LEBESGUE INEQUALITIES FOR CHEBYSHEV THRESHOLDING GREEDY ALGORITHMS

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ABSTRACT. We establish estimates for the Lebesgue parameters of the Chebyshev Weak Thresholding Greedy Algorithm in the case of general bases in Banach spaces. These generalize and slightly improve earlier results in [10], and are complemented with examples showing the optimality of the bounds. Our results also clarify certain bounds recently announced in [19], and answer some questions left open in that paper.

1. INTRODUCTION

Let $\mathbb{X}$ be a Banach space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $\mathbb{X}^*$ be its dual space, and consider a system $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^\infty\subset\mathbb{X}\times\mathbb{X}^*$ with the following properties:

a) $0<\inf_n\{\|\mathbf{e}_n\|,\|\mathbf{e}_n^*\|\}\leq\sup_n\{\|\mathbf{e}_n\|,\|\mathbf{e}_n^*\|\}<\infty$

b) $\mathbf{e}_n^*(\mathbf{e}_m)=\delta_{n,m}$, for all $n,m\geq 1$

c) $\mathbb{X}=\overline{\operatorname{span}\{\mathbf{e}_n : n\in\mathbb{N}\}}$

d) $\mathbb{X}^*=\overline{\operatorname{span}\{\mathbf{e}_n^* : n\in\mathbb{N}\}}^{w^*}$.

Under these conditions $\mathscr{B}=\{\mathbf{e}_n\}_{n=1}^\infty$ is called a *seminormalized Markushevich basis for* $\mathbb{X}$ (or M-basis for short), with *dual system* $\{\mathbf{e}_n^*\}_{n=1}^\infty$. Sometimes we shall consider the following special cases

e) $\mathscr{B}$ is a *Schauder basis* if $K_b:=\sup_N\|S_N\|<\infty$, where $S_Nx:=\sum_{n=1}^N\mathbf{e}_n^*(x)\mathbf{e}_n$ is the N-th partial sum operator

f) $\mathscr{B}$ is a *Cesàro basis* if $\sup_N\|F_N\|<\infty$, where $F_N:=\frac{1}{N}\sum_{n=1}^N S_n$ is the N-th (C,1)-Cesàro operator. In this case we use the constant

$$\beta=\max\big\{\sup_N\|F_N\|,\sup_N\|I-F_N\|\big\}. \tag{1.1}$$

For the latter terminology, see e.g. [21, Def III.11.1]. With every $x\in\mathbb{X}$, we shall associate the formal series $x\sim\sum_{n=1}^\infty\mathbf{e}_n^*(x)\mathbf{e}_n$, where a)-c) imply that $\lim_n\mathbf{e}_n^*(x)=0$. As usual, we denote $\operatorname{supp}x=\{n\in\mathbb{N}:\mathbf{e}_n^*(x)\neq 0\}$.

We recall standard notions about (weak) greedy algorithms; see e.g. the texts [23, 25] for details and historical background. Fix $t\in(0,1]$. We say that A is a *t-greedy set for x of*

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order m, denoted $A \in G(x,m,t)$, if $|A| = m$ and

$$\min_{n\in A} |\mathbf{e}_n^*(x)| \geq t \cdot \max_{n\notin A} |\mathbf{e}_n^*(x)|. \tag{1.2}$$

A *t-greedy operator of order m* is any mapping $\mathscr{G}_m^t : \mathbb{X} \to \mathbb{X}$ which at each $x \in \mathbb{X}$ takes the form

$$\mathscr{G}_m^t(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad \text{for some set} \quad A = A(x, \mathscr{G}_m^t) \in G(x,m,t).$$

We write $\mathbb{G}_m^t$ for the set of all t-greedy operators of order m. The approximation scheme which assigns a sequence $\{\mathscr{G}_m^t(x)\}_{m=1}^\infty$ to each vector $x \in \mathbb{X}$ is called a *Weak Thresholding Greedy Algorithm* (WTGA), see [16, 24]. When $t = 1$ one just says Thresholding Greedy Algorithm (TGA), and drops the super-index t, that is $\mathscr{G}_m^1 = \mathscr{G}_m$, etc.

It is standard to quantify the efficiency of these algorithms, among all possible m-term approximations, in terms of *Lebesgue-type inequalities*. That is, for each $m = 1, 2, \ldots$, we look for the smallest constant $\mathbf{L}_m^t$ such that

$$\|x - \mathscr{G}_m^t(x)\| \leq \mathbf{L}_m^t \sigma_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, \mathscr{G}_m^t \in \mathbb{G}_m^t, \tag{1.3}$$

where

$$\sigma_m(x) := \inf\Big\{ \Big\| x - \sum_{n\in B} b_n \mathbf{e}_n \Big\| \ :\ b_n \in \mathbb{K}, \quad |B| \leq m \Big\}.$$

We call the number $\mathbf{L}_m^t$ the *Lebesgue parameter* associated with the WTGA, and we just write $\mathbf{L}_m$ when $t = 1$. We refer to [25, Chapter 3] for a survey on such inequalities, and to [12, 10, 1, 5, 6] for recent results. It is known that $\mathbf{L}_m^t = O(1)$ holds for a fixed t if and only if it holds for all $t \in (0,1]$, and if and only if $\mathscr{B}$ is unconditional and democratic; see [15] and [23, Thm 1.39]. In this special case $\mathscr{B}$ is called a *greedy basis*.

In this paper we shall be interested in *Chebyshev thresholding greedy algorithms*. These were introduced by Dilworth, Kalton and Kutzarova, see [8, §3], as an enhancement of the TGA. Here, we use the weak version considered in [10]. Namely, for fixed $t \in (0,1]$ we say that $\mathfrak{CG}_m^t : \mathbb{X} \to \mathbb{X}$ is a *Chebyshev t-greedy operator* of order m if for every $x \in \mathbb{X}$ there is a set $A = A(x, \mathfrak{CG}_m^t) \in G(x,m,t)$ such that $\operatorname{supp} \mathfrak{CG}_m^t(x) \subset A$ and moreover

$$\|x - \mathfrak{CG}_m^t(x)\| = \min\Big\{ \|x - \sum_{n\in A} a_n \mathbf{e}_n\| \ :\ a_n \in \mathbb{K} \Big\}.$$

Finally, we define the *weak Chebyshevian Lebesgue parameter* $\mathbf{L}_m^{\mathrm{ch},t}$ as the smallest constant such that

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \mathbf{L}_m^{\mathrm{ch},t} \sigma_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, \mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t},$$

where $\mathbb{G}_m^{\mathrm{ch},t}$ is the collection of all Chebyshev t-greedy operators of order m. As before, when $t = 1$ we shall omit the index t, that is $\mathbf{L}_m^{\mathrm{ch}} := \mathbf{L}_m^{\mathrm{ch},1}$.

When $\mathbf{L}_m^{\mathrm{ch}} = O(1)$ the system $\mathscr{B}$ is called semi-greedy; see [8]. We remark that the first author recently established that a Schauder basis $\mathscr{B}$ is semi-greedy if and only if is quasi-greedy and democratic; see [3].

In this paper we shall be interested in quantitative bounds of $\mathbf{L}_m^{\mathrm{ch},t}$ in terms of the quasi-greedy and democracy parameters of a general M-basis $\mathscr{B}$. Earlier bounds were obtained by Dilworth, Kutzarova and Oikhberg in [10] when $\mathscr{B}$ is a quasi-greedy basis, and very recently, some improvements were also announced by C. Shao and P. Ye in [19, Theorem

3.5]. Unfortunately, various arguments in the last paper seem not to be correct, so one of our goals here is to give precise statements and proofs for the results in [19], and also settle some of the questions which are left open there.

To state our results, we recall the definitions of the involved parameters. Given a finite set $A \subset \mathbb{N}$, we shall use the following standard notation for the indicator sums:

$$\mathbf{1}_A = \sum_{n\in A} \mathbf{e}_n \quad \text{and} \quad \mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n, \quad \varepsilon \in \Upsilon$$

where Υ is the set of all $\varepsilon = \{\varepsilon_n\}_n \subset \mathbb{K}$ with $|\varepsilon_n| = 1$. Similarly, we write

$$P_A(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n.$$

The relevant parameters for this paper are the following:

- Conditionality parameters:

$$k_m := \sup_{|A|\le m} \|P_A\| \quad \text{and} \quad k_m^c = \sup_{|A|\le m} \|I - P_A\|.$$

- Quasi-greedy parameters:

$$g_m := \sup_{\mathscr{G}_k \in \mathbb{G}_k,\, k\le m} \|\mathscr{G}_k\| \quad \text{and} \quad g_m^c := \sup_{\mathscr{G}_k \in \mathbb{G}_k,\, k\le m} \|I - \mathscr{G}_k\|.$$

  Below we shall also use the variant

$$\tilde{g}_m := \sup_{\substack{\mathscr{G}' < \mathscr{G} \\ \mathscr{G}\in\mathbb{G}_k,\, k\le m}} \|\mathscr{G} - \mathscr{G}'\|,$$

  where $\mathscr{G}' < \mathscr{G}$ means that $A(x,\mathscr{G}') \subset A(x,\mathscr{G})$ for all x; see [5].

- Super-democracy parameters:

$$\tilde{\mu}_m = \sup_{\substack{|A|=|B|\le m \\ |\varepsilon|=|\eta|=1}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \quad \text{and} \quad \tilde{\mu}_m^d = \sup_{\substack{|A|=|B|\le m,\, A\cap B=\emptyset \\ |\varepsilon|=|\eta|=1}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}.$$

- Quasi-greedy parameters for constant coefficients (see [5, (3.11)])

$$\gamma_m = \sup_{\substack{|\varepsilon|=1 \\ B\subset A,\, |A|\le m}} \frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|}.$$

Note that $\gamma_m \le g_m \le \tilde{g}_m \le 2g_m$, but in general γ_m may be much smaller than g_m; see e.g. [5, §5.5]. Likewise, in §5 below we show that $\tilde{\mu}_m^d$ may be much smaller than $\tilde{\mu}_m$, except for Schauder bases in which both quantities turn out to be equivalent; see Theorem 5.2.

Our first result is a general upper bound, which improves and extends [19, Theorem 2.4].

Theorem 1.1. *Let $\mathscr{B}$ be an M-basis in $\mathbb{X}$, and let $\mathfrak{K} = \sup_{n,j} \|\mathbf{e}_n^*\|\|\mathbf{e}_j\|$. Then,*

$$\mathbf{L}_m^{\mathrm{ch},t} \le 1 + (1 + \tfrac{1}{t})\,\mathfrak{K} m, \quad \forall\, m \in \mathbb{N},\ t \in (0,1]. \tag{1.4}$$

Moreover, there exists a pair $(\mathbb{X},\mathscr{B})$ where the equality is attained for all m and t.

The second result is a slight generalization of [10, Theorem 4.1], and gives a correct version of [19, Theorem 3.5].

Theorem 1.2. *Let $\mathscr{B}$ be an M-basis in $\mathbb{X}$. Then, for all $m \geq 1$ and $t \in (0,1]$,*

$$\mathbf{L}_m^{\mathrm{ch},t} \;\leq\; g_{2m}^c + \frac{2}{t}\min\big\{\, \tilde{g}_m\tilde{\mu}_m\,,\; \gamma_{2m}\tilde{g}_{2m}\tilde{\mu}_m^d\,\big\}. \tag{1.5}$$

Our next result concerns lower bounds for $\mathbf{L}_m^{\mathrm{ch},t}$, for which we need to introduce weaker versions of the democracy parameters with an additional separation condition. For two finite sets $A,B \subset \mathbb{N}$ and $c \geq 1$, the notation $A > cB$ will stand for $\min A > c \max B$.

- Given an integer $c \geq 2$, we define

$$\vartheta_{m,c} := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \;:\; |\varepsilon| = |\eta| = 1,\ |A| = |B| \leq m \ \text{ with } A > cB \text{ or } B > cA \right\}. \tag{1.6}$$

Theorem 1.3. *If $\mathscr{B}$ is a Cesàro basis in $\mathbb{X}$ with constant β, then for every $c \geq 2$*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta^2}\frac{c-1}{c+1}\vartheta_{m,c}, \quad \forall\, m \in \mathbb{N},\ t \in (0,1].$$

We shall also establish, in Theorem 3.10 below, a similar lower bound valid for more general M-bases (not necessarily of Cesàro type), in terms of a new parameter θ_m which is invariant under rearrangements of $\mathscr{B}$.

Remark 1.4. One may compare the bounds for $\mathbf{L}_m^{\mathrm{ch}}$ above with those for $\mathbf{L}_m$ given in [5]

$$(1)\ \mathbf{L}_m \leq 1 + 3\mathfrak{K}m, \qquad (2)\ \mathbf{L}_m \leq k_{2m}^c + \tilde{g}_m\tilde{\mu}_m, \qquad \text{and} \qquad (3)\ \mathbf{L}_m \geq \tilde{\mu}_m^d,$$

which illustrate a slightly better behavior of the Chebishev TGA. Observe that one also has the trivial inequalities

$$\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t \leq k_m^c\,\mathbf{L}_m^{\mathrm{ch},t}.$$

Indeed, $\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t$ is direct by definition, while $\mathbf{L}_m^t \leq k_m^c\mathbf{L}_m^{\mathrm{ch},t}$ can be proved as follows: take $x \in \mathbb{X}$ and $A = \operatorname{supp}\mathscr{G}_m^t(x)$. Pick a Chebyshev greedy operator $\mathfrak{CG}_m^t$ such that $\operatorname{supp}\mathfrak{CG}_m^t(x) = A$. Then

$$\|x - \mathscr{G}_m^t(x)\| = \|(I-P_A)x\| = \|(I-P_A)(x - \mathfrak{CG}_m^t(x))\| \leq k_m^c\|x - \mathfrak{CG}_m^t(x)\|,$$

so $\mathbf{L}_m^t \leq k_m^c\mathbf{L}_m^{\mathrm{ch},t}$. Hence, when $\mathscr{B}$ is unconditional then $\mathbf{L}_m^t \approx \mathbf{L}_m^{\mathrm{ch},t}$. However for all conditional quasi-greedy and democratic bases we have $\mathbf{L}_m^{\mathrm{ch}} = O(1)$, but $\mathbf{L}_m \to \infty$.

The paper is organized as follows. Section 2 is devoted to preliminary lemmas. In Section 3 we prove Theorems 1.1, 1.2 and 1.3, and also establish the more general lower bound in Theorem 3.10, giving various situations in which it applies. Section 4 is devoted to examples illustrating the optimality of the results; in particular, an optimal bound of $\mathbf{L}_m^{\mathrm{ch}}$ for the trigonometric system in $L^1(\mathbb{T})$, settling a question left open in [19]. In Section 5 we investigate the equivalence between $\tilde{\mu}_m^d$ and $\tilde{\mu}_m$ and show Theorem 5.2. Finally, in Section 6 we study the convergence of $\mathfrak{CG}_m(x)$ and $\mathscr{G}_m(x)$ to x, pointing out the role of a *strong* M-basis assumption for such results.

2. Preliminary results

We recall some basic concepts and results that will be used later in the paper; see [8, 5]. For each $\alpha > 0$ we define the α-truncation of a scalar $y \in \mathbb{K}$ as

$$T_\alpha(y) = \alpha \operatorname{sign} y \ \text{ if } |y| \geq \alpha, \quad \text{and} \quad T_\alpha(y) = y \ \text{ if } |y| \leq \alpha.$$

We extend T_α to an operator in $\mathbb{X}$ by formally assigning $T_\alpha(x) \sim \sum_{n=1}^{\infty} T_\alpha(\mathbf{e}_n^*(x))\mathbf{e}_n$, that is

$$T_\alpha(x) := \alpha \mathbf{1}_{\varepsilon\Lambda_\alpha(x)} + (I - P_{\Lambda_\alpha(x)})(x),$$

where $\Lambda_\alpha(x) = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. Of course, this operator is well defined since $\Lambda_\alpha(x)$ is a finite set. In [5] we can find the following result:

Lemma 2.1. [5, Lemma 2.5] *For all $\alpha > 0$ and $x \in \mathbb{X}$, we have*

$$\|T_\alpha(x)\| \leq g^c_{|\Lambda_\alpha(x)|}\|x\|.$$

We also need a well known property from [8, 9], formulated as follows.

Lemma 2.2. [5, Lemma 2.3] *If $x \in \mathbb{X}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$, then*

$$\min_{n\in G} |\mathbf{e}_n^*(x)|\|\mathbf{1}_{\varepsilon G}\| \leq \tilde{g}_{|G|}\|x\|, \quad \forall G \in G(x,m,1). \tag{2.1}$$

The following version of (2.1), valid even if G is not greedy, improves [10, Lemma 2.2].

Lemma 2.3. *Let $x \in \mathbb{X}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. For every set finite $A \subset \mathbb{N}$, if $\alpha = \min_{n\in A} |\mathbf{e}_n^*(x)|$, then*

$$\alpha\|\mathbf{1}_{\varepsilon A}\| \leq \gamma_{|A\cup\Lambda_\alpha(x)|}\, \tilde{g}_{|A\cup\Lambda_\alpha(x)|}\|x\|, \tag{2.2}$$

where $\Lambda_\alpha(x) = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$.

Proof. Call $G = A \cup \Lambda_\alpha(x)$, and notice that it is a greedy set for x. Then,

$$\alpha\|\mathbf{1}_{\varepsilon A}\| \leq \alpha\, \gamma_{|G|}\|\mathbf{1}_{\varepsilon G}\| \leq \gamma_{|G|}\, \tilde{g}_{|G|}\, \|x\|,$$

using (2.1) in the last step. □

Remark 2.4. The following is a variant of (2.2) with a different constant

$$\min_{n\in A} |\mathbf{e}_n^*(x)|\; \|\mathbf{1}_{\varepsilon A}\| \leq k_{|A|}\, \|x\|. \tag{2.3}$$

A similar proof as the one in Lemma 2.3 can be seen in [4, Proposition 2.5].

Finally, we need the following elementary result, which follows directly from the convexity of the norm; see e.g [25, p. 108] (or [5, Lemma 2.7] if $\mathbb{K} = \mathbb{C}$).

Lemma 2.5. *For all finite sets $A \subset \mathbb{N}$ and scalars $a_n \in \mathbb{K}$ it holds*

$$\Big\|\sum_{n\in A} a_n \mathbf{e}_n\Big\| \leq \max_{n\in A} |a_n| \sup_{|\varepsilon|=1} \big\|\mathbf{1}_{\varepsilon A}\big\|.$$

3. Proof of the main results

3.1. **Proof of Theorem 1.1.** Let $x \in \mathbb{X}$ and $\mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t}$ be a fixed Chebyshev t-greedy operator. Let $A = A(x, \mathfrak{CG}_m^t) \in G(x,m,t)$. Pick any $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $|B| = m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \|x - P_{A\cap B}(x)\| \leq \|P_{B\setminus A}(x)\| + \|x - P_B(x)\|.$$

On the one hand, using (1.2),

$$\|P_{B\setminus A}(x)\| \leq \sup_n \|\mathbf{e}_n\| \sum_{j\in B\setminus A} |\mathbf{e}_j^*(x)| \leq \frac{1}{t} \sup_n \|\mathbf{e}_n\| \sum_{j\in A\setminus B} |\mathbf{e}_j^*(x-z)| \leq \frac{1}{t}\mathfrak{K}m\|x - z\|.$$

On the other hand, using the inequality (3.9) of [5],

$$\|x - P_B(x)\| = \|(I - P_B)(x - z)\| \leq k_m^c\|x - z\| \leq (1 + \mathfrak{K}m)\|x - z\|.$$

Hence, $\mathbf{L}_m^{\mathrm{ch},t} \leq 1 + \big(1+\frac{1}{t}\big)\,\mathfrak{K}m$. Finally, the fact that the equality in (1.4) can be attained is witnessed by Examples 4.1 and 4.2 below.

3.2. **Proof of Theorem 1.2.** The scheme of the proof follows the lines in [8, Theorem 3.2] and [10, Theorem 4.1], with some additional simplifications introduced in [5].

Given $x \in \mathbb{X}$ and $\mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t}$, let $A = A(x, \mathfrak{CG}_m^t) \in G(x,m,t)$. Pick any $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $|B| = m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m^t x\| \leq \|x-p\|, \quad \text{for any } p = \textstyle\sum_{n\in A} a_n \mathbf{e}_n. \tag{3.1}$$

We make the selection of p suggested in [8]. Namely, if $\alpha = \max_{n\notin A} |\mathbf{e}_n^*(x)|$, we let

$$p = P_A(x) - P_A\big(T_\alpha(x-z)\big).$$

It is easily verified that

$$\begin{aligned} x-p &= (I-P_A)\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z) \\ &= P_{B\setminus A}\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z). \end{aligned} \tag{3.2}$$

Since $\Lambda_\alpha(x-z) = \{n \,:\, |\mathbf{e}_n^*(x-z)| > \alpha\} \subset A \cup B$, then Lemma 2.1 gives

$$\big\|T_\alpha(x-z)\big\| \leq g_{2m}^c \|x-z\|. \tag{3.3}$$

Next we treat the first term in (3.2). Observe that $\max_{n\in B\setminus A} |\mathbf{e}_n^*(x - T_\alpha(x-z))| \leq 2\alpha$, so Lemma 2.5 gives

$$\begin{aligned} \big\|P_{B\setminus A}\big(x - T_\alpha(x-z)\big)\big\| &\leq 2\alpha \sup_{|\varepsilon|=1} \big\|\mathbf{1}_{\varepsilon(B\setminus A)}\big\| \\ &\leq \frac{2}{t} \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)| \sup_{|\varepsilon|=1} \big\|\mathbf{1}_{\varepsilon(B\setminus A)}\big\| = (*). \end{aligned} \tag{3.4}$$

At this point we have two possible approaches. Let $\eta_n = \mathrm{sign}\,[e_n^*(x-z)]$. In the first approach we pick a greedy set $\Gamma \in G(x-z, |A\setminus B|, 1)$, and control (3.4) by

$$(*) \leq \frac{2}{t} \min_{n\in\Gamma} |\mathbf{e}_n^*(x-z)|\; \tilde{\mu}_m \big\|\mathbf{1}_{\eta\Gamma}\big\| \leq \frac{2}{t}\, \tilde{\mu}_m\, \tilde{g}_m \|x-z\|, \tag{3.5}$$

using Lemma 2.2 in the last step. In the second approach, we argue as follows

$$(*) \leq \frac{2}{t} \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)|\; \tilde{\mu}_m^d \big\|\mathbf{1}_{\eta(A\setminus B)}\big\| \leq \frac{2}{t}\, \gamma_{2m}\, \tilde{g}_{2m}\, \tilde{\mu}_m^d\, \|x-z\|, \tag{3.6}$$

using in the last step Lemma 2.3 and the fact that, if $\delta = \min_{A\setminus B} |\mathbf{e}_n^*(x-z)|$, then the set $(A\setminus B) \cup \{n \,:\, |\mathbf{e}_n^*(x-z)| > \delta\} \subset A\cup B$ and hence has cardinality $\leq 2m$.

We can now combine the estimates displayed in (3.1)-(3.6) and obtain

$$\|x - \mathfrak{CG}_m^t x\| \leq \Big[g_{2m}^c + \frac{2}{t} \min\big\{\, \tilde{g}_m \tilde{\mu}_m\,,\; \gamma_{2m} \tilde{g}_{2m} \tilde{\mu}_m^d \,\big\}\Big]\, \|x-z\|,$$

which after taking the infimum over all z establishes Theorem 1.2. □

Remark 3.1. In [19, Theorem 3.5] a stronger inequality is stated (for $t=1$), namely

$$\mathbf{L}_m^{\mathrm{ch}} \leq g_{2m}^c + 2\tilde{g}_m \tilde{\mu}_m^d. \tag{3.7}$$

The proof, however, seems to contain a gap, and a missing factor k_m^c should also appear in the last summand. Nevertheless, it is still fair to ask whether the inequality (3.7) asserted in [19] may be true with a different proof.

Remark 3.2. Using Remark 2.4 in place of Lemma 2.3 in (3.6) above leads to an alternative and slightly simpler estimate

$$\mathbf{L}_m^{\mathrm{ch},t} \;\leq\; g_{2m}^c + \frac{2}{t}\, k_m \tilde{\mu}_m^d. \tag{3.8}$$

However, this would not be as efficient as (1.5) when $\mathscr{B}$ is quasi-greedy and conditional.

Remark 3.3. When $\mathscr{B}$ is quasi-greedy with constant $\mathbf{q} = \sup_m g_m < \infty$, then Theorem 1.2 implies the following

$$\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{q} + 4t^{-1}\, \mathbf{q}^2\, \tilde{\mu}_m^d.$$

This is a slight improvement with respect to [10, Theorem 4.1].

3.3. **Proof of Theorem 1.3.** Recall that $S_N = \sum_{n=1}^N \mathbf{e}_n^*(\cdot)\mathbf{e}_n$ and

$$F_N(x) = \frac{1}{N}\sum_{n=1}^N S_n(x) = \sum_{n=1}^N \big(1 - \frac{n-1}{N}\big)\mathbf{e}_n^*(x)\mathbf{e}_n.$$

For $M > N$ we define the operators (of de la Vallée-Poussin type)

$$\begin{aligned} V_{N,M}(x) &= \frac{M}{M-N} F_M(x) - \frac{N}{M-N} F_N(x) \\ &= \sum_{n=1}^N \mathbf{e}_n^*(x)\mathbf{e}_n + \sum_{n=N+1}^M \big(1 - \frac{n-N-1}{M-N}\big)\, \mathbf{e}_n^*(x)\mathbf{e}_n. \end{aligned} \tag{3.9}$$

In particular, observe that, for β as in (1.1) we have

$$\max\big\{\|V_{N,M}\|, \|I - V_{N,M}\|\big\} \;\leq\; \frac{M+N}{M-N}\,\beta. \tag{3.10}$$

We next prove that, if $c \geq 2$, then for all $A, B \subset \mathbb{N}$ such that $B > cA$ with $|A| = |B| \leq m$ it holds

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta}\,\frac{c-1}{c+1}\,\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\, |\varepsilon| = |\eta| = 1. \tag{3.11}$$

Pick any set $C > B$ such that $|B \cup C| = m$, and let

$$x = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_{\eta B} + t\mathbf{1}_C.$$

Then $B \cup C \in G(x, m, t)$, and hence there is a Chebyshev t-greedy operator so that

$$x - \mathfrak{CG}_m^t(x) = \mathbf{1}_{\varepsilon A} + \sum_{n \in B \cup C} a_n \mathbf{e}_n,$$

for some scalars $a_n \in \mathbb{K}$. Clearly,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \mathbf{L}_m^{\mathrm{ch},t}\sigma_m(x) \;\leq\; \mathbf{L}_m^{\mathrm{ch},t}\, \|t\mathbf{1}_{\eta B}\|,$$

using $z = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_C$ an m-term approximant. On the other hand, let $N = \max A$. Since $\min B \cup C > cN$, then (3.9) yields

$$V_{N,cN}(x - \mathfrak{CG}_m^t x) = \mathbf{1}_{\varepsilon A}.$$

Therefore, (3.10) implies that

$$\|x - \mathfrak{CG}_m^t(x)\| \geq \frac{\|V_{N,cN}(x - \mathfrak{CG}_m^t x)\|}{\|V_{N,cN}\|} \geq \frac{c-1}{(c+1)\beta}\, \|\mathbf{1}_{\varepsilon A}\|.$$

We have therefore proved (3.11).

We next show that when $|A| = |B| \le m$ satisfy $A > cB$ then

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{t\beta^2} \frac{c-1}{c+1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\, |\varepsilon| = |\eta| = 1. \tag{3.12}$$

This together with (3.11) is enough to establish Theorem 1.3. We shall actually show a slightly stronger result:

Lemma 3.4. *Let $|A| = |B| \le m$ and let $y \in \mathbb{X}$ be such that $|y|_\infty := \sup_n |\mathbf{e}_n^*(y)| \le 1$ and $A > c(B \cup \operatorname{supp} y)$. Then*

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{t\beta^2} \frac{c-1}{c+1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B} + y\|}, \quad \forall\, |\varepsilon| = |\eta| = 1. \tag{3.13}$$

Observe that the case $y = 0$ in (3.13) yields (3.12). We now show (3.13). Pick a large integer $\lambda > 1$ and a set $C > \lambda A$ such that $|B \cup C| = m$. Let

$$x = \mathbf{1}_{\varepsilon A} + ty + t\mathbf{1}_{\eta B} + t\mathbf{1}_C.$$

As before, $B \cup C \in G(x, m, t)$, and hence for some Chebyshev t-greedy operator we have

$$x - \mathfrak{CG}_m^t(x) = \mathbf{1}_{\varepsilon A} + ty + \sum_{n \in B \cup C} a_n \mathbf{e}_n,$$

for suitable scalars $a_n \in \mathbb{K}$. Choosing $\mathbf{1}_{\varepsilon A} + t\mathbf{1}_C$ as m-term approximant of x we see that

$$\|x - \mathfrak{CG}_m^t(x)\| \le \mathbf{L}_m^{\mathrm{ch},t} \sigma_m(x) \,\le\, \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\eta B} + y\|.$$

On the other hand, calling $N = \max(B \cup \operatorname{supp} y)$ and $L = \max A$ we have

$$(I - V_{N,cN}) \circ V_{L,\lambda L}\big(x - \mathfrak{CG}_m^t x\big) = \mathbf{1}_{\varepsilon A}$$

Thus,

$$\|x - \mathfrak{CG}_m^t(x)\| \ge \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|I - V_{N,cN}\| \|V_{L,\lambda L}\|} \ge \frac{c-1}{(c+1)\beta} \frac{\lambda - 1}{(\lambda+1)\beta} \|\mathbf{1}_{\varepsilon A}\|.$$

Therefore we obtain

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{t\beta^2} \frac{c-1}{c+1} \frac{\lambda-1}{\lambda+1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B} + y\|}$$

which letting $\lambda \to \infty$ yields (3.13). This completes the proof of Lemma 3.4, and hence of Theorem 1.3.

Remark 3.5. When $\mathscr{B}$ is a Schauder basis, a similar proof gives the following lower bound, which is also obtained in [19, Theorem 2.2]

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{(K_b+1)t} \sup\Big\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \;:\; |A| = |B| = m,\ A > B \text{ or } B > A\ ,\ |\varepsilon| = |\eta| = 1 \Big\}.$$

The statement for Cesàro bases, however, will be needed for the applications in §4.3.

3.4. **Lower bounds for general M-bases.** Observe that

$$\vartheta_{m,c} = \sup_{|A| \le m} \vartheta_c(A), \quad \text{where} \quad \vartheta_c(A) = \sup_{\substack{B \,:\, |B|=|A| \\ B > cA \\ \varepsilon,\eta \in \Upsilon}} \max\Big\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \Big\}.$$

We consider a new parameter

$$\vartheta_m = \sup_{|A| \le m} \inf_{c \ge 1} \vartheta_c(A). \tag{3.14}$$

We remark that, unlike $\vartheta_{m,c}$, the parameter ϑ_m depends on $\{\mathbf{e}_n\}_{n=1}^\infty$ but not on the reorderings of the system. We shall give a lower bound for $\mathbf{L}_m^{\mathrm{ch},t}$ in terms of ϑ_m in a less restrictive situation than the Cesàro basis assumption on $\{\mathbf{e}_n\}_{n=1}^\infty$.

Given $\rho \geq 1$, we say that $\{\mathbf{e}_n\}_{n=1}^\infty$ is ρ-admissible if the following holds: *for each finite set $A \subset \mathbb{N}$, there exists $n_0 = n_0(A)$ such that, for all sets B with* $\min B \geq n_0$ *and* $|B| \leq |A|$,

$$\big\| \sum_{n\in A} \alpha_n \mathbf{e}_n \big\| \leq \rho \,\big\| \sum_{n\in A\cup B} \alpha_n \mathbf{e}_n \big\|, \quad \forall\, \alpha_n \in \mathbb{K}. \tag{3.15}$$

Observe that (3.15) implies that

$$\big\| \sum_{n\in B} \alpha_n \mathbf{e}_n \big\| \leq (\rho+1) \,\big\| \sum_{n\in A\cup B} \alpha_n \mathbf{e}_n \big\|, \quad \forall\, \alpha_n \in \mathbb{K}. \tag{3.16}$$

This condition is clearly satisfied by all Schauder and Cesàro bases (with $\rho = K_b$ or $\rho > \beta$), but we shall see below that it also holds in more general situations.

Proposition 3.6. *Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be an M-basis such that $\{\mathbf{e}_n\}_{n=1}^\infty$ is ρ-admissible. Then*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{\vartheta_m}{(\rho+1)t}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0,1]. \tag{3.17}$$

Proof. Fix $A \subset \mathbb{N}$ such that $|A| \leq m$. Choose C disjoint with A such that $|A \cup C| = m$. Let $n_0 = n_0(A \cup C)$ as in the above definition, which we may assume larger than $\max A \cup C$. Pick any B with $\min B \geq n_0$ and $|B| = |A|$, and any $\varepsilon, \eta \in \Upsilon$. Let $x = t\mathbf{1}_{\varepsilon A} + t\mathbf{1}_C + \mathbf{1}_{\eta B}$. Then $A \cup C \in G(x,m,t)$, and there is a Chebyshev t-greedy operator with $\mathfrak{CG}_m^t(x)$ supported in $A \cup C$. Thus,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \mathbf{L}_m^{\mathrm{ch},t}\, \sigma_m(x) \leq \mathbf{L}_m^{\mathrm{ch},t}\, \|x - (\mathbf{1}_{\eta B} + t\mathbf{1}_C)\| = \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\varepsilon A}\|.$$

On the other hand, using the property in (3.16) one obtains

$$\|x - \mathfrak{CG}_m^t(x)\| \geq \frac{\|\mathbf{1}_{\eta B}\|}{\rho+1}.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch},t} \;\geq\; \frac{1}{(\rho+1)t}\, \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}.$$

We now assume additionally that $\min B \geq n_0 + m$, and pick $D \subset [n_0, n_0+m-1]$ such that $|B| + |D| = m$. Let $y = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_{\eta B} + t\mathbf{1}_D$. Then $B \cup D \in G(y,m,t)$ and a similar reasoning gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\rho} \leq \|y - \mathfrak{CG}_m^t(y)\| \leq \mathbf{L}_m^{\mathrm{ch},t}\, \sigma_m(y) \leq \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\eta B}\|.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch},t} \;\geq\; \frac{1}{(\rho+1)t} \max\left\{ \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}, \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \right\},$$

and taking the supremum over all $|B| = |A|$ with $B \geq (n_0+m)A$ and all $\varepsilon, \eta \in \Upsilon$, we see that

$$\mathbf{L}_m^{\mathrm{ch},t} \;\geq\; \frac{\vartheta_{n_0+m}(A)}{(\rho+1)t} \;\geq\; \frac{\inf_{c\geq 1} \vartheta_c(A)}{(\rho+1)t}.$$

Finally, a supremum over all $|A| \leq m$ leads to (3.17). □

We now give some general conditions in $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^\infty$ and $\mathbb{X}$ under which ρ-admissibility holds. We recall a few standard definitions; see e.g. [13]. We use the notation $[\mathbf{e}_n]_{n\in A}=\overline{\text{span}}\{\mathbf{e}_n\}_{n\in A}$, for $A\subset\mathbb{N}$. A sequence $\{\mathbf{e}_n\}_{n=1}^\infty$ is *weakly null* if

$$\lim_{n\to\infty} x^*(\mathbf{e}_n)=0, \quad \forall\, x^*\in\mathbb{X}^*.$$

Given a subset $Y\subset\mathbb{X}^*$, we shall say that $\{\mathbf{e}_n\}_{n=1}^\infty$ is *Y-null* if

$$\lim_{n\to\infty} y(\mathbf{e}_n)=0, \quad \forall\, y\in Y.$$

Given $\kappa\in(0,1]$, we say that a set $Y\subset\mathbb{X}^*$ is κ-norming whenever

$$\sup_{x^*\in Y,\|x^*\|\le 1} |x^*(x)| \,\ge\, \kappa\|x\|, \quad \forall\, x\in\mathbb{X}.$$

We finally introduce a new abstract definition.

Definition 3.7. We say that a biorthogonal system $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^\infty\subset\mathbb{X}\times\mathbb{X}^*$ satisfies the *property $\mathscr{P}(\kappa)$*, for some $0<\kappa\le 1$, if the sequence $\{\|\mathbf{e}_n^*\|\,\mathbf{e}_n\}_{n=1}^\infty\subset\mathbb{X}$ is Y-null, for some subset $Y\subset\mathbb{X}^*$ which is κ-norming.

We remark that in every separable Banach space $\mathbb{X}$ there exists an M-basis $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^\infty$ with the property $\mathscr{P}(1)$; see e.g. [21, Theorem III.8.5][1]. Other examples are given in Remark 3.9 below.

Proposition 3.8. *Let $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^\infty$ be a biorthogonal system in $\mathbb{X}\times\mathbb{X}^*$ with the property $\mathscr{P}(\kappa)$. Then $\{\mathbf{e}_n\}_{n=1}^\infty$ is ρ-admissible for every $\rho>1/\kappa$.*

Proof. Let $Y\subset\mathbb{X}^*$ be the κ-norming set from Definition 3.7. Consider a finite set $A\subset\mathbb{N}$ with say $|A|=m$ and denote

$$E:=[\mathbf{e}_n]_{n\in A}.$$

Given $\varepsilon>0$, one can find a finite set $S\subset Y\cap\{x^*\in\mathbb{X}^*\ :\ \|x^*\|=1\}$ so that

$$\max_{x^*\in S}|x^*(e)|\ge(1-\varepsilon)\kappa\|e\|, \quad \forall\, e\in E. \tag{3.18}$$

Indeed, it suffices to verify the above inequality for e of norm 1. Pick an $\varepsilon\kappa/2$-net $(z_k)_{k=1}^N$ in the unit sphere of E. For any k find a norm one $z_k^*\in Y$ so that $|z_k^*(z_k)|>(1-\varepsilon/2)\kappa$. We claim that $S=\{z_k^*: 1\le k\le N\}$ has the desired properties. To see this, pick a norm one $e\in E$, and find k with $\|e-z_k\|\le\varepsilon\kappa/2$. Then

$$\max_{x^*\in S}|x^*(e)|\ge|z_k^*(e)|\ge|z_k^*(z_k)|-\|e-z_k\|\ge(1-\varepsilon/2)\kappa-\varepsilon\kappa/2=(1-\varepsilon)\kappa.$$

Next, since the sequence $\{\|\mathbf{e}_n^*\|\,\mathbf{e}_n\}$ is Y-null, for each $\delta>0$ we can find an integer $n_0>\max A$ so that

$$\max_{x^*\in S}|x^*(\mathbf{e}_n)|\,\|\mathbf{e}_n^*\|\,\le\,\frac{\delta\kappa}{m}, \quad \forall\, n\ge n_0.$$

Pick any B of cardinality m with $\min B\ge n_0$, and let

$$G:=[\mathbf{e}_n]_{n\in B}.$$

[1]The M-basis constructed in [21] satisfies that $Y=[\mathbf{e}_n^*]_{n\in\mathbb{N}}$ is 1-norming and $\sup_{n\in\mathbb{N}}\|\mathbf{e}_n\|\,\|\mathbf{e}_n^*\|<\infty$. But the latter easily implies that $\{\|\mathbf{e}_n^*\|\,\mathbf{e}_n\}_{n\ge1}$ is Y-null.

For $f = \sum_{n\in B} \mathbf{e}_n^*(f)\mathbf{e}_n \in G$, we have

$$\max_{x^*\in S} |x^*(f)| \le \max_{x^*\in S} \sum_{n\in B} |x^*(\mathbf{e}_n)|\, \|\mathbf{e}_n^*\|\, \|f\| \,\le\, \delta\kappa\|f\|. \tag{3.19}$$

We claim that

$$\|e+f\| \ge \frac{(1-\varepsilon-\delta)\kappa}{1+\delta\kappa}\,\|e\|, \quad \text{for any } e\in E,\, f\in G. \tag{3.20}$$

To show this, we fix $\gamma > 0$ (to be chosen later), and assume first that $\|f\| \ge (1+\gamma)\|e\|$. Then,

$$\|e+f\| \ge \|f\| - \|e\| \ge \gamma\|e\|.$$

Next assume that $\|f\| < (1+\gamma)\|e\|$, then using (3.18) and (3.19) we obtain that

$$\|e+f\| \ge \max_{x^*\in S} |x^*(e+f)| \ge (1-\varepsilon)\kappa\|e\| - \delta\kappa\|f\| > (1-\varepsilon-\delta(1+\gamma))\kappa\|e\|.$$

We now choose γ so that $\gamma = (1-\varepsilon-\delta(1+\gamma))\kappa$, that is,

$$\gamma = \frac{(1-\varepsilon-\delta)\kappa}{1+\delta\kappa},$$

which shows the claim in (3.20). Now, given $\rho > 1/\kappa$, we may pick $\delta = \varepsilon$ sufficiently small so that the above number $\gamma > 1/\rho$. Then, (3.20) becomes

$$\|e+f\| \ge \frac{1}{\rho}\,\|e\|, \quad \text{for any } e\in [e_n]_{n\in A},\, f\in [e_n]_{n\in B},$$

for all B with $\min B \ge n_0$ and $|B| = |A| = m$. Thus, $\{\mathbf{e}_n\}_{n=1}^\infty$ is ρ-admissible. □

Remark 3.9. We give some more examples where property $\mathscr{P}(\kappa)$ holds.

(1) If the sequence $\{\|\mathbf{e}_n^*\|\,\mathbf{e}_n\}_{n=1}^\infty$ is weakly null then $\mathscr{P}(1)$ holds (since $Y = \mathbb{X}^*$ is always 1-norming).

(2) If $\{\mathbf{e}_n\}_{n=1}^\infty$ is a Schauder basis then $\mathscr{P}(\kappa)$ holds with $\kappa = 1/K_b$; see [20, Theorems I.3.1 and I.12.2].

(3) Let $\mathbb{X} = C(K)$, where K is a compact Hausdorff set, and let μ be a Radon probability measure in K with $\operatorname{supp}\mu = K$. Let $\{\mathbf{e}_n\}_{n=1}^\infty$ be a complete system in $\mathbb{X}$ which is orthonormal with respect to μ and uniformly bounded, that is,

$$\int_K \mathbf{e}_n\overline{\mathbf{e}_m}\,d\mu = \delta_{n,m} \quad \text{and} \quad \sup_n \|\mathbf{e}_n\|_\infty < \infty.$$

Then $\{\mathbf{e}_n\}_{n=1}^\infty$ has the property $\mathscr{P}(1)$ in $\mathbb{X} = C(K)$. Indeed, the sequence $\{\mathbf{e}_n\}_{n=1}^\infty$ is $L_1(\mu)$-null in $\mathbb{X}$, while $Y = L_1(\mu)$ is 1-norming in $\mathbb{X}$ (since the natural embedding of $C(K)$ into $L_\infty(\mu)$ is isometric). Specific examples are the trigonometric system in $C[0,1]$ (in the real or complex case), as well as certain polygonal versions of the Walsh system [7, 17, 27], or any reorderings of them (which may cease to be Cesàro bases).

(4) As a dual of the previous, if $\mathbb{X} = L^1(\mu)$ then every system $\{\mathbf{e}_n\}_{n=1}^\infty$ as in (3) is weakly null, and hence case (1) applies.

(5) If $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ is an M-basis such that

$$\varphi(m) := \sup_{|A|\le m} \Big\| \sum_{n\in A} \mathbf{e}_n \Big\| = \mathbf{o}(m), \quad \text{as} \quad m\to\infty,$$

then $\{\mathbf{e}_n\}_{n=1}^\infty$ is weakly null (and in particular, $\mathscr{P}(1)$ holds). Indeed, first note that also $\tilde{\varphi}(m) = \sup\{\|\mathbf{1}_{\eta A}\| : |A| \le m,\ |\eta| = 1\} = \mathbf{o}(m)$. Assume that the system is not weakly null. Then there exist a norm one $x^* \in \mathbb{X}^*$ and $\varepsilon_0 > 0$ so that the set $A = \{n \in \mathbb{N} : |x^*(\mathbf{e}_n)| \ge \varepsilon_0\}$ is infinite. For every $m \ge 1$, pick a set $F \subset A$ with $|F| = m$ and let $\eta_n = \mathrm{sign}[x^*(\mathbf{e}_n)]$; then

$$\tilde{\varphi}(m) \ge \|\mathbf{1}_{\overline{\eta} F}\| \ge |x^*(\sum_{n\in F} \overline{\eta_n}\mathbf{e}_n)| = \sum_{n\in F} |x^*(\mathbf{e}_n)| \ge m\varepsilon_0,$$

contradicting our assumption.

Finally, as a consequence of Propositions 3.6 and 3.8 one obtains

Theorem 3.10. *Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be a seminormalized M-basis with the property $\mathscr{P}(\kappa)$. Then, if ϑ_m is as in* (3.14)*, we have*

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{\kappa\,\vartheta_m}{(\kappa+1)t}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0,1]. \tag{3.21}$$

4. EXAMPLES

The first two examples are variants of those in [5, §5.1] and [6, §8.1].

4.1. **Example 4.1: The summing basis.** Let $\mathbb{X}$ be the closure of the set of all finite sequences $\mathbf{a} = (a_n)_n \in c_{00}$ with the norm

$$\|\mathbf{a}\| = \sup_m \Big|\sum_{n=1}^m a_n\Big|.$$

The canonical system $\mathscr{B} = \{\mathbf{e}_n\}_{n=1}^\infty$ is a Schauder basis in $\mathbb{X}$ with $K_b = 1$ and $\|\mathbf{e}_n\| = 1$ for all n. Also, $\|\mathbf{e}_1^*\| = 1$, $\|\mathbf{e}_n^*\| = 2$ if $n \ge 2$, so $\mathfrak{K} = 2$ in Theorem 1.1; see [5, §5.1]. We now show that, for this example of $(\mathbb{X}, \mathscr{B})$, the bound of Theorem 1.1 is sharp. As in [5, §5.1], we consider the element:

$$x = \Big(\underbrace{\frac12, \frac1t, \frac12}, \dots, \underbrace{\frac12, \frac1t, \frac12}; \frac12; \underbrace{-1, 1}, \dots, \underbrace{-1, 1}, 0, \dots\Big),$$

where we have m blocks of $\left(\frac12, \frac1t, \frac12\right)$ and m blocks of $(-1,1)$. Picking $A = \{n : x_n = -1\}$ as a t-greedy set of x, we see that

$$\begin{aligned}\|x - \mathfrak{CG}_m^t(x)\| &= \min_{a_i, i=1,\dots,m} \Big\|\Big(\frac12, \frac1t, \frac12, \dots, \frac12, \frac1t, \frac12; \frac12; a_1, 1, a_2, 1, \dots, a_m, 1, 0, \dots\Big)\Big\| \\ &\ge \Big\|\Big(\frac12, \frac1t, \frac12, \dots, \frac12, \frac1t, \frac12; \frac12; 0, \dots\Big)\Big\| = m + \frac{m}{t} + \frac12.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_m(x) &\le \Big\|x - \frac{t+1}{t}(0,1,0,\dots,0,1,0;0,\dots)\Big\| \\ &= \Big\|\Big(\frac12, -1, \frac12, \dots, \frac12, -1, \frac12; \frac12; -1, 1, \dots, -1, 1, 0 \dots\Big)\Big\| = \frac12.\end{aligned}$$

Hence, $\mathbf{L}_m^{\mathrm{ch},t} \ge 1 + 2(1 + \frac1t)m$ and we conclude that $\mathbf{L}_m^{\mathrm{ch},t} = 1 + 2(1 + \frac1t)m$ by Theorem 1.1. As a consequence, observe that in this case $\mathfrak{CG}_m^t(x) = 0$.

Remark 4.1. The above example strengthens [19, Theorem 2.4], where the authors are only able to show that $1 + 4m \le \mathbf{L}_m^{\mathrm{ch}} \le 1 + 6m$.

4.2. **Example 4.2: the difference basis.** Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be the canonical basis in $\ell^1(\mathbb{N})$ and define the elements

$$y_1 = \mathbf{e}_1,\ y_n = \mathbf{e}_n - \mathbf{e}_{n-1},\ n = 2,3,...$$

The new system $\mathscr{B} = \{y_n\}_{n=1}^{\infty}$ is called the difference basis of ℓ^1. We recall some basic properties used in [6, §8.1]. If $(b_n)_n \in c_{00}$ then

$$\|\sum_{n=1}^{\infty} b_n y_n\| = \sum_{n=1}^{\infty} |b_n - b_{n+1}|.$$

Also, $\mathscr{B}$ is a monotone basis with $\|y_1\| = 1$, $\|y_n\| = 2$ if $n \geq 2$, and $\|y_n^*\| = 1$ for all $n \geq 1$ (in fact, the dual system corresponds to the summing basis). So, $\mathfrak{K} = 2$ and Theorem 1.1 gives $\mathbf{L}_m^{\mathrm{ch},t} \leq 1 + 2(1+\frac{1}{t})m$ for all $t \in (0,1]$. To show the equality we consider the vector $x = \sum_n b_n y_n$ with coefficients (b_n) given by

$$\Big(1, \underbrace{1, 1, -\tfrac{1}{t}, 1}, ..., \underbrace{1, 1, -\tfrac{1}{t}, 1}, 0, ...\Big),$$

where the block $\Big(1, 1, \frac{-1}{t}, 1\Big)$ is repeated m times. If we take $\Gamma = \{2, 6, ..., 4m-2\}$ as a t-greedy set for x of cardinality m, then

$$\begin{aligned}
\|x - \mathfrak{CG}_m^t(x)\| &= \inf_{(a_j)_{j=1}^m} \|x - \sum_{j=1}^m a_j y_{4j-2}\| \\
&= \inf_{(a_j)_{j=1}^m} \Big\|\Big(1, 1-a_1, 1, \frac{-1}{t}, 1, ..., 1-a_m, 1, \frac{-1}{t}, 1, 0, ...\Big)\Big\| \\
&= \inf_{(a_j)_{j=1}^m} 2\sum_{j=1}^m |a_j| + 2m\Big(1+\frac{1}{t}\Big) + 1 = 2m\Big(1+\frac{1}{t}\Big) + 1.
\end{aligned}$$

Hence, in this case we also have $\mathfrak{CG}_m^t(x) = 0$. On the other hand

$$\sigma_m(x) \leq \Big\|x + (1+\tfrac{1}{t})\sum_{j=1}^m y_{4j}\Big\| = \|(1,1,1,1,1,...,1,1,1,1,0,...)\| = 1.$$

This shows that $\mathbf{L}_m^{\mathrm{ch},t} = 1 + 2(1+\frac{1}{t})m$.

4.3. **Example 4.3: the trigonometric system in $L^p(\mathbb{T})$.** Consider $\mathscr{B} = \{e^{inx}\}_{n\in\mathbb{Z}}$ in $L^p(\mathbb{T})$ for $1 \leq p < \infty$, and in $C(\mathbb{T})$ if $p = \infty$. In [22], Temlyakov showed that

$$c_p m^{|\frac{1}{p}-\frac{1}{2}|} \leq \mathbf{L}_m \leq 1 + 3m^{|\frac{1}{p}-\frac{1}{2}|},$$

for some $c_p > 0$ and all $1 \leq p \leq \infty$. Adapting his argument, Shao and Ye have recently established, in [19, Theorem 2.1], that for $1 < p \leq \infty$ it also holds

$$\mathbf{L}_m^{\mathrm{ch}} \approx m^{|\frac{1}{p}-\frac{1}{2}|}. \tag{4.1}$$

The case $p = 1$ is left as an open question, and only the estimate $\frac{\sqrt{m}}{\ln(m)} \lesssim \mathbf{L}_m^{\mathrm{ch}} \lesssim \sqrt{m}$ is given; see [19, (2.24)]. Moreover, the proof of the case $p = \infty$ seems to contain some gaps and may not be complete.

Here, we shall give a short proof ensuring the validity of (4.1) in the full range $1 \leq p \leq \infty$, with a reasoning similar to [5, §5.4]. More precisely, we shall prove the following.

Proposition 4.2. *Let $1 \le p \le \infty$. Then there exists $c_p > 0$ such that*

$$\mathbf{L}_m^{\mathrm{ch},t} \ge c_p t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0,1]. \tag{4.2}$$

We remark that in the cases $p = 1$ and $p = \infty$ the trigonometric system is not a Schauder basis, but it is a Cesàro basis[2]. So we may use the lower bounds in Theorem 1.3, namely

$$\mathbf{L}_m^{\mathrm{ch},t} \ge c'_p\, t^{-1} \sup_{\substack{|A|=|B|\le m \\ A>2B \text{ or } B>2A}} \sup_{|\varepsilon|=|\eta|=1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}. \tag{4.3}$$

- Case $1 < p \le 2$. Assume that $m = 2\ell + 1$ or $2\ell + 2$ (that is, $\ell = \lfloor \frac{m-1}{2} \rfloor$). We choose $B = \{-\ell, ..., \ell\}$, so that $\mathbf{1}_B = D_\ell$ is the ℓ-th Dirichlet kernel, and hence
$$\|\mathbf{1}_B\|_p = \|D_\ell\|_{L^p(\mathbb{T})} \approx m^{1-\frac{1}{p}}.$$
Next we take a lacunary set $A = \{2^j : j_0 \le j \le j_0 + 2\ell\}$, so that
$$\|\mathbf{1}_A\|_p \approx \sqrt{m}, \tag{4.4}$$
and where j_0 is chosen such that $2^{j_0} \ge m$, and hence $A > 2B$. Then, (4.3) implies
$$\mathbf{L}_m^{\mathrm{ch},t} \ge c_p\, t^{-1} \frac{m^{1/2}}{m^{1-\frac{1}{p}}} = c_p t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}.$$
- Case $2 \le p < \infty$. The same proof works in this case, just reversing the roles of A and B.
- Case $p = \infty$. We replace the lacunary set by a Rudin-Shapiro polynomial of the form
$$R(x) = e^{iNx} \sum_{n=0}^{2^L-1} \varepsilon_n e^{inx}, \quad \text{with } \varepsilon_n \in \{\pm 1\},$$
where L is such that $2^L \le m < 2^{L+1}$; see e.g. [14, p. 33]. Then, $R = \mathbf{1}_{\varepsilon B}$ with $B = N + \{0, 1, \dots, 2^L - 1\}$ and
$$\|\mathbf{1}_{\varepsilon B}\|_\infty = \|R\|_{L^\infty(\mathbb{T})} \approx \sqrt{m}.$$
If we pick $N \ge 2 \cdot 2^L$, then $B > 2A$ with $A = \{\pm 1, \dots, \pm(2^L - 1)\}$. Finally,
$$\|\mathbf{1}_A\|_\infty = \|D_{2^L-1} - 1\|_{L^\infty(\mathbb{T})} \approx m.$$
So, (4.3) implies the desired bound.
- Case $p = 1$. We use the lower bound in Lemma 3.4, namely
$$\mathbf{L}_m^{\mathrm{ch},t} \ge c'_1\, t^{-1} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B + y\|}, \tag{4.5}$$
for all $|A| = |B| \le m$ and all y such that $A > 2(B \cup \operatorname{supp} y)$ and $\sup_n |\mathbf{e}_n^*(y)| \le 1$. As before, let $m = 2\ell + 1$ or $2\ell + 2$, and choose the same sets A and B as in the case $1 < p \le 2$. Next choose y so that the vector
$$V_\ell = \mathbf{1}_B + y$$

[2]We equip $\mathscr{B}$ with its natural ordering $\{1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, \dots\}$.

is a de la Vallée-Poussin kernel as in [14, p. 15]. Then, the Fourier coeffients $\mathbf{e}_n^*(y)$ have modulus ≤ 1 and are supported in $\{n : \ell < |n| \leq 2\ell+1\}$, so the condition $A > 2(B \cup \operatorname{supp} y)$ holds if $2^{j_0} \geq 2m+1$. Finally,

$$\|\mathbf{1}_B + y\|_1 = \|V_\ell\|_{L^1(\mathbb{T})} \leq 3,$$

so the bound $\mathbf{L}_m^{\mathrm{ch},t} \gtrsim t^{-1}\sqrt{m}$ follows from (4.5).

Remark 4.3. Using the trivial upper bound $\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t \lesssim t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}$, we conclude that $\mathbf{L}_m^{\mathrm{ch},t} \approx t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}$ for all $1 \leq p \leq \infty$.

5. Comparison between $\tilde{\mu}_m$ and $\tilde{\mu}_m^d$

In this section we compare the democracy constants $\tilde{\mu}_m$ and $\tilde{\mu}_m^d$ defined in §1 above. Let us first note that

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (\tilde{\mu}_m^d)^2 \tag{5.1}$$

and

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (1+2\kappa)\gamma_m \tilde{\mu}_m^d, \tag{5.2}$$

where $\kappa = 1$ or 2 depending if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Indeed, the left inequality in (5.1) is immediate by definition, and the right one follows from

$$\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} = \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_C\|} \frac{\|\mathbf{1}_C\|}{\|\mathbf{1}_{\varepsilon A}\|} \leq (\tilde{\mu}_m^d)^2,$$

for any $|A| = |B| \leq m$ and any C disjoint with $A \cup B$ with $|C| = |A| = |B|$. Concerning the right inequality in (5.2), we use that if $|A| = |B| \leq m$ then

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\| + \|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\|}{\|\mathbf{1}_{\eta(B\setminus A)}\|} + \frac{\|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m \tilde{\mu}_m^d + 2\kappa\gamma_m,$$

using in the last step [5, Lemma 3.3]. From (5.2) we see that $\tilde{\mu}_m \approx \tilde{\mu}_m^d$ when $\mathscr{B}$ is quasi-greedy for constant coefficients.

In the next subsection we shall show that $\tilde{\mu}_m \approx \tilde{\mu}_m^d$ for all Schauder bases, a result which seems new in the literature.

5.1. Equivalence for Schauder bases.

We begin with a simple observation.

Lemma 5.1.

$$\tilde{\mu}_m^d = \sup\left\{\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} : |B| \leq |A| \leq m,\ A \cap B = \emptyset,\ |\varepsilon| = |\eta| = 1\right\}. \tag{5.3}$$

Proof. Let $|\varepsilon| = |\eta| = 1$ and $|B| \leq |A| \leq m$ with $A \cap B = \emptyset$. We must show that $\|\mathbf{1}_{\eta B}\|/\|\mathbf{1}_{\varepsilon A}\| \leq \tilde{\mu}_m^d$. Pick any set C disjoint with $A \cup B$ such that $|B| + |C| = |A|$. We now use the elementary inequality

$$\|x\| = \left\|\frac{x+y}{2} + \frac{x-y}{2}\right\| \leq \max\{\|x+y\|, \|x-y\|\}, \tag{5.4}$$

with $x = \mathbf{1}_{\eta B}$ and $y = \mathbf{1}_C$. Let $\eta' \in \Upsilon$ be such that $\eta'|_B = \eta|_B$ and $\eta'|_C = \pm 1$, according to the sign that reaches the maximum in (5.4). Then $\|\mathbf{1}_{\eta B}\| \leq \|\mathbf{1}_{\eta'(B\cup C)}\| \leq \tilde{\mu}_m^d \|\mathbf{1}_{\varepsilon A}\|$, and the result follows. □

Theorem 5.2. *If K_b is the basis constant and $\varkappa = \sup_n \|\mathbf{e}_n^*\| \|\mathbf{e}_n\|$, then*

$$\tilde{\mu}_m \leq 2(K_b+1)\tilde{\mu}_m^d + \varkappa K_b. \tag{5.5}$$

Proof. Let $|A| = |B| \leq m$, and $|\varepsilon| = |\eta| = 1$. Then

$$\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \leq \frac{\|\mathbf{1}_{\eta(B\setminus A)}\|}{\|\mathbf{1}_{\varepsilon A}\|} + \frac{\|\mathbf{1}_{\eta(B\cap A)}\|}{\|\mathbf{1}_{\varepsilon A}\|} = I + II.$$

Lemma 5.1 implies $I \leq \tilde{\mu}_m^d$. We now bound II. Pick an integer n_0 such that $A_1 = \{n \in A \,:\, n \leq n_0\}$ and $A_2 = A \setminus A_1$ satisfy

$$|A_1| = |A_2| \quad \text{(if } |A| \text{ is even)}, \quad \text{or} \quad |A_1| = \frac{|A|-1}{2} = |A_2| - 1 \quad \text{(if } |A| \text{ is odd)}.$$

Then

$$\begin{aligned} II &\leq \frac{\|\mathbf{1}_{\eta(B\cap A_1)}\|}{\|\mathbf{1}_{\varepsilon A}\|} + \frac{\|\mathbf{1}_{\eta(B\cap A_2)}\|}{\|\mathbf{1}_{\varepsilon A}\|} \\ &\leq (K_b+1)\frac{\|\mathbf{1}_{\eta(B\cap A_1)}\|}{\|\mathbf{1}_{\varepsilon A_2}\|} + K_b \frac{\|\mathbf{1}_{\eta(B\cap A_2)}\|}{\|\mathbf{1}_{\varepsilon A_1}\|} \;=\; II_1 + II_2, \end{aligned}$$

using in the second line the basis constant bound for the denominator. Since $|B\cap A_1| \leq |A_1| \leq |A_2|$, we see that

$$II_1 \leq (K_b+1)\tilde{\mu}_m^d.$$

On the other hand, picking any number $n_1 \in B\cap A_2$, and using $\|\mathbf{e}_{n_1}^*\| \|\mathbf{1}_{\varepsilon A}\| \geq |\mathbf{e}_{n_1}^*(\mathbf{1}_{\varepsilon A})| = 1$, we see that

$$II_2 \leq K_b \frac{\|\mathbf{1}_{\eta(B\cap A_2\setminus\{n_1\})}\|}{\|\mathbf{1}_{\varepsilon A_1}\|} + K_b \|\mathbf{e}_{n_1}\| \|\mathbf{e}_{n_1}^*\| \leq K_b \tilde{\mu}_m^d + \varkappa K_b,$$

the last bound due to $|B\cap A_2 \setminus \{n_1\}| \leq |A_2| - 1 \leq |A_1|$ and Lemma 5.1. Putting together the previous bounds easily leads to (5.5). □

Remark 5.3. A similar argument shows the equivalence of the standard (unsigned) democracy parameters

$$\mu_m = \sup_{|A|=|B|\leq m} \frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|} \quad \text{and} \quad \mu_m^d = \sup_{\substack{|A|=|B|\leq m \\ A\cap B=\emptyset}} \frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|}. \tag{5.6}$$

Indeed, in this case, the analog of (5.3) takes the weaker form

$$\mu_m^d \leq \sup_{\substack{|B|\leq|A|\leq m \\ A\cap B=\emptyset}} \frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|} \leq K_b \mu_m^d. \tag{5.7}$$

Then, (5.7) and the same proof we gave for Theorem 5.2 (with $\eta = \varepsilon \equiv 1$) leads to

$$\mu_m \leq 2(K_b+1)K_b\, \mu_m^d + \varkappa K_b. \tag{5.8}$$

5.2. **An example where $\tilde{\mu}_m$ grows faster than $\tilde{\mu}_m^d$.** The following example also seems to be new in the literature. As in (5.6), we denote by μ_m, μ_m^d the democracy parameters corresponding to constant signs.

Theorem 5.4. *There exists a Banach space $\mathbb{X}$ with an M-basis $\mathscr{B}$ such that*

$$\limsup_{m\to\infty}\frac{\tilde{\mu}_m}{[\tilde{\mu}_m^d]^{2-\varepsilon}}=\limsup_{m\to\infty}\frac{\mu_m}{[\mu_m^d]^{2-\varepsilon}}=\infty,\quad \forall\,\varepsilon>0.$$

Proof. Let $N_0=1$, and define recursively $N_k=2^{2^{N_{k-1}}}$, and $N_k'=N_1+\ldots+N_{k-1}$. Consider the blocks of integers

$$S_k=\big\{N_k'+1,\ldots,N_k'+N_k\big\},$$

and denote the tail blocks by $T_k=\cup_{j\ge k+1}S_j$. Finally, let

$$\mathfrak{N}_k=\big\{(\sigma_j)_{j\in S_k}\;:\;\sigma_j\in\{\pm1\}\quad\text{and}\quad\sum_{j\in S_k}\sigma_j=0\big\}.$$

We define a real Banach space $\mathbb{X}$ as the closure of c_{00} with the norm

$$\|x\|\;=\;\max\Big\{\;\|x\|_\infty,\;\sup_{k\ge1}\alpha_k\sup_{\sigma\in\mathfrak{N}_k}\big|\langle\mathbf{1}_{\sigma S_k},x\rangle\big|,\;\sup_{k\ge1}\beta_k\sup_{\substack{S\subset T_k\\|S|=N_k}}\sum_{j\in S}|x_j|\;\Big\},$$

where the weights α_k and β_k are chosen as follows:

$$\alpha_k=2^{-N_{k-1}}=\frac{1}{\log_2 N_k}\quad\text{and}\quad\beta_k=\frac{1}{\sqrt{N_k}}.$$

Observe that

$$N_k'=N_1+\ldots+N_{k-1}\le 2N_{k-1}=2\log_2\log_2 N_k\quad\text{and}\quad\frac{\alpha_k}{\beta_k}=\frac{\sqrt{N_k}}{\log_2 N_k}.$$

Claim 1: $\quad \tilde{\mu}_{N_k}\ge\mu_{N_k}\ge\dfrac{N_k/2}{(\log_2 N_k)\sqrt{\log_2\log_2 N_k}}$, for all $k\ge1$.

Proof. Pick any $A\subset S_k\cup S_{k+1}$ such that $|A|=N_k$ and $|A\cap S_k|=|A\cap S_{k+1}|=N_k/2$. Then

$$\|\mathbf{1}_A\|\ge\alpha_k N_k/2=\frac{N_k/2}{\log_2 N_k}.$$

Next, pick $B=S_k$, so that $|B|=|A|=N_k$ and

$$\|\mathbf{1}_B\|=\max\Big\{\;1,\;\alpha_k\cdot 0,\;\sup_{n\le k-1}\;\beta_n N_n\;\Big\}=\beta_{k-1}N_{k-1}=\sqrt{N_{k-1}}=\sqrt{\log_2\log_2 N_k}.$$

Then $\mu_{N_k}\ge\|\mathbf{1}_A\|/\|\mathbf{1}_B\|\ge\frac{N_k/2}{(\log_2 N_k)\sqrt{\log_2\log_2 N_k}}$. ◻

Claim 2: $\quad \mu_{N_k}^d\le\tilde{\mu}_{N_k}^d\le\sqrt{N_k}$, for all $k\ge2$.

Proof. Let A,B be any pair of disjoint sets with $|A|=|B|\le N_k$, and let $|\varepsilon|=|\eta|=1$. If $|A|=|B|\le\sqrt{N_k}$, then the trivial bounds $\|\mathbf{1}_{\varepsilon A}\|\le|A|$ and $\|\mathbf{1}_{\eta B}\|\ge1$ give

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}\le\sqrt{N_k}.$$

So, it remains to consider the cases $\sqrt{N_k}<|A|=|B|\le N_k$. We split A into three parts

$$A_0=A\cap S_k,\quad A_+=A\cap T_k,\quad A_-=A\cap[S_1\cup\ldots\cup S_{k-1}].$$

Then, we have the following upper bound

$$\begin{aligned}\|\mathbf{1}_{\varepsilon A}\| &\leq \max\Big\{1,\ \sup_{n<k}\alpha_n|A_-|,\ \alpha_k|A_0|,\ \sup_{n>k}\alpha_n N_k,\ \sup_{n<k}\beta_n N_n,\ \sup_{n\geq k}\beta_n|A|\ \Big\}\\ &\leq \max\Big\{\ N_k',\ \alpha_k|A_0|,\ \beta_k|A|\ \Big\},\end{aligned}$$

due to the elementary inequalities

- $\sup_{n<k}\alpha_n|A_-|\leq|A_-|\leq N_k'$
- $\sup_{n>k}\alpha_n N_k=\alpha_{k+1}N_k=N_k 2^{-N_k}\leq 1$
- $\sup_{n<k}\beta_n N_n=\sqrt{N_{k-1}}\leq N_{k-1}\leq N_k'$
- $\sup_{n\geq k}\beta_n|A|=\beta_k|A|$.

Moreover, since $\beta_k|A|\leq\min\{\beta_k N_k=\sqrt{N_k},\ \alpha_k|A|\ \}$, we derive

$$\|\mathbf{1}_{\varepsilon A}\|\leq\max\{\sqrt{N_k},\alpha_k|A_0|\}\quad\text{and}\quad\|\mathbf{1}_{\varepsilon A}\|\leq\max\{N_k',\alpha_k|A|\}. \tag{5.9}$$

We now give a lower bound for $\|\mathbf{1}_{\eta B}\|$. The key estimate will rely on the following

Lemma 5.5. *Let $B_0=B\cap S_k$ and $B_0^c=S_k\setminus B_0$. Then*

$$\sup_{\sigma\in\mathfrak{N}_k}\big|\langle\mathbf{1}_{\sigma S_k},\mathbf{1}_{\eta B_0}\rangle\big|\ \geq\ \min\{|B_0|,|B_0^c|\}. \tag{5.10}$$

Proof. If $|B_0|\leq N_k/2$, then we may select any $\sigma\in\mathfrak{N}_k$ such that $\sigma|_{B_0}=\eta$ (which is possible since $|B_0^c|\geq|B_0|$), which gives

$$|\langle\mathbf{1}_{\sigma S_k},\mathbf{1}_{\eta B_0}\rangle|=|B_0|=\min\{|B_0|,|B_0^c|\}.$$

Assume now that $|B_0|>N_k/2$. Pick any $S\subset B_0$ with $|S|=|B_0^c|=N_k-|B_0|$. Choose $\nu\in\{-1,1\}^{B_0^c}$ so that $\sum_{i\in S}\eta_i+\sum_{i\in B_0^c}\nu_i=0$. Choose $\tau\in\{-1,1\}^{B_0\setminus S}$ so that $\sum_{i\in B_0\setminus S}\tau_i=0$. Replacing τ by $-\tau$, if necessary, we may assume that $\sum_{i\in B_0\setminus S}\tau_i\eta_i\geq 0$. Finally, define $\sigma\in\mathfrak{N}_k$ by setting

$$\sigma|_S=\eta|_S,\quad\sigma|_{B_0^c}=\nu|_{B_0^c},\quad\sigma|_{B_0\setminus S}=\tau|_{B_0\setminus S}.$$

Then,

$$|\langle\mathbf{1}_{\sigma S_k},\mathbf{1}_{\eta B_0}\rangle|=\sum_{i\in S}\eta_i^2+\sum_{i\in B_0\setminus S}\tau_i\eta_i\geq|S|=|B_0^c|=\min\{|B_0|,|B_0^c|\}.\quad\square$$

From the lemma and the definition of the norm we see that

$$\|\mathbf{1}_{\eta B}\|\geq\max\Big\{\ 1,\ \alpha_k\min\{|B_0|,|B_0^c|\},\ \beta_k|B_+|\ \Big\}. \tag{5.11}$$

We shall finally combine the estimates in (5.9) and (5.11) to establish Claim 2. We distinguish two cases

Case 1: $\min\{|B_0|,|B_0^c|\}=|B_0^c|$. Then, since $A_0\subset B_0^c$, we see that

$$\alpha_k|A_0|\leq\alpha_k|B_0^c|\leq\|\mathbf{1}_{\eta B}\|,$$

and therefore the first estimate in (5.9) gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}\leq\frac{\max\{\sqrt{N_k},\|\mathbf{1}_{\eta B}\|\}}{\|\mathbf{1}_{\eta B}\|}\leq\sqrt{N_k}.$$

Case 2: $\min\{|B_0|,|B_0^c|\}=|B_0|$. Then, (5.11) reduces to

$$\|\mathbf{1}_{\eta B}\|\geq\max\{\ \alpha_k|B_0|,\ \beta_k|B_+|\ \}\geq\beta_k\frac{|B_0|+|B_+|}{2}=\beta_k\frac{|B|-|B_-|}{2}\geq\beta_k|B|/4,$$

since $|B_-| \leq N'_k \leq \sqrt{N_k}/2 \leq |B|/2$, if $k \geq 2$. Also, the second bound in (5.9) reads

$$\|\mathbf{1}_{\varepsilon A}\| \leq \alpha_k |A|,$$

since $N'_k \leq \sqrt{N_k}/\log_2 N_k = \alpha_k \sqrt{N_k} \leq \alpha_k |A|$, if $k \geq 2$. Thus

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\alpha_k |A|}{\beta_k |B|/4} = \frac{4\alpha_k}{\beta_k} = \frac{4\sqrt{N_k}}{\log_2 N_k} \leq \sqrt{N_k}.$$

This establishes Claim 2. □

From Claims 1 and 2 we now deduce that

$$\frac{\mu_{N_k}}{[\tilde{\mu}^d_{N_k}]^{2-\varepsilon}} \geq \frac{N_k^{\varepsilon/2}/2}{(\log_2 N_k)\sqrt{\log_2 \log_2 N_k}} \to \infty,$$

and therefore

$$\limsup_{N\to\infty} \frac{\mu_N}{[\mu^d_N]^{2-\varepsilon}} = \limsup_{N\to\infty} \frac{\tilde{\mu}_N}{[\tilde{\mu}^d_N]^{2-\varepsilon}} = \infty. \qquad \square$$

6. Norm convergence of $\mathfrak{CG}^t_m x$ and $\mathscr{G}^t_m x$

In this section we search for conditions on $\mathscr{B} = \{\mathbf{e}_n\}_{n=1}^\infty$ under which it holds

$$\|x - \mathfrak{CG}_m(x)\| \to 0, \quad \forall\, x \in \mathbb{X}. \tag{6.1}$$

In [19, Theorem 1.1] this convergence is asserted for every basis in $\mathbb{X}$. Here we investigate whether (6.1) may be true for a general M-basis, as defined in §1.

The solution to this question requires the notion of *strong M-basis*; see [21, Def 8.4]. We say that $\mathscr{B}$ is a strong M-basis if additionally to the conditions (a)-(d) in §1 it also holds

$$\overline{\operatorname{span}\{\mathbf{e}_n\}_{n\in A}} = \big\{x \in \mathbb{X} \ : \ \operatorname{supp} x \subset A\big\}, \quad \forall\, A \subset \mathbb{N}. \tag{6.2}$$

Clearly, all Schauder or Cesàro bases (in some ordering) are strong M-bases; see e.g. [18] for further examples. However, there exist M-bases which are not strong M-bases, see e.g. [21, p. 244], or [11][1] for seminormalized examples in Hilbert spaces.

Lemma 6.1. *If $\mathscr{B}$ is an M-basis which is not a strong M-basis, then there exists an $x_0 \in \mathbb{X}$ such that, for all Chebyshev greedy operators $\mathfrak{CG}_m$,*

$$\liminf_{m\to\infty} \|x_0 - \mathfrak{CG}_m x_0\| > 0. \tag{6.3}$$

Proof. If $\mathscr{B}$ is not a strong M-basis there exists some set $A \subset \mathbb{N}$ (necessarily infinite) and some $x_0 \in \mathbb{X}$ with $\operatorname{supp} x_0 \subset A$ such that

$$\delta = \operatorname{dist}(x_0, [\mathbf{e}_n]_A) > 0.$$

Since $\operatorname{supp} \mathfrak{CG}_m x_0$ is always a subset of A, this implies (6.3). □

Remark 6.2. The above reasoning also implies that $\liminf_m \|x_0 - \mathscr{G}_m x_0\| > 0$, for all greedy operators $\mathscr{G}_m$. In particular, if there exists a not strong M-basis with the quasi-greedy condition

$$C_q := \sup_{\substack{\mathscr{G}_m \in \mathbb{G}_m \\ m\in\mathbb{N}}} \|\mathscr{G}_m\| < \infty, \tag{6.4}$$

[1]We thank V. Kadets for kindly providing this reference.

it will not occur that $\mathscr{G}_m x$ converges to x for all $x \in \mathbb{X}$. This observation suggests that in the characterization of quasi-greedy biorthogonal systems given in [28, Theorem 1] one may need to assume that $\mathscr{B}$ is a *strong M-basis*, or else clarify if this property could be a consequence of (6.4)[2].

Here we show that under the strong M-basis assumption, the conclusions of [19, Theorem 1.1] (and also of "3 ⇒ 1" in [28, Theorem 1]) hold.

Proposition 6.3. *If $\mathscr{B}$ is a strong M-basis then, for all Chebyshev t-greedy operators $\mathfrak{CG}^t_m$,*

$$\lim_{m\to\infty} \|x - \mathfrak{CG}^t_m x\| = 0, \quad \forall\, x \in \mathbb{X}. \tag{6.5}$$

If additionally $C_q < \infty$, then for all t-greedy operators $\mathscr{G}^t_m$,

$$\lim_{m\to\infty} \|x - \mathscr{G}^t_m x\| = 0, \quad \forall\, x \in \mathbb{X}. \tag{6.6}$$

Proof. Given $x \in \mathbb{X}$ and $\varepsilon > 0$, by (6.2) there exists $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $\|x - z\| < \varepsilon$, for some finite set $B \subset \operatorname{supp} x$. Let $\alpha = \min_{n\in B} |\mathbf{e}^*_n(x)|$ and

$$\bar{\Lambda}_\alpha = \{n \,:\, |\mathbf{e}^*_n(x)| \ge \alpha\}.$$

Since $\alpha > 0$, this is a finite greedy set for x which contains B. Moreover, we claim that

$$\bar{\Lambda}_\alpha \subset \operatorname{supp} \mathfrak{CG}^t_m x =: A, \quad \forall\, m > |\bar{\Lambda}_{t\alpha}|. \tag{6.7}$$

Indeed, if this was not the case there would exist $n_0 \in \bar{\Lambda}_\alpha \setminus A$, and since A is a t-greedy set for x, then $\min_{n\in A} |\mathbf{e}^*_n(x)| \ge t|\mathbf{e}^*_{n_0}(x)| \ge t\alpha$. So, $A \subset \bar{\Lambda}_{t\alpha}$, which is a contradiction since $m = |A| > |\bar{\Lambda}_{t\alpha}|$. Therefore, (6.7) holds and hence

$$\|x - \mathfrak{CG}^t_m x\| \le \|x - \sum_{n\in B} b_n \mathbf{e}_n\| < \varepsilon, \quad \forall\, m > |\bar{\Lambda}_{t\alpha}|.$$

This establishes (6.5).

We now prove (6.6). As above, let $z = \sum_{n\in B} b_n \mathbf{e}_n$ with $B \subset \operatorname{supp} x$ and $\|x - z\| < \varepsilon$. Performing if necessary a small perturbation in the b_n's, we may assume that $b_n \ne \mathbf{e}^*_n(x)$ for all $n \in B$. Let now

$$\alpha_1 = \min_{n\in B} |\mathbf{e}^*_n(x)|, \quad \alpha_2 = \min_{n\in B} |\mathbf{e}^*_n(x - z)|, \quad \text{and} \quad \alpha = \min\{\alpha_1, \alpha_2\} > 0.$$

Consider the sets

$$\bar{\Lambda}_{t\alpha} = \{n \,:\, |\mathbf{e}^*_n(x)| \ge t\alpha\} = \{n \,:\, |\mathbf{e}^*_n(x - z)| \ge t\alpha\},$$

which for all $t \in (0, 1]$ are greedy sets for *both* x and $x - z$, and contain B. We claim that,

$$\text{if } m > |\bar{\Lambda}_{t\alpha}| \text{ and } A := \operatorname{supp} \mathscr{G}^t_m x, \quad \text{then} \quad \bar{\Lambda}_\alpha \subset A \quad \text{and} \quad A \in G(x - z, m, t). \tag{6.8}$$

The assertion $\bar{\Lambda}_\alpha \subset A$ is proved exactly as in (6.7). Next, we must show that

$$\text{if } n \in A \quad \text{then} \quad |\mathbf{e}^*_n(x - z)| \;\ge\; t \max_{k\notin A} |\mathbf{e}^*_k(x - z)| \;=\; t \max_{k\notin A} |\mathbf{e}^*_k(x)|.$$

[2] After this manuscript was completed, this question has been considered and settled in [2, Corollary 3.2]. There it is shown that a complete seminormalized biorthogonal system with the property (6.4) is necessarily a strong M-basis.

This is clear if $n \in A \setminus B$ since $\mathbf{e}_n^*(x-z) = \mathbf{e}_n^*(x)$, and $A \in G(x,m,t)$. On the other hand, if $n \in B$, then $|\mathbf{e}_n^*(x-z)| \geq \alpha_2 \geq \alpha \geq \max_{k\in A^c} |\mathbf{e}_k^*(x)|$, the last inequality due to $\bar{\Lambda}_\alpha \subset A$. Thus (6.8) holds true, and therefore

$$\mathscr{G}_m^t(x) - z = \sum_{n\in A} \mathbf{e}_n^*(x-z)\mathbf{e}_n \, = \bar{\mathscr{G}}_m^t(x-z),$$

for some $\bar{\mathscr{G}}_m^t \in \mathbb{G}_m^t$. Thus,

$$\|\mathscr{G}_m^t(x) - x\| \, = \, \|(I - \bar{\mathscr{G}}_m^t)(x-z)\| \, \leq \, (1 + \|\bar{\mathscr{G}}_m^t\|)\,\varepsilon,$$

and the result follows from $\sup_m \|\bar{\mathscr{G}}_m^t\| \leq (1+4C_q/t)C_q$, by [10, Lemma 2.1].

□

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