CONDITIONALITY CONSTANTS OF QUASI-GREEDY BASES IN SUPER-REFLEXIVE BANACH SPACES

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ABSTRACT. We show that in a super-reflexive Banach space, the conditionality constants $k_N(\mathscr{B})$ of a quasi-greedy basis $\mathscr{B}$ grow at most like $O(\log N)^{1-\varepsilon}$ for some $0 < \varepsilon < 1$. This extends results by the third-named author and Wojtaszczyk [13], where this property was shown for quasi-greedy bases in L_p for $1 < p < \infty$. We also give an example of a quasi-greedy basis $\mathscr{B}$ in a reflexive Banach space with $k_N(\mathscr{B}) \approx \log N$.

1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space with a Schauder basis $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^{\infty}$, which we shall assume semi-normalized, i.e., $c_1 \le \|\mathbf{e}_j\| \le c_2$ for all j, for some constants $c_2 \ge c_1 > 0$. For $x \in \mathbb{X}$ we write the corresponding series expansion in terms of the basis $\mathscr{B}$ as $x = \sum_{j=1}^{\infty} a_j(x)\mathbf{e}_j$.

Associated with $\mathscr{B}$, for each finite $A \subset \mathbb{N}$ we consider the projection operators

$$x \in \mathbb{X} \longmapsto S_A(x) := \sum_{j\in A} a_j(x)\mathbf{e}_j,$$

and define the sequence

$$k_N = k_N(\mathscr{B}, \mathbb{X}) := \sup_{|A|\le N} \big\|S_A\big\|, \quad N = 1, 2, \ldots$$

Notice that $\mathscr{B}$ is unconditional if and only if $k_N = O(1)$. In general, k_N may grow as fast as $O(N)$, and this sequence may be used to quantify the conditionality of the basis $\mathscr{B}$ in $\mathbb{X}$. It is a consequence of a classical result of Gurarii-Gurarii [14] and James [18] that if $\mathbb{X}$ is a super-reflexive Banach space (i.e., every Banach space finitely representable in $\mathbb{X}$ is reflexive), then

$$k_N = O(N^{1-\varepsilon}), \quad \text{for some } 0 < \varepsilon < 1.$$

In this paper we shall be interested in bases $\mathscr{B}$ which are *quasi-greedy* [20, 28], that is, their expansions converge when the summands are rearranged

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in decreasing order. More precisely, for any choice of greedy operators

$$x \in \mathbb{X} \longmapsto G_N(x) = \sum_{j \in \Lambda_N(x)} a_j(x)\mathbf{e}_j, \tag{1.1}$$

where $\Lambda_N(x)$ is a set of cardinality N such that

$$\min_{j\in\Lambda_N(x)} |a_j| \geq \max_{j\notin\Lambda_N(x)} |a_j|,$$

it holds that $G_N(x) \to x$, for all $x \in \mathbb{X}$. We refer to [25] for background and applications of quasi-greedy bases in the study of non-linear N-term approximation in Banach spaces.

It follows from a result of Dilworth, Kalton and Kutzarova [6, Lemma 8.2] that quasi-greedy bases cannot be "too conditional" in the sense that they satisfy the estimate

$$k_N(\mathscr{B}, \mathbb{X}) = O(\log N). \tag{1.2}$$

See also [9, 12]. Moreover, there are examples of quasi-greedy bases in certain Banach spaces for which the logarithmic growth is actually attained ([12, §6]).

More recently, it was noticed in [13] that (1.2) can be improved to $k_N = O(\log N)^{1-\varepsilon}$ for some $0 < \varepsilon < 1$, at least when $\mathbb{X} = L_p$ and $1 < p < \infty$. The purpose of this note is to show that this improvement continues to hold for any super-reflexive Banach space, while it is not necessarily true for reflexive spaces.

Theorem 1.1. *Let $\mathbb{X}$ be a super-reflexive Banach space, and $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$ a quasi-greedy basis. Then there exists $0 < \varepsilon = \varepsilon(\mathscr{B}, \mathbb{X}) < 1$ such that*

$$k_N(\mathscr{B}, \mathbb{X}) \;=\; O(\log N)^{1-\varepsilon}.$$

Theorem 1.2. *There exists a reflexive Banach space $\mathbb{X}$ and a quasi-greedy basis $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$ such that*

$$k_N(\mathscr{B}, \mathbb{X}) \;\approx\; \log N, \quad N = 2, 3, \ldots$$

In fact the result holds for the infinite direct sum $\mathbb{X} = (\oplus_{n=1}^\infty \ell_1^n)_p$, where $1 < p < \infty$.

We note that bounds on the sequence (k_N) are useful in N-term approximation. In particular, if $\mathscr{B}$ is an *almost-greedy basis*, i.e., quasi-greedy and democratic (see [7]) in $\mathbb{X}$, then (k_N) quantifies the performance of the greedy algorithm versus the best N-term approximation. More precisely, if $\Sigma_N = \{\sum_{\lambda\in\Lambda} c_\lambda \mathbf{e}_\lambda : \ \operatorname{Card}\Lambda \leq N\}$, we have the following:

Corollary 1.3. *Let $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^{\infty}$ be an almost-greedy basis in a super-reflexive Banach space $\mathbb{X}$. Then there exists $0 < \varepsilon = \varepsilon(\mathscr{B}, \mathbb{X}) < 1$ and $c > 0$ such that for all $x \in \mathbb{X}$ and $N = 2, 3, \ldots$,*

$$\|x - G_N x\| \leq c\,(\log N)^{1-\varepsilon}\,\mathrm{dist}(x, \Sigma_N),$$

where $\mathrm{dist}(x, \Sigma_N) = \inf\{\|x - y\| \colon y \in \Sigma_N\}$.

This is a direct consequence of Theorem 1.1 and [26, Thm 2.1] (or [12, Thm 1.1]).

We conclude by recalling some examples of *super-reflexive* Banach spaces. This notion, introduced by James in [18], has several equivalent formulations, one of which being the existence of an equivalent norm which is either uniformly convex or uniformly smooth [18, 11]. In particular, this is the case for $L^p(\mu)$ with $1 < p < \infty$ over any measure space, but also for most examples of reflexive Banach spaces arising in harmonic and functional analysis. Here we list some of them:

(i) Bochner-Lebesgue spaces $L_p(\mu, X)$ over any measure space are uniformly convex if X is uniformly convex and $1 < p < \infty$, [10]. As a consequence, a space $L_p(\mu, X)$ and its subspaces inherit the super-reflexivity from X. That covers the classical families of Sobolev, Besov and Triebel-Lizorkin spaces in $\mathbb{R}^n$ for a wide range of parameters, exactly the ones making them reflexive. The isomorphic embedding into a space of the form $L_p(\mu, L_q(\nu))$ comes from their very definition, see [27], but it is also possible to show isomorphisms with the help of special bases, see for instance [4] for certain Sobolev and Besov spaces.

(ii) Orlicz spaces satisfying Luxemburg's characterizations of reflexivity [23] are super-reflexive; see [1]. We note that Luxembourg assumptions on the measure cover the most usual cases, as Orlicz sequence spaces or function spaces on $\mathbb{R}^n$ with the Lebesgue measure.

(iii) Super-reflexivity has also been studied in Lorentz-type spaces, where its characterization is very close to reflexivity, see for instance [15, 19, 16].

(iv) Uniformly non-square Banach spaces are also super-reflexive. These spaces, introduced by James in [17], are those that satisfy

$$\sup\{\min\{\|x + y\|, \|x - y\|\} : \|x\| = \|y\| = 1\} < 2.$$

(v) Super-reflexivity is preserved as well by certain operations to produce new spaces such as finite products, quotients, ultrapowers and interpolation spaces. In fact, if one of the spaces of the interpolation pair is super-reflexive then all the intermediate spaces are super-reflexive, either with the real [2] or the complex method [5].

2. Proof of Theorem 1.1

All we shall need below from the space $\mathbb{X}$ is the existence of an equivalent norm $\|\!| \cdot |\!\|$ in $\mathbb{X}$ which is *uniformly convex*. That such a norm exists in any super-reflexive space is a classical result of Enflo, see [11]. For more properties of super-reflexive Banach spaces see [3] or [24].

For simplicity, we assume that the original norm $\|\cdot\|$ in $\mathbb{X}$ is uniformly convex. There is no loss of generality in this assumption since the property of $\mathscr{B}$ being quasi-greedy is preserved under equivalent norms in $\mathbb{X}$ and $k_N(\mathscr{B}, \|\cdot\|) \le C k_N(\mathscr{B}, \|\!|\cdot|\!\|)$ with C independent of N.

Recall that a norm $\|\cdot\|$ is uniformly convex in $\mathbb{X}$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad 1 - \left\|\frac{x+y}{2}\right\| < \delta \quad \Longrightarrow \quad \|x-y\| < \varepsilon. \tag{2.1}$$

We denote by $\delta(\varepsilon)$ the largest δ such that (2.1) holds, and call the mapping $\varepsilon \mapsto \delta(\varepsilon)$ the *modulus of convexity* of the norm.

The proof of Theorem 1.1 will partly follow the scheme developed in [13, §5]. We shall denote $\kappa = \kappa(\mathscr{B}, \mathbb{X}) > 0$ the smallest constant such that for all $x \in \mathbb{X}$ and $N = 1, 2, \dots,$

$$\max\big\{\|G_N x\|, \|x - G_N x\|\big\} \le \kappa\, \|x\|, \tag{2.2}$$

for all operators G_N as in (1.1). The existence of such constant is actually equivalent to the quasi-greediness of $\mathscr{B}$ (see [28, Thm 1]).

We will write $x \succcurlyeq y$ when $x = \sum_{j\in A} x_j \mathbf{e}_j$ and $y = \sum_{k \in B} y_k \mathbf{e}_k$ have disjoint supports (i.e., $A \cap B = \emptyset$) and $\min_{j\in A} |x_j| \ge \max_{k\in B} |y_k|$.

We first establish the following lemma, which is the analogue of [13, Lemma B.2.ii]. The proof is based on a result that can be found in Beauzamy's textbook [3, p. 190].

Lemma 2.1. *Let $\mathbb{X}$ be equipped with a uniformly convex norm with modulus of convexity $\delta(\cdot)$, and let $\mathscr{B}$ be a quasi-greedy basis with constant κ. Then, for each $1 < p < \infty$, there exists a constant $\gamma = \gamma(p, \kappa, \delta) < 2^{p-1}$ such that*

$$\big\|x + y\big\|^p \le \gamma\,\Big(\|x\|^p + \|y\|^p\Big), \quad \forall\, x \succcurlyeq y. \tag{2.3}$$

Proof. In Proposition 1 of [3, p.190] it is stated that if $\mathbb{X}$ is uniformly convex,

$$\left\|\frac{x+y}{2}\right\|^p \le \Big(1 - \delta_p\big(\tfrac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\big)\Big)\, \frac{\|x\|^p + \|y\|^p}{2}, \quad \forall\, x, y \in \mathbb{X},$$

for every $p > 1$ and a suitable function δ_p with the property $\delta_p(\varepsilon) \ge c_p \delta(\varepsilon)$ for some $c_p > 0$ depending only on p (see pp. 193–194). Therefore we have,

$$\|x+y\|^p \le \Big(1 - c_p \delta\big(\tfrac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\big)\Big)\, 2^{p-1}\big(\|x\|^p + \|y\|^p\big), \quad \forall\, x, y \in \mathbb{X}.$$

Notice that the quasi-greediness assumption yields

$$\|x-y\| \;\geq\; \frac{1}{\kappa}\,\max\big\{\|x\|,\|y\|\big\},$$

whenever $x \succcurlyeq y$. Since $\delta(\varepsilon)$ is an increasing function we deduce

$$\|x+y\|^p \leq \big(1-c_p\delta(1/\kappa)\big)2^{p-1}\big(\|x\|^p+\|y\|^p\big).$$

Take $\gamma = \big(1-c_p\delta(1/\kappa)\big)2^{p-1}$. Since $\delta(1/\kappa)>0$ we conclude $\gamma<2^{p-1}$ as desired. □

Remark 2.2. One could give a different proof of (2.3) using a weak parallelogram inequality

$$\|\!|x+y|\!\|^p+\eta\,\|\!|x-y|\!\|^p \leq 2^{p-1}\big(\|\!|x|\!\|^p+\|\!|y|\!\|^p\big), \quad x,y\in\mathbb{X}, \tag{2.4}$$

for some $\eta>0$, and arguing as in [13, Lemma B.2.ii]. It is a known result of Pisier [24, Thms 3.1 and 3.2] that every super-reflexive space posseses an equivalent norm $\|\!|\cdot|\!\|$ with the property (2.4), at least *for some* $p<\infty$. We have preferred the give version of Lemma (2.1), since it is valid *for all* $1<p<\infty$, and does not depend on the deeper result of Pisier.

Iterating this result one easily proves the following (see [13, Lemma 2.4]).

Lemma 2.3. *With the assumptions of Lemma 2.1, if* $x_1\succcurlyeq x_2\succcurlyeq\ldots\succcurlyeq x_m$ *have pairwise disjoint supports, then*

$$\big\|x_1+\ldots+x_m\big\|^p \;\leq\; \gamma^{\lceil\log_2 m\rceil}\sum_{j=1}^m\|x_j\|^p, \tag{2.5}$$

where $\gamma<2^{p-1}$ *is the same constant as in* (2.3).

We are now in the position to prove Theorem 1.1. To that end we must show that for $A\subset\mathbb{N}$ with $|A|=N\geq 2$, and every $x=\sum_i a_i\mathbf{e}_i\in\mathbb{X}$ we have

$$\|S_A(x)\| \leq\; C\,(\log N)^{1-\varepsilon}\,\|x\|, \tag{2.6}$$

for a suitable $0<\varepsilon<1$ (independent of x and N) to be determined. By scaling we may assume that $\max_i|a_i|=1$, so that by (2.2),

$$\|x\|\geq\frac{1}{\kappa}\|G_1x\|\geq\frac{c_1}{\kappa}.$$

Let $m=\lceil\log_2 N\rceil$, so that $2^{m-1}<N\leq 2^m$. For $\ell=1,\ldots,m$, we define

$$F_\ell=\{j\;:\;\; 2^{-\ell}<|a_j|\leq 2^{-(\ell-1)}\}\quad\text{and}\quad F_{m+1}=\{j\;:\;\;|a_j|\leq 2^{-m}\}.$$

Next write A as a disjoint union of the sets $A_\ell=A\cap F_\ell$, $\ell=1,\ldots,m+1$. Clearly

$$\|S_{A_{m+1}}x\|\leq\sum_{i\in A_{m+1}}|a_i|\|\mathbf{e}_i\|\leq c_2\,2^{-m}N\leq c_2\leq\frac{\kappa c_2}{c_1}\,\|x\|. \tag{2.7}$$

For the other terms we appeal to Lemmas 5.2 and 5.3 in [12], which use the quasi-greedy property and the fact that $A_\ell \subset \{j \,:\, 2^{-\ell} < |a_j| \leq 2^{-(\ell-1)}\}$ to obtain

$$\|S_{A_\ell} x\| \leq C\,\|x\|,$$

for a positive constant C (independent of x and ℓ). Lemma 2.3 gives

$$\left\|\sum_{\ell=1}^m S_{A_\ell} x\right\|^p \leq \gamma^{\lceil \log_2 m\rceil} \sum_{\ell=1}^m \|S_{A_\ell} x\|^p \leq C^p\,\gamma^{\lceil \log_2 m\rceil}\, m\,\|x\|^p. \tag{2.8}$$

Now we can write

$$\gamma^{\log_2 m}\, m = 2^{\log_2 m\,\log_2\gamma}\, m = m^{1+\log_2\gamma} = m^{p\,\alpha},$$

if we set $\alpha = (1+\log_2\gamma)/p$. Notice that $\alpha < 1$ since $\gamma < 2^{p-1}$, by Lemma 2.1. Thus, combining (2.7) with (2.8) we obtain

$$\|S_A x\| \leq C'\, m^\alpha\,\|x\| \leq C''\,(\log N)^\alpha\,\|x\|,$$

which implies (2.6) if we set

$$\varepsilon = 1-\alpha = 1-(1+\log_2\gamma)/p > 0.$$

Notice that ε only depends on p, κ and the modulus of convexity $\delta(\cdot)$. $\square$

3. Proof of Theorem 1.2

The construction in the proof below is a variation of a standard procedure (cf. [28, Corollary 5] and [8]) that was communicated to one of the authors by P. Wojtaszczyk.

Proof. Let $1<p<\infty$. Take a Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ with a basis $\mathscr{B}_{\mathbb{E}} = \{\mathbf{x}_j\}_{j=1}^\infty$ such that $k_N(\mathscr{B}_{\mathbb{E}}, \mathbb{E}) \approx \log N$, $N=2,3,\dots$ (see e.g. [12, §6] or [13, §3.4]). For each $n=1,2,\dots$, let $E_n = \operatorname{span}\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$, and consider the Banach space $\mathbb{X} = (\oplus_{n=1}^\infty E_n)_p$ consisting of all vectors of the form $x = \{\sum_{j=1}^n c_{n,j}\mathbf{x}_j\}_{n=1}^\infty$ for which the norm given by

$$\|x\| = \left(\sum_{n=1}^\infty \left\|\sum_{j=1}^n c_{n,j}\mathbf{x}_j\right\|_{\mathbb{E}}^p\right)^{1/p}$$

is finite. Being the ℓ_p-sum of finite dimensional spaces, $\mathbb{X}$ is clearly reflexive. It is straightforward to verify that the natural basis $\mathscr{B}_{\mathbb{X}} = \cup_{n=1}^\infty\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$ of $\mathbb{X}$ satisfies $\kappa(\mathscr{B}_{\mathbb{X}}, \mathbb{X}) = \kappa(\mathscr{B}_{\mathbb{E}}, \mathbb{E})$, as defined in (2.2), so the basis is quasi-greedy. The same applies to the identity $k_N(\mathscr{B}_{\mathbb{X}}, \mathbb{X}) = k_N(\mathscr{B}_{\mathbb{E}}, \mathbb{E})$, $N=1,2,\dots$ and the first statement of the theorem follows.

To see the second statement, take the basic sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ in ℓ_1 constructed by Lindenstrauss in [21]. This sequence is defined as

$$\mathbf{x}_n = \mathbf{e}_n - \frac{1}{2}(\mathbf{e}_{2n+1} + \mathbf{e}_{2n+2}), \quad n = 1, 2, \ldots,$$

where $\{\mathbf{e}_j\}_{j=1}^{\infty}$ denotes the canonical basis of ℓ_1. $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a conditional quasi-greedy basic sequence [8] with $k_N \approx \log N$ [12]. In this case, the space E_n verifies that the Banach-Mazur distance between E_n and ℓ_1^n is uniformly bounded above by 2 as proved by Lindenstrauss and Pełczyński in [22, Example 8.1]. We infer that $(\oplus_{n=1}^{\infty} E_n)_p$ is isomorphic to $(\oplus_{n=1}^{\infty} \ell_1^n)_p$ as wished. $\square$

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