Padova, 21-6-2021

Bergman projections and decoupling inequalities

- Thanks to organizers / participants

- <u>Topic</u> : Bergman projections in tubes over cones

↳ · project started by Békollé - Bonami ~ 95

· further developed in various papers by

Bekolle, Bonami, Peloso, Ricci, GG, Debertol, Gonessa, Nana, Sehba, ...

· So far, has only been settled in <u>light-cones</u>, after the proof of the <u>sharp decoupling inequality</u> by Bourgain-Demeter '2015

Attempted program

<u>L1</u> : Bergman spaces, Bergman projections, cones

<u>L2</u> : Bdedness of $\mathbb{P}$ in $L^{2,q}$ spaces

<u>L3</u> : Bddness of $\mathbb{P}$ in $L^{p,q}$ spaces

<u>L4</u> : Comments on decoupling inequalities

Remarks

- This mini-course is a more detailed elaboration of some earlier talks → Cortona '18 and Indam '21.

- Not possible to be completely self-contained, so references will occasionally be given

  → Cones : Faraut-Koranyi

  → Bergman projection : lecture notes from 2004 (BBGNPR)

  → Decoupling ineq. : book of Demeter (chp 9, 10).

I.- Bergman spaces and projections

1.1.- Definitions

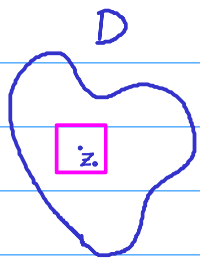
Let $D \subseteq \mathbb{C}^n$ open set, $d\mu(z) = w(z)\,dz$ positive meas on D

cont's

Def 1 For $1 \le p < \infty$, the Bergman space $A^p_\mu(D)$ consists of all $f \in \mathcal{H}(D)$:

$$\|f\|_{A^p_\mu} = \left(\int_D |f(z)|^p \, d\mu(z) \right)^{1/p} < \infty .$$

Lemma 1 $\boxed{\text{If } f \in A^p_\mu(D) \text{ and } z_0 \in D \text{ then } |f(z_0)| \le C_{z_0} \cdot \|f\|_{A^p_\mu}}$



P/ Use MV Thm for holom functions

$$|f(z_0)| = \left| \fint_{D_\varepsilon(z_0)} f(\zeta)\,d\zeta \right| \overset{\text{Hölder}}{\le} \left(\fint_{D_\varepsilon(z_0)} |f(\zeta)|^p \, d\zeta \right)^{1/p}$$

$$\le \left(\int_{D_\varepsilon(z_0)} |f(\zeta)|^p w(\zeta)\,d\zeta \right)^{\frac{1}{p}} \frac{1}{|D_\varepsilon(z_0)| \cdot \min\limits_{D_\varepsilon(z_0)} w(\zeta)}$$

$$\le \|f\|_{A^p_\mu} \cdot C_{z_0} .$$

∎

Note : C_{z_0} depends continuous in z_0 (so it is uniform if $z_0 \in K \subset\subset D$)

As a consequence

Corollary : $\boxed{A^p_\mu \text{ is a closed subspace of } L^p_\mu(D)}$ (hence a B. space)

When $p=2 \Rightarrow A^2_\mu$ = Hilbert space $\hookrightarrow L^2_\mu(D)$

Def 2 The <u>Bergman projection</u> (associated with (D,μ)) is the orthogonal projection

$$\underline{P} = P^D_\mu : L^2_\mu(D) \longrightarrow A^2_\mu .$$

It is given by a "Kernel"

$$P^D_\mu f(z) = \int_D B_\mu(z,w) f(w)\, d\mu(w), \qquad z \in D$$

which satisfies

- $B_\mu(z,\cdot) \in L^2_\mu, \qquad \forall z \in D$
- $\overline{B_\mu(z,w)} = \overline{B_\mu(w,z)}$
- $B_\mu(z,w)$ is holomorphic in z and antiholom in w

$\Rightarrow B_\mu(z,w)$ is called <u>Bergman Kernel associated with (D,μ)</u>

In particular

$$\left. \begin{array}{l} f(z) = \int_D B_\mu(z,w) f(w)\, d\mu(w) \\ \qquad \forall f \in A^2_\mu \end{array} \right\} \Rightarrow B_\mu = \text{reproducing Kernel for } A^2_\mu .$$

<u>Exercise 1</u> Use Lemma 1 + Riesz Thm to prove the previous assertions

$\longrightarrow$ can check [FK, Ch 9.2] or [Krantz, SCV, Ch 1.4]

Question: When does P_μ extend as bdd operator

$$P_\mu : L^p_\mu(D) \longrightarrow L^p_\mu(D) \; ?$$

- Here "extend" means from the dense set $L^p_\mu \cap L^2_\mu$ to all L^p_μ.
- Q is equivalent to $P_\mu : L^p_\mu \longrightarrow A^p_\mu$ (check)
- If Q holds (and $A^2_\mu \cap A^p_\mu$ dense in A^p_μ) $\Longrightarrow$ $P_\mu^2 = P_\mu$ in L^p_μ.

Notes

- Q is an important question in SCV, Operator theory and H.A., which gives info about:

  ① Geometric properties of D

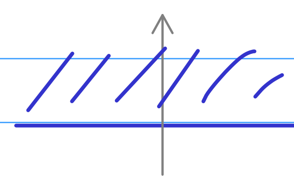
  ② Structure of $A^p_\mu(D)$ $\longrightarrow$ duality, atomic decomp, interpolation,...

  ③ Regularity of $\bar\partial$-problem $\longrightarrow$ $\bar\partial u = f$

  ④ Interesting "model" operator in H.A.

Examples

① D = upper half-plane $= \mathbb{R} + i(0,\infty) = \mathbb{C}_+ \subseteq \mathbb{C}$



Let $d\mu(z) = y^\gamma \, dx \frac{dy}{y}$, $\gamma > 0$.

Then one has:

$$(*) \qquad B_\gamma(z,w) = \frac{c_\gamma}{(z-\bar{w})^{1+\gamma}} = \int_0^\infty e^{i(z-\bar{w})\cdot\xi} \cdot \xi^\gamma \, d\xi$$

and one can show:

Propos: $\boxed{P_\gamma : L^p_\gamma(\mathbb{C}_+) \longrightarrow A^p_\gamma(\mathbb{C}_+), \text{ for all } \begin{cases} 1<p<\infty \\ \gamma>0 \end{cases}.}$

__Note__ Bddness fails at $p=1$. E.g. if $d\mu(z) = dz$ $(\gamma=1)$

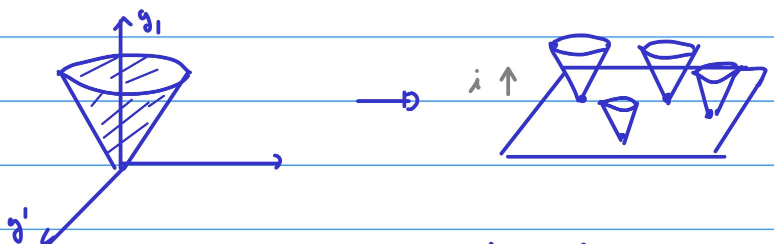
$$\Rightarrow \quad P\left(1_{D_\varepsilon(i)}\right)(z) = \int_{D_\varepsilon(i)} B(z,w)\,dw = B(z,i) = \frac{c}{(z+i)^2} \notin L^1(\mathbb{C}_+).$$

__Example 2__ Tube domain over light-cones

$D = \mathbb{R}^n + i\,\Omega$ where

Lorentz form

$$\Omega = \Lambda_n = \Big\{ (y_1, y') \in \mathbb{R}^n \;/\; \Delta(y) = \overbrace{y_1^2 - |y'|^2}^{} > 0, \quad y_1 > 0 \Big\}$$



A natural family of measures is

$$d\mu(z) = \Delta^\gamma(y)\, \frac{dx\,dy}{\Delta(y)^{n/2}}, \qquad \gamma > \frac{n}{2} - 1 \quad \left(\text{so that } A^2_\gamma \neq \{0\}\right)$$

Then one can show

$$(*) \qquad B_\gamma(z,w) = \frac{c_\gamma}{\Delta(z-\bar{w})^{\gamma+\frac{n}{2}}} = \int_\Omega e^{i(z-\bar{w}|\xi)}\, \Delta^\gamma(\xi)\, d\xi.$$

¡explicit Bergman kernel!

But ... this time cannot expect P bdd L^p_γ $\forall\ 1<p<\infty$!!

As before, if say $d\mu(z) = dz$ $(\gamma = n/2)$

$$P\left(1_{D_\varepsilon(ie_1)}\right)(z) = B(z, ie_1) = \frac{1}{\Delta(z+ie_1)^n} \notin L^p(D) \quad \text{if } p \le 1 + \frac{n-2}{2n} \Bigg\} \; !!$$

$e_1 = (1,0,\dots,0)$

Conjecture 1 (BB'95)

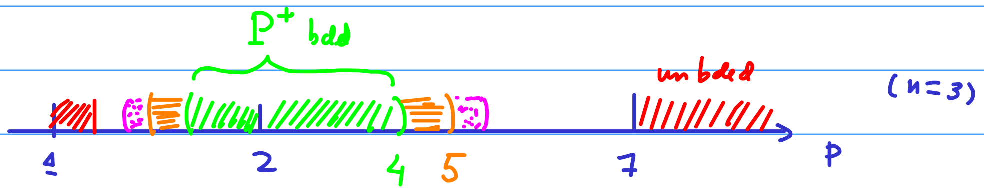
For $D = \mathbb{R}^n + i\Lambda_n$,

$$P : L^p \longrightarrow L^p \text{ bdd} \iff 1 + \frac{n-2}{2n} < p < 1 + \frac{2n}{n-2}$$

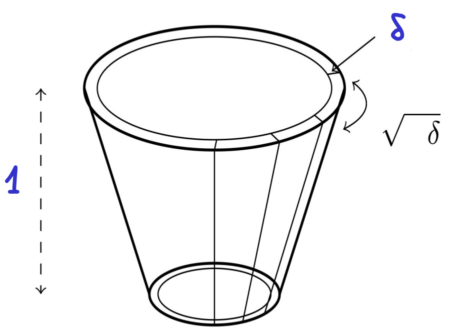
The main goal of this lectures is to show the validity of Conj 1.

The historic progress in this question has been

- BB'95
- BBPR'98
- BBGR'09 + GaSe '07-10
- BoDe'15 (+ BBGR'09) $\longrightarrow$ solution to full conjecture linked ("equivalent") to progress in decoupling inequalities.



Statement of decoupling inequalities



Split a δ-nbd of (truncated) cone into $\sqrt{\delta}$ plates

- $\Gamma_\delta = \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1} \,/\, \begin{array}{l} 1 \le \tau \le 2 \\ |\tau - |\xi|| \le \delta \end{array} \right\}$
- $\{\omega_j\} \subseteq S^{n-2}$ $\sqrt{\delta}$-separated points

$$\Rightarrow \quad \Pi_j^{(\delta)} = \left\{ (\tau, \xi) \in \Gamma_\delta \;\middle|\; \left| \frac{\xi}{|\xi|} - \omega_j \right| \le \sqrt{\delta} \right\}$$

$\hookrightarrow$ these are $1 \times \delta \times (\sqrt{\delta} \times \dots \times \sqrt{\delta})$ "rectangles".

Thm (Bourgain-Demeter, 2015)

For all $2 \le p \le \frac{2n}{n-2}$, and all $\varepsilon > 0$,

$$\left\| \sum f_j \right\| \le_\varepsilon \delta^{-\varepsilon} \left(\sum_j \| f_j \|_p^2 \right)^{1/2}$$

for all δ, and all f_j : $\operatorname{supp} \hat{f}_j \subseteq \Pi_j^{(\delta)}$.

Notes:
- $p = 2 \to$ trivial with $\varepsilon = 0$ by OG.
- $2 < p \le \frac{2n}{n-2} \longrightarrow$ proved in sharp form by [BD, Annals Math' 2015].

[earlier partial results by Wolff'00, Łaba'02, GaSe 07-10]

1.2.- Symmetric cones

We need some definitions, examples and properties.

$\longrightarrow$ a more complete theory in the book Faraut-Korányi.

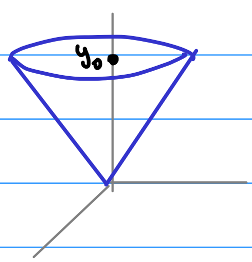
Def: Ω is a symmetric cone in $(V, \langle\cdot|\cdot\rangle) \equiv \mathbb{R}^n$ if

n-dim vector sp.

(i) Ω cone : $\left.\begin{array}{l} x \in \Omega \\ \lambda > 0 \end{array}\right\} \Longrightarrow \lambda x \in \Omega$

(ii) Ω open convex + homogeneous + self-dual

here



- homogeneous : the group $G(\Omega) = \{ g \in Gl(V) \;/\; g(\Omega) = \Omega \}$ acts transitively in Ω :

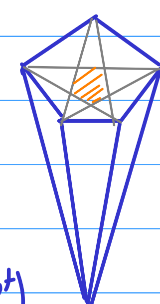
if $y_0, y_1 \in \Omega \Longrightarrow \exists\, g \in G(\Omega) \;/\; y_0 = g(y_1)$.

- self-dual : Ω coincides with its dual cone

$$\Omega^* = \{ \xi \in V^* \;/\; (\xi | y) > 0 \;,\; \forall y \in \bar{\Omega} \setminus \{0\} \}$$

Exercise: A pentagonal cone is not homogeneous.

Hint: linear maps send inner pentagonal cone to itself.



Note: We assume all cones are "proper" (no full straight lines)

Examples:

① $n = 1 \longrightarrow \Omega = (0, \infty)$ only 1 symmetric cone in $\mathbb{R}$.

② $\Omega = \Lambda_n$ = light-cone in $\mathbb{R}^n \longrightarrow$ easy to check that it's symmetric

here $G(\Omega) = \mathbb{R}_+ \cdot O_+(1, n-1)$

$\longleftarrow$ $g \in Gl(n,\mathbb{R}) \;/\; \Delta(gx) = \Delta(x)$ $(g_{11} > 0)$

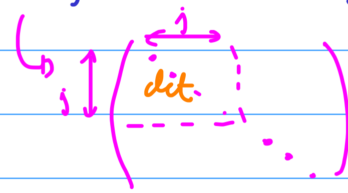
③ Cones of posit definite symmetric $r\times r$ matrices

Let $V = \mathrm{Sym}(r, \mathbb{R}) \longrightarrow$ vector space dim $n = \frac{r\cdot(r+1)}{2}$.

Let

$$\Omega = \mathrm{Sym}_+(r, \mathbb{R}) = \{ y \in V \ / \ \text{all eigenvalues } \lambda_j > 0,\ j=1,\dots,r \}$$

$$= \{ y \in V \ / \ \text{all princ minors } \Delta_j(y) > 0,\ j=1,\dots,r \}$$



In this example

$$G(\Omega) \equiv \{ g \in M_{r\times r}(\mathbb{R}) \ / \ \det g \neq 0 \}$$

with the action given by

$$y \in V \longmapsto g\bullet y = g y g^t \in V$$

Homogeneity of Ω follows from diagonalization theorem:

if $y \in \mathrm{Sym}_+(r, \mathbb{R}) \Longrightarrow y = k a^2 k^t = ka \bullet I$, $\quad \begin{cases} k \in SO(r) \\ a = \text{diag.} \end{cases}$

Exercise $\quad \Delta_3 \equiv \mathrm{Sym}_+(2, \mathbb{R})$

Hint: Use

$$\mathbb{R}^3 \longrightarrow \mathrm{Sym}(2, \mathbb{R}) \qquad \text{(isomorphism)}$$

$$(y_1, y_2, y_3) \longmapsto Y = \begin{pmatrix} y_1 - y_2 & y_3 \\ y_3 & y_1 + y_2 \end{pmatrix}$$

and check $\left.\begin{aligned} \det Y &= \Delta(y) \\ \mathrm{Tr}\, Y &= 2y_1 \end{aligned}\right\} \rightarrow \quad y \in \Delta_3 \Leftrightarrow Y$ posit def

④ Cones of posit def hermitian matrices, over $\mathbb{K} = \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$

(only if $r \leq 3$)

$V = \mathrm{Herm}(r, \mathbb{K})$ and $\Omega = \mathrm{Herm}_+(r, \mathbb{K})$

These examples are actually <u>ALL</u> irreducible symmetric cones !!

[see FK, chap V].

We shall mainly focus on Δ_n and $\mathrm{Sym}_+(r, \mathbb{R})$.

rank 2

rank r

Moral: The matrix language gives the correct tools to work with Δ_n.

↳ in general, sym cones → Jordan algebras

homog cones → T-algebras.

Three properties of symmetric cones

1.- Group identification

Prop1: $\exists$ subgroup $H \subseteq G(\Omega)$ acting simply transitively in Ω, ie

$$\begin{aligned} H &\longrightarrow \Omega \qquad \text{homeomorphism.} \\ h &\longmapsto h \cdot e \end{aligned}$$

Moreover, $H = NA$

↳ this will give the right "coordinates" to work in Ω

Examples

① $\Omega = (0, \infty) \longrightarrow H = \mathbb{R}_+$

② $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$

If $y \in \Omega \longrightarrow$ we Gauss (Cholesky) decompos

$$y = h h^t = h \cdot I, \quad \text{where} \quad h = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_r \end{pmatrix}$$

$\Rightarrow H =$ group of lower triang matrices with $t_j > 0$.

This group factors as $H = NA$ with

$$N = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix} \mid a_j > 0 \right\}$$

↳ nilpotent ↳ abelian

As an explicit example, if $r=2$

$$\xi = \begin{pmatrix} \xi_1 & \xi_3 \\ \xi_3 & \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi_3/\xi_1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 - \frac{\xi_3^2}{\xi_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \xi_3/\xi_1 \\ 0 & 1 \end{pmatrix}$$

$$= \Bigg[\underbrace{\begin{pmatrix} 1 & 0 \\ \xi_3/\xi_1 & 1 \end{pmatrix}}_{\in N} \underbrace{\begin{pmatrix} \sqrt{\xi_1} & 0 \\ 0 & \frac{\sqrt{\xi_1 \xi_2 - \xi_3^2}}{\sqrt{\xi_1}} \end{pmatrix}}_{\in A} \Bigg] \cdot I$$

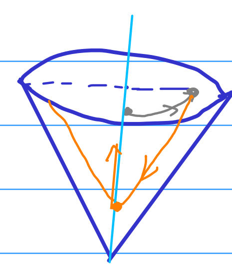
<u>Note</u> the "A-coordinates" of ξ are given by

$$\xi = n \cdot \begin{pmatrix} \Delta_1(\xi) & 0 \\ 0 & \frac{\Delta_2(\xi)}{\Delta_1(\xi)} \end{pmatrix}$$

↳ so expect an important role of pral minors...

③ $\Omega = \Lambda_n =$ light-cone

Recall that $G(\Omega) = \mathbb{R}_+ O_+(1, n-1)$.



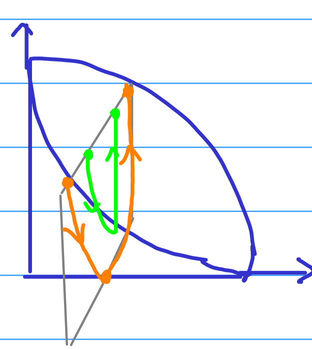
One can factor $G(\Omega) = KAK$ with

$$A = \left\{ \left(\begin{array}{cc|c} r\operatorname{ch} t & r \operatorname{sh} t & 0 \\ r \operatorname{sh} t & r\operatorname{ch} t & 0 \\ \hline 0 & 0 & I \end{array} \right) \;/\; r>0,\ t\in\mathbb{R} \right\} \quad \text{and} \quad K = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k \end{array} \right) \;/\; k \in SO(n-1) \right\}$$

However, it is often more useful to work with $G(\Omega) = NAK$ where

$$N = \left\{ \left(\begin{array}{cc|c} 1+\frac{|v|^2}{2} & \frac{|v|^2}{2} & v^t \\ -\frac{|v|^2}{2} & 1-\frac{|v|^2}{2} & -v^t \\ \hline v & v & I_{n-2} \end{array} \right) \;/\; v \in \mathbb{R}^{n-2} \right\}$$

↳ N-orbits = parabolas



Property 2 Generalized powers in Ω

Def: If Δ_j = j^{th} principal minor, then we define

$$\Delta^{\underline{\alpha}}(\xi) = \prod_{j=1}^{r} \underbrace{\left(\frac{\Delta_j(\xi)}{\Delta_{j-1}(\xi)}\right)}_{j^{th}\text{- diagonal entry in } NA\text{- coordinates}}^{\alpha_j}, \quad \text{if } \underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{C}^r.$$

These functions have 2 important properties

1. Homogeneity: $\Delta^{\underline{\alpha}}(h \cdot \xi) = \Delta^{\underline{\alpha}}(h \cdot e) \cdot \Delta^{\underline{\alpha}}(\xi)$, $\forall h \in H, \xi \in \Omega$

   $\Delta^{\underline{\alpha}}(n \cdot \xi) = \Delta^{\underline{\alpha}}(\xi)$, $\forall n \in N$

2. $\{\Delta^{\underline{\alpha}}\}_{\underline{\alpha} \in \mathbb{C}^r}$ are the "characters" of the group H.

This can be seen in [FK, chpVI] for general sym cones, and verify by hand in our examples 1, 2, 3.

Example ① $\Omega = (0, \infty) \Longrightarrow \Delta^{\alpha}(y) = y^{\alpha}$.

② $\Omega = \Lambda_n \longrightarrow \Delta^{(\alpha_1, \alpha_2)}(y) = (y_1 - y_2)^{\alpha_1 - \alpha_2} \cdot \Delta^{\alpha_2}(y)$

③ $\Omega = Sym_+(r) \longrightarrow \Delta^{(\alpha_1, \dots, \alpha_r)}(y) = \Delta_1^{\alpha_1 - \alpha_2}(y) \cdot \Delta_2^{\alpha_2 - \alpha_3}(y) \cdot \dots \cdot \Delta_r^{\alpha_r}(y)$

Note: If $\underline{\alpha} = (\alpha, \dots, \alpha)$ = scalar power $\longrightarrow \Delta^{\underline{\alpha}}(y) = \Delta_r^{\alpha}(y) = \det(y)^{\alpha}$.

(most frequent power)

Property 3 Gamma integrals in Ω $\longrightarrow$ [FK, ch. VII]

Def If $\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$, we let

$$\Gamma_\Omega(\underline{\alpha}) := \int_\Omega e^{-(\underline{e}|\xi)} \Delta^{\underline{\alpha}}(\xi) \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)}$$

invariant measure

$(d = n-2 \text{ in } \Lambda_n)$

Lemma 1: $\Gamma_\Omega(\underline{\alpha}) = c \cdot \prod_{j=1}^{r} \Gamma\left(\alpha_j - (j-1)\frac{d}{2}\right)$ where $d = \dim \mathbb{K}$

So, $\Gamma_\Omega(\underline{\alpha}) < \infty \iff \underline{\alpha} > g_0 = \left(0, \frac{d}{2}, \dots, (r-1)\frac{d}{2}\right)$

[Key exponent for $\Delta^p_{\underline{\alpha}} \neq \{0\}$]

As an application of P1 + P2 + P3

Lemma 2

$$\int_\Omega e^{-(y|\xi)} \Delta^{\underline{\alpha}}(\xi) \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)} = \Gamma_\Omega(\underline{\alpha}) \cdot \Delta^{\underline{\alpha}}(y^{-1})$$

Note: One can write

$$\Delta_j^* = \left(\boxed{\det}\right)$$

$$\Delta^{\underline{\alpha}}(y^{-1}) = \Delta_*^{-\underline{\alpha}_*}(y) \quad \text{where } \underline{\alpha}_* = (\alpha_r, \dots, \alpha_1) \text{ and } \Delta_* = \text{minors from below}$$

[For example if $y = \mathrm{diag}(t_1, \dots, t_r)$,

$$\Delta^{\underline{\alpha}}(t^{-1}) = \left(\frac{1}{t_1}\right)^{\alpha_1} \cdots \left(\frac{1}{t_r}\right)^{\alpha_r} = \left(\Delta_1^*(t)\right)^{-\alpha_r} \cdots \left(\frac{\Delta_r^*(t)}{\Delta_{r-1}^*(t)}\right)^{-\alpha_1} = \Delta_*^{-\underline{\alpha}_*}(t)$$

]

Application 1 : Hilbert type integrals

Lemma 3

$$\boxed{\int_\Omega \frac{\Delta^{\alpha}(y)}{\Delta^{\alpha+\beta}(y+e)}\, \underbrace{\frac{dy}{\Delta^{\frac{n}{r}}(y)}}_{d\sigma(y)} < \infty \iff \alpha > g_0\,,\quad \beta > g_0^*}$$

Proof :

$$I = \int_\Omega \Delta^{\alpha}(y) \int_\Omega e^{-(y+e|\xi)}\, \Delta_*^{\alpha_*+\beta_*}(\xi)\, d\sigma(\xi)\, d\sigma(y)$$

$$\overset{\text{Fub}}{=} \int_\Omega e^{-(e|\xi)}\, \Delta_*^{\alpha_*+\beta_*}(\xi)\, \underbrace{\int_\Omega e^{-(y|\xi)}\, \Delta^{\alpha}(y)\, d\sigma(y)}_{\overset{(L2)}{=}\ c_1 \cdot \Delta_*^{-\alpha_*}(\xi) \quad \text{iff } \alpha > g_0}\, d\sigma(\xi)$$

$$= c_1 \int_\Omega e^{-(e|\xi)}\, \Delta_*^{\beta_*}(\xi)\, d\sigma(\xi) = c_1 \cdot c_2 \quad \text{iff } \beta_* > g_0$$

∎

Application 2 Paley-Wiener for A^2_γ and explicit $B_\gamma(z,w)$

Consider $A^p_\mu(D)$ with $D = T_\Omega = \mathbb{R}^n + i\Omega$

and

$$d\mu(z) = \Delta^{\gamma}(y)\, dx\, d\sigma(y) \qquad \left[d\sigma(y) = dy / \Delta^{\frac{n}{r}}(y) \right]$$

For simplicity we write $A^p_\mu(T_\Omega) = A^p_\gamma$

Theorem Paley-Wiener for A^2_γ

$$F \in A^2_\gamma \iff \begin{cases} F(z) = \int_\Omega e^{i(z|\xi)} f(\xi)\, d\xi, & z \in T_\Omega \\ \text{for some } f \in L^2(\Omega, \Delta^{-\gamma}(\xi)\, d\xi) \end{cases}$$

Moreover

$$(*) \qquad \|F\|_{A^2_\gamma} = c \cdot \left[\int_\Omega |f(\xi)|^2 \frac{d\xi}{\Delta^{\gamma}(\xi)} \right]^{1/2}$$

Proof: See [FK, chap IX.3].

more details?

We only prove the formula (**), assuming the integral representation

$$F(z) = \int_\Omega e^{i(z|\xi)} f(\xi)\, d\xi \quad , \quad z \in T_\Omega.$$

By Plancherel, if $z = x+iy$,

$$\| F(\cdot + iy) \|_{L^2(\mathbb{R}^n)} = \left[\int_\Omega \left| e^{-(y|\xi)} f(\xi) \right|^2 d\xi \right]^{1/2}$$

So

$$\| F \|^2_{A^2_\gamma} = \int_\Omega \int_\Omega e^{-2(y|\xi)} |f(\xi)|^2 d\xi \; \Delta^\gamma(y)\, d\sigma(y)$$

$$\stackrel{\text{Fub}}{=} \int_\Omega |f(\xi)|^2 \underbrace{\int_\Omega e^{-2(y|\xi)} \Delta^\gamma(y)\, d\sigma(y)}_{c_\gamma \cdot \Delta^{-\gamma}(2\xi)}\, d\xi \quad \leftarrow \gamma > g_0$$

$$= c_\gamma' \cdot \| f \|^2_{L^2(d\xi / \Delta^\gamma(\xi))}$$

Exercise: Adapt previous statement and proof to vector powers $\Delta^{\underline{\gamma}}(y)$, $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$.

Corollary 1

$$\boxed{B_\gamma(z,\omega) = c_\gamma \int_\Omega e^{i(z-\bar\omega|\xi)} \Delta^\gamma(\xi)\, d\xi = \frac{c_\gamma'}{\Delta(z-\bar\omega)^{\gamma+\frac{n}{r}}}}$$

P/ Fix $\omega \in T_\Omega$. By Paley-Wiener $\exists\, b_\omega \in L^2(\Omega, d\xi/\Delta^\gamma(\xi))$ st

$$B_\gamma(z,\omega) = \int_\Omega e^{i(z|\xi)} b_\omega(\xi)\, d\xi \qquad (\text{since } B_\gamma(\cdot,\omega) \in A^2_\gamma)$$

Since B_γ is a Repr Kernel, $\forall F \in A^2_\gamma$ we have

$$F(\omega) = \langle F, B_\gamma(\cdot,\omega)\rangle \overset{PW}{=} c_\gamma \int_\Omega f(\xi)\,\overline{b_\omega(\xi)}\, \frac{d\xi}{\Delta^\gamma(\xi)}$$

and
$$F(\omega) = \int_\Omega e^{i(\omega|\xi)} f(\xi)\,d\xi$$

$$\xrightarrow{\forall f} \quad b_\omega(\xi) = \Delta^\gamma(\xi)\, e^{-i(\bar\omega|\xi)}$$

Thus

$$\Rightarrow B(z,\omega) = \int_\Omega e^{i(z|\xi)} e^{-i(\bar\omega|\xi)} \Delta^\gamma(\xi) \overset{LZ}{=} \frac{c_\gamma}{\Delta(z-\bar\omega)^{\gamma+\frac{n}{r}}}. \quad \blacksquare$$

Corollary 2

$$\boxed{B_\gamma(z, ie) \in L^p_\gamma(T_\Omega) \iff p > 1 + \frac{\frac{n}{r}-1}{\gamma+\frac{n}{r}}}$$

P/
$$\int_{\mathbb{R}^n} |B_\gamma(z, ie)|^p\,dx = c\int_{\mathbb{R}^n} \frac{dx}{|\Delta(z+ie)|^{(\gamma+\frac{n}{r})p}} = c\cdot \left\| \frac{1}{\Delta(x+i(y+e))^{\underbrace{(\gamma+\frac{n}{r})\frac{p}{2}}_{\text{call } \delta+\frac{n}{r}}}}\right\|^2_{L^2(dx)}$$

$$\overset{\text{Planch}}{=} \int_\Omega \left|e^{-(y+e|\xi)}\Delta^\delta(\xi)\right|^2 d\xi$$

$$\overset{LZ}{=} c\cdot \Delta^{-2\delta-\frac{n}{r}}(y+e) \longrightarrow \text{Need } 2\delta+\frac{n}{r} > g_0$$

Next
$$\int_\Omega \cdots \Delta^\gamma(y)\,d\nu(y) = c\cdot\int_\Omega \frac{\Delta^\gamma(y)}{\Delta^{2\delta+\frac{n}{r}}(y+e)}\,d\nu(y) < \infty$$

$$\overset{\text{Appl}}{\iff} \gamma > g_0 \text{ and } 2\delta+\frac{n}{r} > g_0^* + \gamma$$

Here all powers are scalar $\Gamma=(\gamma,\dots,\gamma) \to$ so WMAT $g_0 = g_0^* = (r-1)\frac{d}{2} = \frac{n}{r}-1$

$$\Rightarrow \text{need } 2\delta + \cancel{\tfrac{n}{r}} > \cancel{\tfrac{n}{r}} - 1 + \gamma \Rightarrow \delta > \frac{\gamma-1}{2} \Rightarrow \overset{\delta=\dots}{\cdots\cdots} \Rightarrow p > 1+\frac{\frac{n}{r}-1}{\gamma+\frac{n}{r}} \quad \blacksquare$$