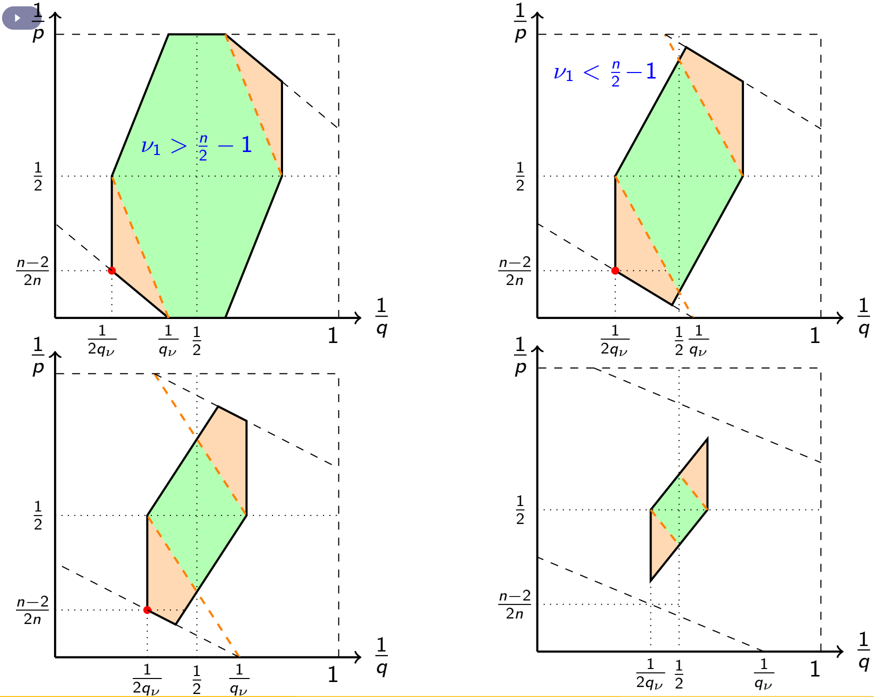
- Considered by D. Debertol (2005)

$$\begin{aligned} \Delta^{\boldsymbol{\nu}}(y) &= \Delta_1^{\nu_1-\nu_2}(y)\Delta_2^{\nu_2-\nu_3}(y)\cdots\Delta(y)^{\nu_r} \\ &= (y_1-y_2)^{\nu_1-\nu_2}\,(y_1^2-|y'|^2)^{\nu_2} \end{aligned}$$

- For light-cones Λ_n, the condition $\boldsymbol{\nu} > \mathbf{g}_0$ becomes

$$\nu_1 > 0 \quad \text{and} \quad \nu_2 > \tfrac{n}{2} - 1$$

- ...new difficulties appear when $\nu_1 \in (0, n/2-1)$!!


$\nu_1 > \frac{n}{2} - 1$
$\nu_1 < \frac{n}{2} - 1$
$\frac{1}{p}$
$\frac{1}{q}$
$\frac{1}{2}$
$\frac{n-2}{2n}$
$\frac{1}{2q_\nu}$
$\frac{1}{q_\nu}$
1

- Debertol results (in green) are based in the simpler inequality

$$\|\sum_\lambda g * \psi_\lambda\|_p \lesssim \Big[\sum_\lambda \|g * \psi_\lambda\|_p^s\Big]^{\frac{1}{s}}, \quad \text{with } s = p \wedge p' \qquad (*)$$

- **Q1:** Can do better in light-cones with BD-decoupling inequality?
- When $r \geq 3$ and $\boldsymbol{\nu} \neq (\nu, \ldots, \nu)$, a new gap appears for

$$P_{\boldsymbol{\nu}} : L^{2,q}_{\boldsymbol{\nu}}(T_\Omega) \to L^{2,q}_{\boldsymbol{\nu}}(T_\Omega)$$

- **Q2:** Find sharp range of bdedness for $P_{\boldsymbol{\nu}}$ in $L^{2,q}_{\boldsymbol{\nu}}$.
- possibly related with similar problem for Hilbert operator $S_{\boldsymbol{\nu}}$ in $L^q_{\boldsymbol{\nu}}(\Omega)$...

The most recent results are due to Békollé, Gonessa, Nana from 2019

THEOREM (BGN'2019)

P_ν bded in $L^{p,q}_\nu(T_{\Lambda_n})$ in the full conjectured region, for all $\nu > \mathbf{g}_0$

- $2 \leq p \leq \frac{2n}{n-2} \rightarrow$ use BD-decoupling + previous approach
- $p > \frac{2n}{n-2}$: new situation arises if $\nu_1 \in (0, \frac{n}{2} - 1) \rightarrow$ cannot use interpolation! figures
- one posibility could be the (sharp) decoupling estimate

$$\| \sum_{\lambda \in S_\delta} g * \psi_\lambda \|_p \lesssim \delta^{-(\frac{n-2}{4} - \frac{n}{2p} + \varepsilon)} \Big(\sum_\lambda \| g * \psi_\lambda \|_p^2 \Big)^{\frac{1}{2}}, \quad p > \tfrac{2n}{n-2}$$

  but... it does not give optimal results for P_ν!
- the right substitute is the inequality

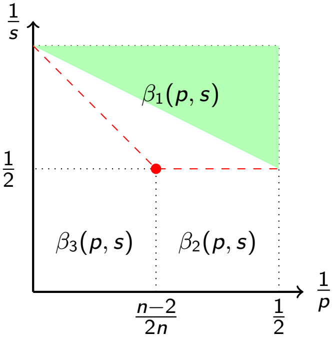
$$\| \sum_{\lambda \in S_\delta} g * \psi_\lambda \|_p \lesssim \delta^{-\varepsilon} \Big(\sum_\lambda \| g * \psi_\lambda \|_p^s \Big)^{\frac{1}{s}}, \quad p > \tfrac{2n}{n-2}$$

  with the **optimal** $s = p/(p - \frac{n}{n-2})$!!

General decoupling-type inequalities:

$$\|\sum_{\lambda\in S_\delta} g*\psi_\lambda\|_p \lesssim \delta^{-(\beta(p,s)+\varepsilon)}\Big(\sum_\lambda \|g*\psi_\lambda\|_p^s\Big)^{\frac{1}{s}}, \quad p\geq 2$$

with $\beta(p,s)$ given in the figure below

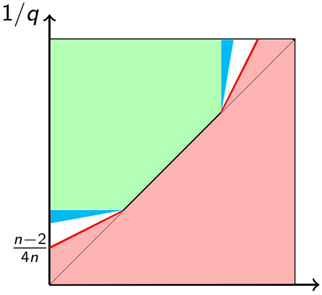


- $\beta_1(p,s) = 0$
- $\beta_2(p,s) = (\frac{1}{2}-\frac{1}{s})\frac{n-2}{2}$
- $\beta_3(p,s) = \frac{n-2}{2s'} - \frac{n}{2p}$

The dashed line gives the optimal estimate in the Bergman projection problem

Open questions:

1. if $r \geq 3$, adapt decoupling ideas to prove bdedness of $P_{\boldsymbol{\nu}}$ in $L^{p,q}_{\boldsymbol{\nu}}$ **beyond** the green hexagon
2. if $r \geq 3$ and $\boldsymbol{\nu} \neq (\nu, \ldots, \nu)$ characterize bdedness of
$$S_{\boldsymbol{\nu}} : L^q_{\boldsymbol{\nu}}(\Omega) \to L^q_{\boldsymbol{\nu}}(\Omega) \quad \text{and} \quad P_{\boldsymbol{\nu}} : L^{2,q}_{\boldsymbol{\nu}}(T_\Omega) \to L^{2,q}_{\boldsymbol{\nu}}(T_\Omega)$$
3. **Local inequalities:** if $D =$ bded symm domain equivalent to T_Ω then bdedness makes sense for
$$P_D : L^p(D) \to L^q(D) \quad \text{when } p > q$$
trivial range from $p = q$ + Hölder, but ... larger range expected [BB95]



- Present results (in blue) are from [BoGaNa14]
- equivalent to "local" bdedness of $P_{\boldsymbol{\nu}}$ in T_Ω

$$\Big\| \chi_{\mathbb{B}} P_{\boldsymbol{\nu}}(f \chi_{\mathbb{B}}) \Big\|_{L^q_{\boldsymbol{\nu}}(T_\Omega)} \leq \|f\|_{L^p_{\boldsymbol{\nu}}(T_\Omega)}$$

Thanks for listening!