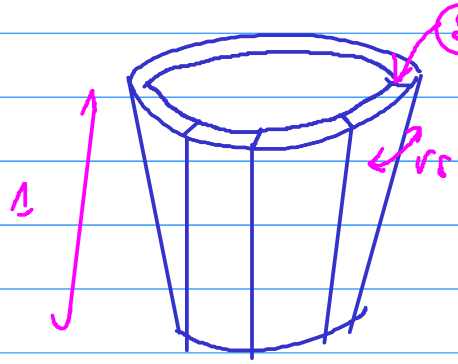
III. Decoupling Inequalities in $\mathbb{R}^n$

Light-cones in $\mathbb{R}^n$



Let $\Gamma = \partial\Lambda_n \cap \{1 \le \tau \le 2\}$ and $\boxed{\Gamma(\delta) = \Gamma + O(\delta)}$

Let $\{\omega_j\} \subseteq S^{n-2}$ be $\sqrt{\delta}$-separated and

$$\Pi_j^{(\delta)} = \{ (\tau, \xi) \in \Gamma(\delta) \;/\; |\tfrac{\xi}{|\xi|} - \omega_j| \le c\sqrt{\delta} \}$$

↳ covering of $\Gamma(\delta)$ by $1 \times \delta \times (\sqrt{\delta} \times \overset{n-2}{\dots} \times \sqrt{\delta})$ – plates

Thm 1

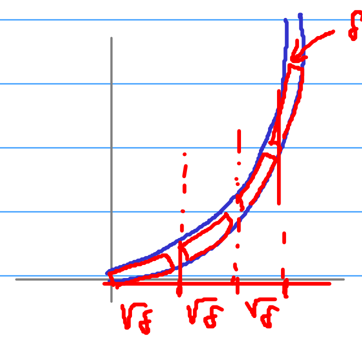
If $2 \le p \le \frac{2n}{n-2}$ and $\varepsilon > 0$, then

$$\|\sum f_j\|_p \lesssim_\varepsilon (1/\delta)^\varepsilon \Big(\sum_j \|f_j\|_p^2\Big)^{1/2}$$

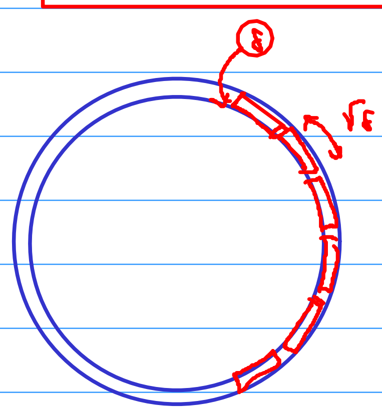
for all $\operatorname{supp} \hat{f}_j \subseteq \Pi_j^{(\delta)}$

It turns out to be a special case of

$d = n-1$



Spheres (or paraboloids) in $\mathbb{R}^d$



Let $S(\delta) = S^{d-1} + O(\delta)$ in $\mathbb{R}^d$

Let $\{\omega_j\} \subseteq S^{d-1}$ be $\sqrt{\delta}$-separated and

$$\Pi_j^{(\delta)} = \{ \xi \in S(\delta) \;/\; |\tfrac{\xi}{|\xi|} - \omega_j| \le c\sqrt{\delta} \}$$

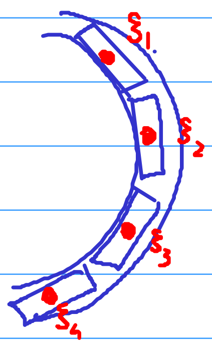
a covering of $S(\delta)$ by $\delta \times (\sqrt{\delta} \times \overset{d-1}{\dots} \times \sqrt{\delta})$ plates

Thm 2

> If $2 \le p \le \frac{2(d+1)}{d-1}$ and $\varepsilon > 0$, then
> $$\Big\| \sum f_j \Big\|_p \lesssim_\varepsilon (1/\delta)^\varepsilon \Big(\sum_j \|f_j\|_p^2 \Big)^{1/2}$$
> for all $\operatorname{supp} \hat f_j \subseteq \Pi_j^{(\delta)}$

Examples

① $\|\sum f_i\|_p \le \left(\sum \|f_i\|_p^s\right)^{1/s}$ can only hold if $s \le 2$.



Pick φ_j / $\operatorname{supp} \hat\varphi_j \subseteq \Pi_j \cap B_\delta(\xi_j)$

ie $\hat\varphi_j(\xi) = \hat\varphi(\xi - \xi_j)$ for a fixed $\hat\varphi \in C_c^\infty(B_{\delta}(0))$

Rademacher

Let $f^\omega(x) = \sum a_j r_j(\omega) \varphi_j(x) = \sum f_j^\omega$

indep of ω!!

Now $\left(\sum \|f_j^\omega\|_p^s\right)^{1/s} = \left(\sum |a_j|^s \|\varphi_j\|_p^s\right)^{1/s} = \left(\sum |a_j|^s\right)^{1/s} \|\varphi\|_p$

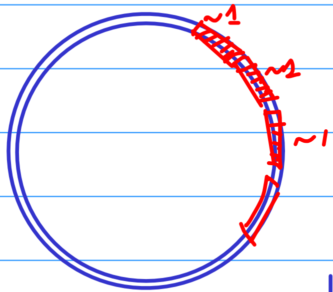
while
$$\mathbb{E}\left[\Big\|\sum f_j^\omega\Big\|_p^p\right] \overset{\text{Fub}}{=} \int \mathbb{E}\Big|\sum a_j \varphi_j(x) r_j(\omega)\Big|^p dx$$
$$\overset{\text{Khintchine}}{\simeq} \int \left(\sum |a_j|^2 |\varphi_j(x)|^2\right)^{\frac{p}{2}} dx = \left(\sum |a_j|^2\right)^{\frac{p}{2}} \|\varphi\|_p^p$$

$\varphi_j(x) = e^{ix\xi_j}\varphi(x)$

So (p,s)-ineq $\implies \left(\sum |a_j|^2\right)^{1/2} \lesssim \left(\sum |a_j|^s\right)^{1/s} \implies \underline{s \le 2}$.

Note: A variation of this examples shows that the Bergman proj. P_δ is not bdd on L^{pq}_δ when $q \ge 2q_p$

Example 2 why $2 \le p \le \frac{2(d+1)}{d-1}$?



Pick now $\hat{f}_j = \psi_{\pi_j}$ ← smoothing of $\mathbb{1}_{\pi}$ ie

$$\left[\psi_\pi \in C_c^\infty(\pi),\ 0 \le \psi \le 1 \text{ and } \psi|_{\pi/2} \equiv 1\right]$$

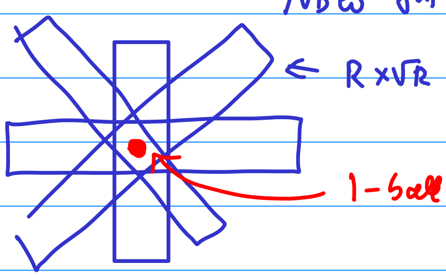
Then

$$\|f_j\|_p \approx \|\hat{f}_j\|_{p'} \simeq |\pi_j|^{1/p'} \simeq \delta^{\frac{d+1}{2}\cdot\frac{1}{p'}}$$

$$\boxed{\begin{array}{c}\text{Convention}\\ R := 1/\delta\end{array}}$$

$$\Rightarrow \left(\sum \|f_j\|_p^2\right)^{\frac12} \simeq \left(R^{\frac{d-1}{2}}\cdot\left(\delta^{\frac{d+1}{2}\cdot\frac{1}{p'}}\right)^2\right)^{\frac12} = \delta^{\frac{d+1}{2p'}-\frac{d-1}{4}}$$

Now $f(x) = \sum f_j(x)$ makes "largest" contribution at $|x| \le 2$ ...



Indeed,

$$|f(x)| = \left|\int \hat{f}(\xi)\left(e^{ix\cdot\xi} - 1 + 1\right) d\xi\right|$$

$$\ge \int \hat{f}(\xi)\, d\xi - \int \hat{f}(\xi)\underbrace{\left|e^{ix\cdot\xi} - 1\right|}_{\ll 1 \text{ if } |x| \le 1} d\xi$$

$$\gtrsim \int \hat{f}(\xi) = |S(\delta)| \simeq \delta$$

So if (p,S)-dec holds then

$$\delta \lesssim \delta^{\frac{d+1}{2p'} - \frac{d-1}{4}} \Leftrightarrow 1 \ge \frac{d+1}{2p'} - \frac{d-1}{4}$$

$$\Leftrightarrow \dots \Leftrightarrow p \le \frac{2(d+1)}{d-1}$$

Exercise 1. Show that, for $p > \frac{2(d+1)}{d-1}$, if it holds

$$\left\|\sum f_j\right\|_p \le R^\alpha \left(\sum \|f_j\|_p^2\right)^{1/2}$$

then necessarily $\alpha \ge \frac{d-1}{4} - \frac{d+1}{2p}$.

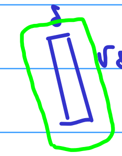
Some facts about this inequalities

FACT 1 Dec ineq are "localized in scale R", ie, if one knows

$$\Big\|\sum f_j\Big\|_{L^p(Q)} \le \Big(\sum \|f_j\|^2_{L^p(\psi_Q)}\Big)^{1/2} \quad \forall\ R\text{-cubes } Q$$

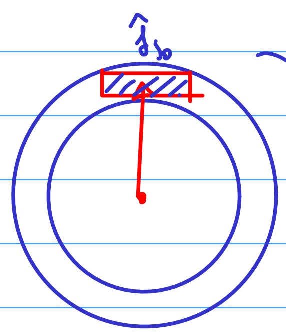
$$\Longrightarrow \quad \Big\|\sum f_j\Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum \|f_j\|^2_{L^p(\mathbb{R}^d)}\Big)^{1/2}.$$

[Idea: $\|\sum_j f_j\|_{L^p(Q)} \lesssim \|\sum_j f_j\psi_Q\|_{L^p(Q)}$



and $\widehat{f_j\psi_Q} = \hat{f}_j * \hat{\psi}_Q \xrightarrow{\text{supp}} \Pi_j + O(\delta) \subseteq 2\Pi_j$]

FACT 2 : space-frequency decomposition

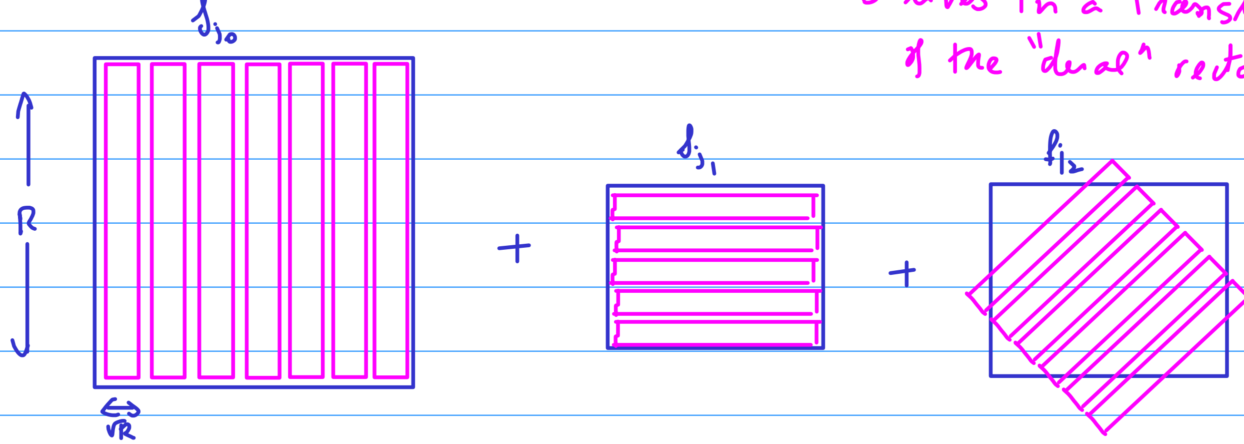


expand $\hat{f}_{j_0}(\xi) = \Big(\sum_{\lambda\in\Lambda_{j_0}} c_\lambda e^{i\lambda\cdot\xi}\Big)\cdot \psi_{\Pi_j}$

$$\Longrightarrow \quad f_{j_0}(x) = \sum_{\lambda\in\Lambda_{j_0}} c_\lambda \underbrace{\hat{\psi}_{\Pi_{j_0}}(x-\lambda)}$$

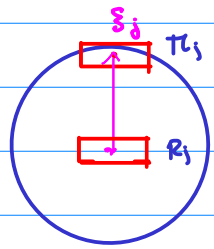
→ lives in a translation of the "dual" rectangle $\Pi_{j_0}^*$

$R\times\sqrt{R}$ - tube



So $f = \sum f_j$ = sum of train waves living in a $\sqrt{\delta}$-separated family of $R\times\sqrt{R}$ - tubes

Note: Each tube has an oscillation tied to the plate location



$$\psi_{T_j} = \psi_{R_j}(\cdot - \xi_j) \longrightarrow \widehat{\psi}_{T_j}(x) = e^{i x \cdot \xi_j}\, \widehat{\psi}_{R_j}(\xi)$$

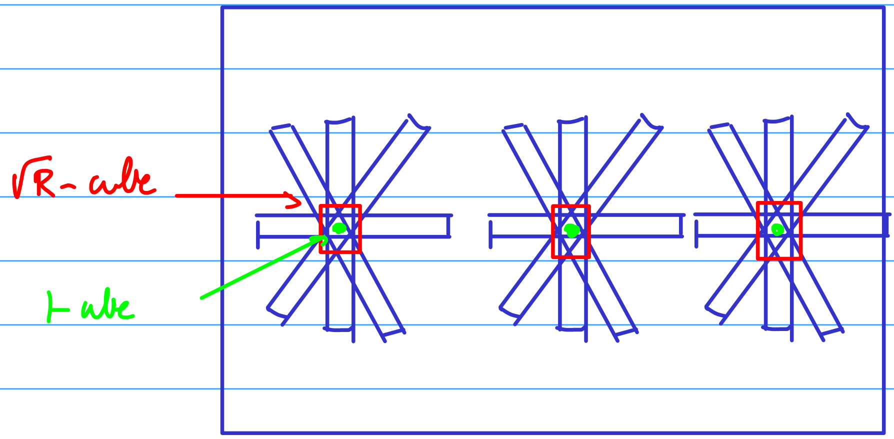
oscillation of freq $\simeq 1$
in longest direction of tube

Moral: Decoupling inequalities should preserve the main contributions of these collections of oscillating waves...

Example 3 An extremizer function

Suppose $f = \sum f_j$ is such that

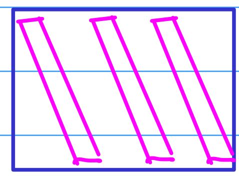
- # directions $= R^{\frac{d-1}{2}}$
- # tubes / direction $= N$ (fixed)
- tubes are arranged in "disjoint bushes"



Because of oscillation of these waves, we expect the main contribution of each "bush" to occur in a 1-ball !!

↳ Let us see how the decoupling ineq justifies this fact (so contributions outside 1-balls are negligible)

Each f_j = sum of N tubes $\left[\text{i.e. } \hat{f}_j = \Big(\sum_{\lambda\in\Lambda_j} e^{i\lambda\cdot\xi}\Big)\,\psi_{\pi_j}\ ,\ |\Lambda_j| = N\right]$



$$\Rightarrow \|f_j\| \simeq (N\cdot|\pi_j^*|)^{1/p} = N^{1/p}\cdot R^{\frac{d+1}{2}\cdot\frac{1}{p}}$$

So $\left(\sum \|f_j\|_p^2\right)^{\frac12} = (\#)^{\frac12}\cdot N^{\frac1p}\cdot R^{\frac{d+1}{2}\cdot\frac1p} = R^{\frac{d-1}{4}}\cdot R^{\frac{d+1}{2}\cdot\frac1p}\cdot N^{\frac1p}$

On the other hand, the contribution at the small 1-balls leads to

$$\Big|\sum f_j(x)\Big| \simeq \# = R^{\frac{d-1}{2}} \quad \text{for } x\in B_1$$

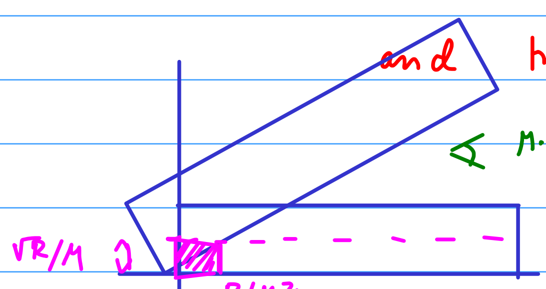
Thus $\|\sum f_j\|_p \gtrsim \left(\sum_{|B|=1}\int_B |\sum f_j|^p\right)^{1/p} \gtrsim R^{\frac{d-1}{2}}\cdot \overset{\#\{B_1\}}{N}{}^{1/p}$

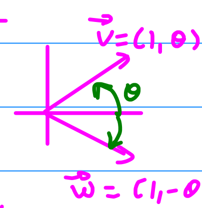
Comparing both expressions

$$\text{LHS} \simeq \text{RHS} \quad \text{iff} \quad p = \frac{2(d+1)}{d-1}\ !!$$

Moral: Optimal decoupling ineq preserve the main "constructive interferences" in these highly overlapping tubes.

Exercise: In the previous example, assume that only M-consequtive directions are kept (say in $\mathbb{R}^2$). ($d=2$) Show that constructive interference happens in an $\left(\frac{R}{M^2}\times\frac{\sqrt{R}}{M}\right)$-rectangle and that these tubes are also optimizers of ($p=6$, $s=2$).



$\Big[$ 

$$e^{i\vec{v}\cdot\vec{x}} + e^{i\vec{w}\cdot\vec{x}} = e^{ix}\cdot 2\cos(\theta y)$$

≈ 1 when $|y| \leq 1/\theta$ $\Big]$

Note: All these examples are discussed in Demeter's book (Chapter 10.1)

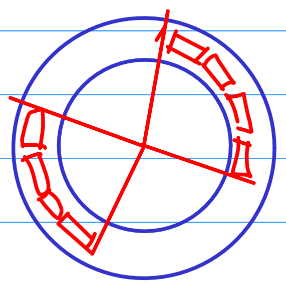
Ideas in proof of Thm 2 (case $d=2$ circle / parabola)

TOOL 1 **Bilinear method** TFAE

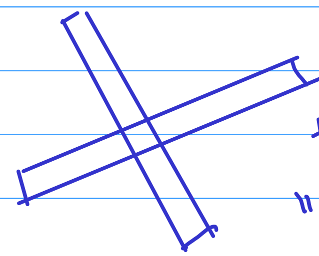
(a) $\| \sum f_j \|_p \lesssim R^{\alpha} \cdot \left(\sum \|f_j\|_p^2 \right)^{1/2}$

(b) $\| (\sum f_j)(\sum g_j) \|_{p/2} \le R^{2\alpha} \left(\sum \|f_j\|_p^2 \right)^{1/2} \cdot \left(\sum \|g_j\|_p^2 \right)^{1/2}$

provided $\mathrm{dist}(\mathrm{Supp}\, \hat{f}_j, \mathrm{Supp}\, \hat{g}_\ell) \gtrsim 1 \quad \forall j, \ell$

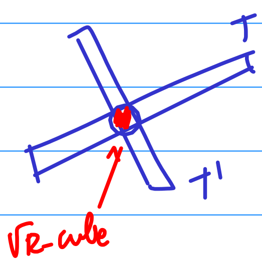


← frequencies 1-separated } $\Longrightarrow$



tubes are "transversal"

The inequality (b) can profit from this "transversality", via Kakeya-type integral estimates:



$$\frac{1}{|T \cap T'|} \int \Big(\sum_{T \in \mathcal{T}_1} c_T 1_T \Big) \Big(\sum_{T' \in \mathcal{T}_2} c_{T'} 1_{T'} \Big) \lesssim \frac{1}{|T|} \Big[\int \Big(\sum_{\mathcal{T}_1} c_T 1_T \Big) \Big] \cdot \Big[\int \Big(\sum_{\mathcal{T}_2} c_{T'} 1_{T'} \Big) \Big]$$

for every $\mathcal{T}_1, \mathcal{T}_2$ families of "transversal" tubes (and $c_T, c_{T'} \ge 0$)

Note: When $d \ge 3 \to$ one needs the deeper d-linear Kakeya estimates

Tool 2 : Induction on scales

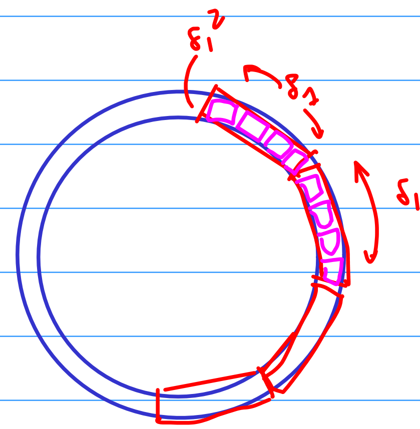
Roughly means that:

if $\quad \|(\sum f_j)(\sum g_j)\|_{p/2} \le R^{2\alpha} \left(\sum \|f_j\|_p^2\right)^{1/2} \cdot \left(\sum \|g_j\|_p^2\right)^{1/2}$ holds $\forall \delta$

$\Longrightarrow \quad \|(\sum f_j)(\sum g_j)\|_{p/2} \le R^{2(\alpha-\epsilon_0)} \left(\sum \|f_j\|_p^2\right)^{1/2} \cdot \left(\sum \|g_j\|_p^2\right)^{1/2}$ also holds

$\to$ so iterating the process can obtain a power of R as small as desired.

$\boxed{\text{Tool 3}}$ Passage from δ_1-arcs to δ_1^2-arcs



This is a <u>key step</u>. It produces a <u>GAIN</u> using Kakeya-estimate at $L^{p/2}$

+ interpolation $\|\cdot\|_{L^{p/2}} \le \|\cdot\|_{L^2}^{1-u} \cdot \|\cdot\|_{L^p}^{u}$
+ L^2-OG
+ Bernstein principle $|f_\theta| \simeq$ const in $(1/\delta^2)$-cube if $\operatorname{supp} \hat{f}_\theta \subseteq \delta^2$-cube
+ pigeon-hole : uniformization of sets $\mathbb{I}$ of directions s.t. $\|f_\theta\|_{L^{p/2}} \simeq$ const $\forall \theta \in \mathbb{I}$

$\hookrightarrow$ <u>Details</u> : See Appendix to this lecture notes (or chap 10 in Demeter's book).