II. Bdedness of Bergman projections

1. – The positive operators P_γ^+

Let $D = T_\Omega = \mathbb{R}^n + i\Omega$, Ω = sym cone (eg $\Omega = \Lambda_n$)

Let $A_\gamma^p = \{ f \mid \iint_{T_\Omega} |f(x+iy)|^p \, \underbrace{dx\, \Delta^\gamma(y)\, d\sigma(y)}_{d_\gamma z} < \infty \}$, $\quad d\sigma(y) = \frac{dy}{\Delta^{\frac{n}{r}}(y)}$

(here $1 \le p < \infty$, $\gamma > g_0$).

Let $P_\gamma^+ f(z) = \iint_{T_\Omega} |B_\gamma(z,w)| \, f(w)\, d_\gamma w$

Q | When is $P_\gamma^+ : L_\gamma^p(T_\Omega) \longrightarrow L^p(T_\Omega)$?

Recall that
$$|B_\gamma(z,w)| = \frac{1}{|\Delta(z-\bar w)|^{\gamma+\frac{n}{r}}} = \frac{1}{|\Delta(x-u+i(y+v))|^{\gamma+\frac{n}{r}}}$$

A first (naive) approach would be $\quad \left[\begin{array}{l} z = x+iy \\ w = u+iv \end{array}\right]$

$$\| P_\gamma^+ f(\cdot+iy)\|_{L^p(dx)} \overset{\text{Mink}}{\le} \int_\Omega \int_{\mathbb{R}^n} \underbrace{\|f(\cdot+iv)\|}_{g(v)} \, B_\gamma(iy, u+iv)\, du\, d_\gamma v$$

$$= \int_\Omega g(v) \left[\int_{\mathbb{R}^n} \frac{du}{|\Delta(u+i(v+y))|^{\gamma+\frac{n}{r}}} \right] d_\gamma v$$

Exercise
$$\int_{\mathbb{R}^n} \frac{du}{|\Delta(u+i\eta)|^{\gamma+\frac{n}{r}}} = c_\gamma \cdot \frac{1}{\Delta(\eta)^\gamma} \qquad \forall \eta \in \Omega, \quad \gamma > g_0^*$$

($\gamma > \frac{n}{r} - 1$ (scalar powers))

Hint
- By homogeneity, reduce to $\eta = e$
- Use Plancherel to transform in Γ-integral.

So $\quad \| P_\gamma^+ f(\cdot + iy) \|_{L^p(dx)} \leq \int_\Omega \frac{g(v)}{\Delta^\gamma(y+v)} \, d_\gamma v =: S_\gamma g(y)$

← Hilbert-type operator

Lemma 1 Let $\gamma > g_0$, then

$$S_\gamma : L^p_\gamma(\Omega) \longrightarrow L^p_\gamma(\Omega) \iff p_\gamma' < p < p_\gamma := 1 + \frac{\gamma}{\frac{n}{r}-1}$$

Proof: "⇐" Since S_γ = positive operator ⟶ use **Schur test**

Key: use as test functions $\Delta^{\underline{\alpha}}(\xi)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$.

$$S g(y) = \int_\Omega \frac{g(v)\, \Delta^{\underline{\alpha}}(v)}{\Delta(y+v)^{\frac{\gamma}{p}}} \frac{\Delta^{-\underline{\alpha}}(v)}{\Delta(y+v)^{\frac{\gamma}{p'}}} \, d_\gamma v$$

$$\overset{\text{Hölder}}{\leq} \left(\int_\Omega \frac{| g(v)\, \Delta^{\underline{\alpha}}(v)|^p}{\Delta(y+v)^\gamma} \, d_\gamma v \right)^{\frac{1}{p}} \cdot \underbrace{\left(\int_\Omega \frac{\Delta^{-\underline{\alpha} p'}(v)}{\Delta(y+v)^\gamma} \, d_\gamma v \right)^{\frac{1}{p'}}}_{\Delta^{-\underline{\alpha}}(y) \text{ if } \begin{cases} \gamma - \underline{\alpha} p' > g_0 \\ \underline{\alpha} p' > g_0^* \end{cases}}$$

Thus

$$\| S_\gamma g \|^p_{L^p_\gamma} \leq \int_\Omega \int_\Omega \frac{|g(v)|^p \Delta^{\underline{\alpha} p}(v)}{\Delta(y+v)^\gamma} \, d_\gamma v \; \Delta^{-\underline{\alpha} p}(y) \, d_\gamma y$$

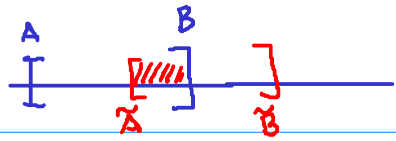
$$\overset{\text{Fub}}{=} \int_\Omega |g(v)|^p \Delta^{\underline{\alpha} p}(v) \underbrace{\left(\int_\Omega \frac{\Delta^{-\underline{\alpha} p}(y)}{\Delta(y+v)^\gamma} \, d_\gamma y \right)}_{= \Delta^{-\underline{\alpha}p}(v) \text{ if } \begin{cases} \gamma - \underline{\alpha} p > g_0 \\ \underline{\alpha} p > g_0^* \end{cases}} d_\gamma v$$

$$= \| g \|^p_{L^p_\gamma(\Omega)}.$$

⟶ the index $\underline{\alpha}$ must satisfy simultaneously

$$\frac{g_0^*}{p} < \underline{\alpha} < \frac{\gamma - g_0}{p} \quad \text{and} \quad \frac{g_0^*}{p'} < \underline{\alpha} < \frac{\gamma - g_0}{p'}$$

Numerology



$\neq \emptyset \iff \tilde{A} < B$ and $A < \tilde{B}$

$$\longrightarrow \quad \frac{g_0^+}{p'} < \frac{\gamma - g_0}{p} \quad \text{and} \quad \frac{g_0^+}{p} < \frac{\gamma - g_0}{p'}$$

$$\Longrightarrow \quad \frac{p}{p'} = p-1 < \frac{\gamma - g_0}{g_0^+} \quad \text{and} \quad \frac{p'}{p} = \frac{1}{p-1} < \frac{\gamma - g_0}{g_0^+}$$

$$\Longrightarrow \quad 1 + \frac{g_0^+}{\gamma - g_0} < p < 1 + \frac{\gamma - g_0}{g_0^+}$$

$$\iff \quad \boxed{1 + \frac{\frac{n}{r} - 1}{\gamma} < p < 1 + \frac{\gamma}{\frac{n}{r} - 1}} \longrightarrow P_\gamma$$

Conclusion

Using the lemma,

$[g = \|f(\cdot + iy)\|_{L^p(dx)}]$

$$\|P_\gamma^+ f\|_{L^p_\gamma(T_\Omega)} \leq \|S_\gamma g\|_{L^p_\gamma(\Omega)} \lesssim \|g\|_{L^p_\gamma(\Omega)} = \|f\|_{L^p_\gamma(T_\Omega)}$$

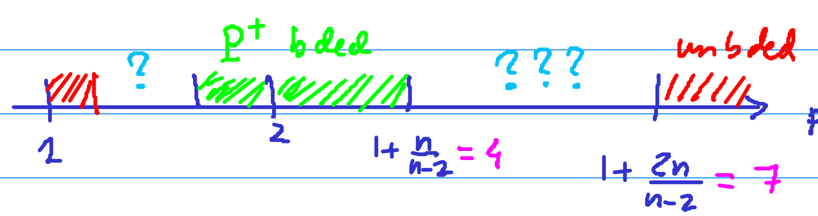
So $\boxed{S_\gamma \text{ bded } L^p_\gamma(\Omega) \Longrightarrow P_\gamma^+ \text{ bded } L^p_\gamma(T_\Omega)}$

Notes :

① One can also show "$\Longleftarrow$"

② But ... the range $p_\gamma' < p < p_\gamma$ is MUCH SMALLER than the conjectured for P_γ ...

Example $\Omega = \Lambda_n$ and $\gamma = n/2$ (unweighted)



$n = 3$

This was the result proved in BeBo '95.

Question : Find optimal range of bddness for $S_{\underline{\gamma}}$ in $L^p_{\underline{\gamma}}(\Omega)$ when $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$.

↳ If $r \geq 3$ and γ = vector $\Rightarrow$ optimality of $p'_\gamma < p < p_\gamma$ is still open !!

See recent work in [GaNa '20].

2.- Bddness of P_γ in $L^{2,q}_\gamma$

Def : Consider the mixed normed spaces $L^{pq}_\gamma(T_\Omega)$, given by

$$\|f\|_{L^{pq}_\gamma} = \left[\int_\Omega \|f(\cdot + iy)\|^q_{L^p(dx)} \underbrace{\Delta^\gamma(y)\, d\sigma(y)}_{d_\gamma y} \right]^{1/q} < \infty.$$

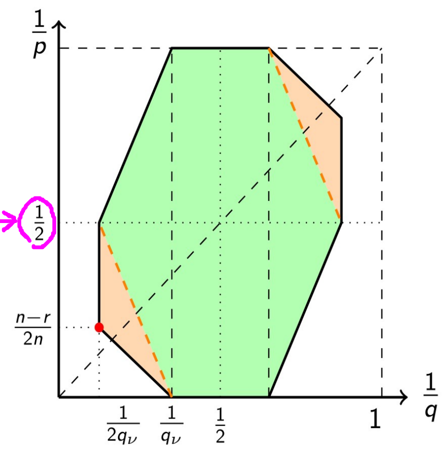
Theorem 1 BBPR '98, BBG '00, BBGR '04

$$P_\gamma \text{ bdd in } L^{2,q}_\gamma(T_\Omega) \iff (2q_\gamma)' < q < 2q_\gamma$$

NOTE : ① Here $q_\gamma = p_\gamma = 1 + \frac{\gamma}{\frac{n}{r}-1}$ as before.

② The previous proof actually gives $P^+_\gamma : L^{p,q}_\gamma \longrightarrow L^{pq}_\gamma$ $\forall\, 1 \le p \le \infty$, $q'_\gamma < q < q_\gamma$

③ So by interpolation one obtains



Comments

① When restricted to $p = q$, it gives bddness of P_γ for

$$1 + q'_\gamma < p < 1 + q_\gamma$$

($p=2$) → $\frac{1}{2}$

↑ is better than P^+_γ !!

② • BBPR'98 ⟶ case $\Omega = \Lambda_n$

• BBG'00 ⟶ general Ω

• BBGR'04 ⟶ proper general context + optimal counterexamples
+ new improvements (for Λ_n)
↳ relation with "decoupling ineq".

③ The key new tool in BBPR'98 is the "spectral decomposition of Ω"

3.- The spectral decomposition of Ω

This is explained in detail in Lecture Notes BBGNPR'2004, § 2.4

Using the identification

$$H \longrightarrow \Omega$$
$$h \longmapsto h\cdot e$$

one defines a riemannian H-invariant metric in Ω.

Namely, if $y_0 = h\cdot e \in \Omega$

$$\langle \vec{\xi}, \vec{\eta} \rangle_{y_0} := \langle h^{-1}\vec{\xi} \mid h^{-1}\vec{\eta} \rangle, \quad \vec{\xi}, \vec{\eta} \in \mathbb{R}^n.$$

This generates a distance in Ω

$$d(y_0, y_1) = \inf \left\{ \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)} \, dt \;\Big/\; \begin{array}{l} \gamma : [0,1] \xrightarrow{\text{smooth}} \Omega \\ \gamma(0) = y_0 \\ \gamma(1) = y_1 \end{array} \right\}$$

Exercise ① $d(h\cdot y_0, h\cdot y_1) = d(y_0, y_1)$, $\forall h \in H$

② Show that (1) also holds if $h \in G = HK$.

Hint Show first that $\langle h\vec{\xi}, h\vec{\eta} \rangle_{h\cdot y_0} = \langle \vec{\xi}, \vec{\eta} \rangle_{y_0}$ $\forall h \in G$.

From the G-invariant distance $d(\cdot,\cdot)$ in Ω one constructs

① $\{\xi_\lambda\}$ = 1-separated set in Ω

② $\{B_\lambda = B_1(\xi_\lambda)\}$ = covering of Ω with FIP

③ $\hat{\varphi}_\lambda \in C_c^\infty(B_\lambda^*)$ s.t. $\sum \hat{\varphi}_\lambda(\xi) \equiv 1$, $\xi \in \Omega$

(smooth partition of unity in Ω)

[NOTE: So far, we do not worry so much about the "shape" of B_λ's..]

The key properties will be

Proposition 1 If $d(\xi, \xi_\lambda) \leq 1$, then

① $\Delta_j(\xi) \simeq \Delta_j(\xi_\lambda)$, $j = 1, \dots, r$

② $\dfrac{(y|\xi)}{(y|\xi_\lambda)} \approx 1$, unif in $y \in \Omega$.

Proof: See details in Lecture Notes 2004.

① Case $\xi_\lambda = e$

Since $\overline{B_1(e)}$ = fixed compact set in Ω,

$$\frac{1}{C_1} \leq \frac{\Delta_j(\xi)}{\Delta_j(e)} \leq C_1 \qquad \forall\, \xi \in \overline{B_1(e)}\,; \quad \forall\, j = 1, \dots, r.$$

General $\xi_\lambda = h \cdot e$ ⟶ use H-invariance of d and homogeneity of Δ_j (details in Lecture Notes 2004).

② Case $\xi_\lambda = e$

$$\frac{(y|\xi)}{(y|e)} = \frac{(y|\xi)}{|y|} \quad \text{and} \quad \xi \in \overline{B_1(e)},\ \frac{y}{|y|} \in S^{n-1} \Big\} \text{ cupt sets}.$$

General ξ_λ ⟶ homogeneity. ∎

Note The previous proposition allows to justify

$$\| \mathcal{F}^{-1}(\hat{\psi}_\lambda(\xi)\Delta^{\alpha}(\xi)\, e^{-(y|\xi)}\hat{f})\|_{L^p(dx)} \simeq \Delta^{\alpha}(\xi_\lambda)\, e^{-(y|\xi_\lambda)} \|\psi_\lambda * f\|_{L^p(dx)}$$

↑ modulo consts

↳ this will be used often in the proofs ...

Proof of Theorem 1

Recall, need to determine when

$$P_\gamma : L^{2,q}_\gamma(T_\Omega) \longrightarrow L^{2,q}_\gamma(T_\Omega).$$

- The proof works for all Ω = sym cone, $\gamma = (\gamma_1, \dots, \gamma)$ scalar weight
- I use a general approach which will be valid later for $L^{p,q}_\gamma$...

Step 1

$$P_\gamma f(x+iy) = \int_\Omega \int_{\mathbb{R}} B_\gamma(x-u+i(y+v))\, f(u+iv)\, du\, d_\gamma v$$

$f_v(u) = f(u+iv)$

$$= \int_\Omega \mathcal{F}^{-1}_{\mathbb{R}^n}\left[e^{-(y+v|\xi)} \Delta^{\gamma}(\xi)\, \hat{f}_v(\xi)\, 1_\Omega(\xi)\right](x)\, d_\gamma v$$

recall $B_\gamma(z) = \int_\Omega e^{i(z|\xi)} \Delta^{\gamma}(\xi)\, d\xi$

$$= \mathcal{F}^{-1}_{\mathbb{R}^n}\left[e^{-(y|\xi)} \Delta^{\gamma}(\xi)\, 1_\Omega(\xi) \cdot \underbrace{\int_\Omega \hat{f}_v(\xi)\, e^{-(v|\xi)}\, d_\gamma v}_{\hat{G}(\xi)} \right](x)$$

$$= \sum_\lambda \mathcal{F}^{-1}_{\mathbb{R}^n}\left[e^{-(y|\xi)} \Delta^{\gamma}(\xi)\, \hat{\psi}_\lambda(\xi)\, \hat{G}(\xi)\right](x)$$

↖ disjoint freq supports

So, by orthogonality

$$\Big\| \sum_\lambda g * \psi_\lambda \Big\|_{L^2} \simeq \Big(\sum_\lambda \|g * \psi_\lambda\|^2_{L^2} \Big)^{1/2}$$

A1 **(p, s) - assumption** Suppose that, for each $\{\psi_\lambda\}$ we could prove

$$(*) \quad \Big\| \sum_\lambda g * \psi_\lambda \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\sum_\lambda \| g * \psi_\lambda \|^s_{L^p(\mathbb{R}^n)} \Big)^{1/s}$$

for some $s = s(p)$ and all $\hat{g} \in C_c^\infty(\Omega)$.

Example

① If $p = 2 \longrightarrow (*)$ holds with $s = 2$ (OG)

② $\forall \; 1 \leq p \leq \infty \longrightarrow (*)$ holds trivially $s = 1$ (triangle ineq)

③ An interpolation argument will show that

$(*)$ holds always with $s = \min\{p, p'\}$

Note: The "decoupling" inequalities are precisely improvements over these trivial cases of the exponent s

↳ we will discuss them later...

So, continuing the proof

$$\| P_\gamma f(\cdot + iy) \|_{L^p(\mathbb{R}^n)} = \Big\| \sum_\lambda \mathcal{F}^{-1}(\Delta^\gamma e^{-(y|\cdot)} \hat{\psi}_\lambda \hat{G}) \Big\|_{L^p(\mathbb{R}^n)}$$

(p, s) - ineq

$$\lesssim \Big(\sum_\lambda \| \mathcal{F}^{-1}(\Delta^\gamma e^{-(y|\cdot)} \hat{\psi}_\lambda \hat{G}) \|^s_{L^p} \Big)^{1/s}$$

$$\lesssim \Big(\sum_\lambda \Delta^{\gamma s}(\xi_\lambda) e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|^s_p \Big)^{1/s}$$

Now interpolate in $L^q_\gamma(dy)$

Step 2

$$\| P_\gamma f \|^q_{L^{p,q}_\gamma(T_\Omega)} \lesssim \int_\Omega \Big[\sum_\lambda \Delta^{\gamma s}(\xi_\lambda) e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|^s_p \Big]^{q/s} d_\gamma y$$

A 2 Suppose we can pass $\ell^{q/s}$ – norm inside of $\sum_\lambda$, ie

$$\int_\Omega \Big[\sum_\lambda \Delta^{\gamma s}(\xi_\lambda)\, e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|_p^s \Big]^{q/s} d_\gamma y \lesssim$$

omit irrelevant constants

$$\overset{(A2)}{\lesssim} \int_\Omega \sum_\lambda \Delta^{\gamma q}(\xi_\lambda)\, e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|_p^q \, d_\gamma y$$

$$= \sum_\lambda \Delta^{\gamma q}(\xi_\lambda) \| G * \psi_\lambda \|_p^q \underbrace{\int_\Omega e^{-(y|\xi_\lambda)} d_\gamma y}_{\Delta^{-\gamma}(\xi_\lambda)} \qquad \gamma > g_0$$

$$\approx \sum_\lambda \Delta^{\gamma(q-1)}(\xi_\lambda) \| G * \psi_\lambda \|_p^q$$

$$=: \| G \|^q_{B^{-\gamma(q-1)}_{pq}}$$

¡ Besov type norm!

Justification of A 2

- If $q/s \le 1 \longrightarrow$ trivial
- If $q/s > 1 \longrightarrow$ must use Schur test (ie Hölder + test Δ^α)

¡ new restriction!

Lemma If $1 \le q < s\, q_\gamma$, then (A 2) holds

P/

$$\Big[\sum_\lambda \Delta^{\gamma s}(\xi_\lambda)\, e^{-(y|\xi_\lambda)} \| G * \psi_\lambda \|_p^s \, \Delta^{-\alpha}(\xi_\lambda)\, \Delta^{\alpha}(\xi_\lambda) \Big]^{q/s} \le$$

$$\overset{\text{Hölder } q/s}{\le} \Big(\sum_\lambda \Delta^{\gamma q}(\xi_\lambda) \| G_\lambda \|_p^q\, \Delta^{-\alpha \frac{q}{s}}(\xi_\lambda) \cdot e^{-(y|\xi_\lambda)} \Big) \cdot \Big(\underbrace{\sum_\lambda \Delta^{\alpha (q/s)'}(\xi_\lambda)\, e^{-(y|\xi_\lambda)}} \Big)^{\frac{q/s}{(q/s)'}}$$

$$\approx \int_\Omega \Delta^{\alpha (q/s)'}(\xi)\, e^{-(y|\xi)} d\sigma(\xi)$$

$$\simeq \Delta_*^{-\alpha_* (q/s)'}(y) \quad \text{if } \alpha \left(\tfrac{q}{s}\right)' > g_0$$

So, taking

$$\int_\Omega [\dots] d_\gamma y \lesssim \sum_\lambda \Delta^{\gamma q}(\xi_\lambda) \| G_\lambda \|_p^q\, \Delta^{-\alpha \frac{q}{s}}(\xi_\lambda) \int_\Omega e^{-(y|\xi_\lambda)}\, \Delta_*^{-\alpha_*(q/s)}(y)\, d_\gamma y$$

The last Ω-integral equals $\Delta^{\underline{s}\cdot\frac{q}{s}-\gamma}(\xi_\lambda)$ if $\gamma - \alpha_\gamma(\frac{q}{s}) > g_0$

$$\simeq \sum_\lambda \Delta^{\gamma q}(\xi_\lambda)\, \|G_\lambda\|_p^q\, \Delta^{-\underline{s}\frac{q}{s}}(\xi_\lambda)\, \Delta^{\underline{s}\frac{q}{s}-\gamma}(\xi_\lambda)$$

$$= \sum_\lambda \Delta^{\gamma(q-1)}(\xi_\lambda)\cdot \|G * \varphi_\lambda\|_p^q = \|G\|^q_{B^{-\gamma(q-1)}_{pq}}.$$

This argument needs $\frac{g_0}{(q/s)'} < \underline{s} < \frac{\gamma - g_0^*}{q/s}$

which is only possible if

$$\frac{g_0}{(q/s)'} < \frac{\gamma - g_0^*}{q/s} \iff \left(\frac{q}{s}-1\right) g_0 < \gamma - g_0^* \iff \frac{q}{s} < 1 + \frac{\gamma - g_0^*}{g_0}$$

If $\gamma = (\gamma, \dots, \gamma)$ this is the same as $\frac{q}{s} < 1 + \frac{\gamma}{\frac{n}{r}-1} = q_\gamma$. ∎

Step 3 Recall that $\hat{G}(\xi) = \int_\Omega \hat{f}_v(\xi)\, e^{-(v|\xi)}\, d_\gamma v$.

It remains to show that

$$\|G\|_{B^{-\gamma(q-1)}_{pq}} = \left(\sum_\lambda \Delta^{\gamma(q-1)}(\xi_\lambda)\, \|G * \varphi_\lambda\|_p^q\right)^{\frac{1}{q}} \lesssim \|f\|_{L^{pq}_\gamma(T_\Omega)}$$

This step is actually easier, by a "dualization" of the previous arguments.

Lemma 3 Assume the $(p', \tilde{s})$ ineq holds. Then (i.e. $\tilde{s} = s(p')$)

$$\|G\|_{B^{-\gamma(q-1)}_{pq}} \lesssim \|f\|_{L^{pq}_\gamma} \quad \text{if } q' < \tilde{s}\cdot q_\gamma$$

Proof If $B = B^{-\gamma q/q'}_{pq} \Rightarrow B^* = B^{\gamma}_{p'q'}$ (exercise: check duality pairing)

Then $\|G\|_B = \sup_{\|h\|_{B^*}=1} |\langle G, h\rangle|$

Now $\langle G, h\rangle = \int_\Omega \hat{G}(\xi)\,\overline{\hat{h}(\xi)}\,d\xi$

def $\hat{G}$ $\rightarrow$ $= \int_\Omega \left[\int_\Omega \hat{f}_v(\xi)\, e^{-(v|\xi)}\, d_\gamma v\right] \overline{\hat{h}(\xi)}\, d\xi$

$\overset{\text{Fub}}{=} \int_\Omega \int_\Omega \hat{f}_v(\xi)\, \overline{\hat{h}(\xi)\, e^{-(v|\xi)}}\, d\xi\, d_\gamma v$

Now we find $\Phi \in A^2_\gamma$: $\widehat{(P_\gamma \Phi)_v}(\xi) = \hat{h}(\xi)\, e^{-(v|\xi)}$

Exercise : Check that $\Phi(z) = \int_\Omega e^{i(z|\xi)}\, \hat{h}(\xi)\, d\xi$ has this property

So $\langle G, h\rangle \overset{\text{Pl}}{=} \int_\Omega \int_{\mathbb{R}^n} f_v(u)\, \overline{(P_\gamma \Phi)_v(u)}\, du\, d_\gamma v$

$= \langle f, P_\gamma \Phi \rangle_{L^2_\gamma(T_\Omega)}$

Thus

$$|\langle G, h\rangle| \le \|f\|_{L^{pq}_\gamma} \cdot \|P_\gamma \Phi\|_{L^{p'q'}_\gamma}$$

Now, by steps 1 + 2, needs $q' < \tilde{s}\cdot q_\gamma$

$$\|P_\gamma \Phi\|_{L^{p'q'}_\gamma} \lesssim \|G_\Phi\|_{B^{-\gamma(q'-1)}_{p'q'}},$$

where

$$\hat{G}_\Phi(\xi) = \int_\Omega \hat{\Phi}_v(\xi)\, e^{-(v|\xi)}\, d_\gamma v$$

$$= \hat{h}(\xi) \int_\Omega e^{-2(v|\xi)}\, \Delta^\gamma(v)\, d\sigma(v) \simeq \hat{h}(\xi)\, \Delta^{-\gamma}(\xi)$$

So,

$$\|G_\Phi\|_{B^{-\gamma(q'-1)}_{p'q'}} = \left(\sum_\lambda \Delta^{\gamma(q'-1)}(\xi_\lambda)\, \|\mathcal{F}^{-1}[\hat{\psi}_\lambda \hat{h} \Delta^{-\gamma}]\|^{q'}_{p'}\right)^{1/q'}$$

$$\simeq \left(\sum_\lambda \Delta^{-\gamma}(\xi_\lambda)\, \|\psi_\lambda * h\|^{q'}_{p'}\right)^{1/q'} = \|h\|_{B^\gamma_{p'q'}} = 1.$$

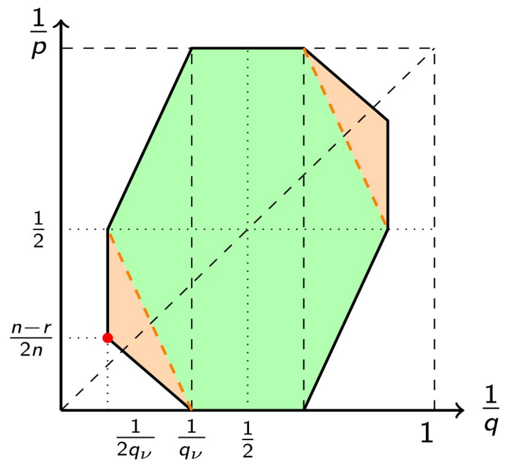
■

Overall: Using $s = \min\{p, p'\}$, we have the restrictions ($s = \tilde{s}$)

$$(s\, q_\gamma)' < q < s \cdot q_\gamma$$

- For $p = 2 \Rightarrow s = 2 \Rightarrow$ $\boxed{(2q_\gamma)' < q < 2q_\gamma}$
- For $p \neq 2 \longrightarrow s = p \wedge p' \longrightarrow$ full hexagon !!

  ¡¡ no need of interpolation with P_γ^t !!



Note:

① Counterexamples can be given for the various border lines

↳ see [BBGR'04].

② What happens in orange region?

Key observation:

If we could prove

$$(\ast)\quad \Big\| \sum_\lambda g * \varphi_\lambda \Big\|_p \le \Big(\sum_\lambda \|g * \varphi_\lambda\|_p^2 \Big)^{1/2}, \quad 2 \le p \le \frac{2n}{n-r}$$

$\Longrightarrow$ we would get P_γ bded L^{pq}_γ $\quad 2 \le q \le 2q_\gamma$

↳ ¡¡ full conjectured region !!

Q | Is $(\ast)$ true when $2 \le p \le \frac{2n}{n-r}$?

↳ We need MORE about the geometry of the balls B_λ (not just the OG)

... so far only for light-cones Λ_n we know something...