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LITTLEWOOD–PALEY DECOMPOSITIONS RELATED TO SYMMETRIC CONES AND BERGMAN PROJECTIONS IN TUBE DOMAINS

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1. *Introduction*

Let Ω be an *irreducible symmetric cone* in a Euclidean vector space V of dimension n, endowed with an inner product $(\cdot\,|\,\cdot)$ for which the cone Ω is self-dual. We can identify V with $\mathbb{R}^n$, by endowing the latter with such an inner product. We denote by $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexification of V, which we may also identify with $\mathbb{C}^n$. As in the text [**8**], we write the *rank* and *determinant* associated with a cone by

$$r = \operatorname{rank}\Omega, \quad \text{and} \quad \Delta(x) = \det x, \ \text{ for } x \in V.$$

The precise meaning of these notions is explained in some more detail in § 2.

A typical example is the light-cones in $\mathbb{R}^n$, with $n \geqslant 3$, defined by

$$\Lambda_n = \{y = (y_1, y') \in \mathbb{R}^n : y_1^2 - |y'|^2 > 0,\ y_1 > 0\}.$$

These are symmetric cones of rank 2 with determinant given by the *Lorentz form* $\Delta(y) = y_1^2 - |y'|^2$. A second example is the cones $\mathrm{Sym}_+(r,\mathbb{R})$ of positive-definite symmetric matrices, which have rank r and the usual determinant for matrices. In this last case, the underlying vector space V is the space of symmetric matrices $\mathrm{Sym}(r,\mathbb{R})$, with dimension $n = \frac{1}{2}r(r+1)$, and with a Euclidean norm defined by the Hilbert–Schmidt inner product (which does not coincide with the canonical inner product in the usual identification between V and $\mathbb{R}^n$). These two are the most characteristic examples in the classification of symmetric cones. The reader less familiar with the general theory should keep these cases in mind, and refer occasionally to the text [**8**] (or the more informal lecture notes [**3**]).

The goal of this paper is to present, in the general setting of symmetric cones, a special Littlewood–Paley decomposition adapted to the geometry of Ω. This will be applied to analytic problems, such as the boundedness of Bergman projectors and the characterization of boundary values for Bergman spaces in the tube domain T_Ω. To describe our setting, let us denote by $\mathcal{S}_\Omega$ the space of Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{Supp}\widehat{f} \subset \overline{\Omega}$, and normalize the Fourier transform by

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i(x|\xi)} f(x)\,dx, \quad \text{for } \xi \in \mathbb{R}^n.$$

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Our basic tool will be a special decomposition for functions in $\mathcal{S}_\Omega$,

$$f = \sum_j f * \psi_j, \quad \text{for all } f \in \mathcal{S}_\Omega, \tag{1.1}$$

where $\widehat{\psi}_j$ are supported on 'suitable frequency blocks' B_j which form a *Whitney covering* of the cone Ω. More precisely, and in analogy with the dyadic decomposition of the half-line $(0, \infty)$ (that is, the 1-dimensional cone), we let B_j be the 'balls' $B_j = \{\xi \in \Omega : d(\xi, \xi_j) < 1\}$, obtained from the homogeneous structure of the cone via an invariant distance d and a d-lattice $\{\xi_j\}$. These will turn out to be the right sets for the discretization of many operators related to Ω, since functions typically appearing in multiplier expressions (such as $\Delta(\xi)$ or $e^{-(\xi|y)}$, for fixed $y \in \Omega$) remain essentially constant when $\xi \in B_j$.

A characteristic example of this situation is the *generalized wave operator on the cone*:

$$\Box = \Delta\left(\frac{1}{i}\frac{\partial}{\partial x}\right),$$

which is the differential operator of degree r defined by the equality

$$\Delta\left(\frac{1}{i}\frac{\partial}{\partial x}\right)[e^{i(x|\xi)}] = \Delta(\xi) e^{i(x|\xi)}, \quad \text{where } \xi \in \mathbb{R}^n. \tag{1.2}$$

This corresponds, in cones of rank 1 and 2, to

$$\Box = \frac{1}{i}\frac{d}{dx} \ \text{ in } (0,\infty), \quad \text{and} \quad \Box = -\frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}\right) \ \text{ in } \Lambda_n.$$

The Littlewood–Paley decomposition (1.1) provides a formal 'discretization' of the action of $\Box$ on functions with spectrum in Ω:

$$\Box f = \mathcal{F}^{-1}(\Delta(\xi)\widehat{f}(\xi)) = \sum_j \Delta(\xi_j)\, f * \psi_j * \mathcal{F}^{-1}(m_j), \quad \text{for } f \in \mathcal{S}_\Omega,$$

where $\{m_j\}$ is a uniformly bounded family of multipliers.

From these facts it is natural to introduce a new family of Besov-type spaces, $B^{p,q}_\nu$, adapted to the Littlewood–Paley decomposition (1.1). These are defined as the equivalence classes of tempered distributions which have finite seminorms

$$\|f\|_{B^{p,q}_\nu} = \left[\sum_j \Delta^{-\nu}(\xi_j)\, \|f * \psi_j\|_p^q\right]^{1/q}. \tag{1.3}$$

Our first result shows that these spaces satisfy properties analogous to those of the one-dimensional homogeneous Besov spaces, with the role of the usual derivation played now by the wave operator $\Box$. We have chosen a special normalization of indices in (1.3) which is convenient for the applications that follow, but which is slightly different from the standard notation in, for example, [**9**]. (When $n = 1$ and $\xi_j = 2^j$, the norm in (1.3) corresponds to the classical Besov space $\dot{B}^{-\nu/q}_{p,q}(\mathbb{R})$.)

THEOREM 1.4. *Let $\nu \in \mathbb{R}$ and $1 \leqslant p, q < \infty$. Then*

(1) *$B^{p,q}_\nu$ is a Banach space and does not depend on the choice of $\{\psi_j\}$ and $\{\xi_j\}$;*
(2) *$\Box : B^{p,q}_\nu \to B^{p,q}_{\nu+q}$ is an isomorphism of Banach spaces;*
(3) *if $p, q > 1$, then $(B^{p,q}_\nu)^*$ is isomorphic to $B^{p',q'}_{-\nu q'/q}$ with the usual duality pairing.*

The rest of the paper is devoted to applications of this theory to two open problems involving the class of *Bergman spaces*. In this paper, a weighted mixed-norm version of these spaces is defined by the following integrability condition:

$$\|F\|_{L^{p,q}_\nu} := \left[\int_\Omega \left(\int_{\mathbb{R}^n} |F(x+iy)|^p\,dx\right)^{q/p} \Delta^{\nu-n/r}(y)\,dy\right]^{1/q} < \infty. \tag{1.5}$$

Thus, when $1 \leqslant p, q < \infty$ and $\nu \in \mathbb{R}$, we denote by $A^{p,q}_\nu(T_\Omega)$ the closed subspace of $L^{p,q}_\nu$ consisting of holomorphic functions in the tube T_Ω. We observe that these spaces are non-null only when $\nu > n/r - 1$ (see, for example, [**3**]). The usual A^p space corresponds to $p = q$ and $\nu = n/r$. To simplify notation we shall write $A^p_\nu = A^{p,\,p}_\nu$, and similarly $L^p_\nu = L^{p,\,p}_\nu$.

Two main questions concerning these spaces will be studied here:

(1) the characterization of boundary values of functions in $A^{p,q}_\nu$, as distributions in the Besov spaces $B^{p,q}_\nu$;

(2) the boundedness of Bergman projectors P_ν in $L^{p,q}_\nu$ spaces, where P_ν is the usual orthogonal projection from L^2_ν onto A^2_ν.

Regarding the first question, the general idea is to write a holomorphic function F in the tube T_Ω in terms of its *Fourier–Laplace transform*:

$$F(z) = \mathcal{L}g(z) = \int_\Omega e^{i\,(z|\xi)} g(\xi)\,d\xi, \quad \text{for } z \in T_\Omega, \tag{1.6}$$

for some distribution g supported in $\overline{\Omega}$. Roughly speaking, the new distribution $f = \mathcal{F}^{-1}g$ plays the role of the 'Shilov boundary value' for F, while the condition $F \in A^{p,q}_\nu$ is naturally related with $f \in B^{p,q}_\nu$. Observe that we must exclude some indices, since by Theorem 1.4 we can only give a meaning to (1.6) when $\mathcal{F}^{-1}(e^{-(y|\cdot)}\chi_\Omega)$ has a finite $B^{p',q'}_{-\nu q'/q}$-norm. As we shall see, this can only happen when q is below the critical index

$$\widetilde{q}_{\nu,p} = \frac{\nu + n/r - 1}{((n/r)(1/p') - 1)_+}$$

(with $\widetilde{q}_{\nu,p} = \infty$, if $n/r \leqslant p'$). A detailed justification of these facts will be presented in §§ 3.4 and 4.1, leading to the following theorem.

Theorem 1.7. *Let $\nu > n/r - 1$, $1 \leqslant p < \infty$ and $1 \leqslant q < \widetilde{q}_{\nu,p}$. Then, for every $F \in A^{p,q}_\nu$ there exists a (unique) tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j f * \psi_j$ in $\mathcal{S}'(\mathbb{R}^n)$, $\|f\|_{B^{p,q}_\nu} < \infty$ and $F = \mathcal{L}\widehat{f}$. Moreover we have*

(1) $\lim_{y\to 0,\, y\in\Omega} F(\cdot + iy) = f$, in both $\mathcal{S}'(\mathbb{R}^n)$ and $B^{p,q}_\nu$;

(2) $\|f\|_{B^{p,q}_\nu} \leqslant C\,\|F\|_{A^{p,q}_\nu}$, for all $F \in A^{p,q}_\nu$.

The converse result is more interesting, and turns out to be equivalent to the second of the questions posed above. We only have a partial answer, for which we need to introduce two new critical indices

$$q_\nu = \frac{\nu + n/r - 1}{n/r - 1}, \quad q_{\nu,p} = \min\{p,\ p'\}\, q_\nu\,.$$

Observe that in 1-dimension the three indices are equal to ∞, while in general we have the ordering

$$2 < q_\nu \leqslant q_{\nu,p} \leqslant \widetilde{q}_{\nu,p}.$$

The role of these new indices will be clarified later in relation to the Bergman projectors.

THEOREM 1.8. *Let $\nu > n/r - 1$, $1 \leqslant p < \infty$ and $1 \leqslant q < q_{\nu,p}$. Given a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j f * \psi_j$ and $\|f\|_{B_\nu^{p,q}} < \infty$, then the holomorphic function $F = \mathcal{L}\widehat{f}$ belongs to $A_\nu^{p,q}$, and moreover, there exists a constant $C > 0$ so that*

$$\frac{1}{C}\,\|f\|_{B_\nu^{p,q}} \leqslant \|\mathcal{L}\widehat{f}\|_{A_\nu^{p,q}} \leqslant C\,\|f\|_{B_\nu^{p,q}}, \quad \textit{for } f \in B_\nu^{p,q}.$$

This theorem is sharp for $1 \leqslant p \leqslant 2$, in the sense that for each $q \geqslant q_{\nu,p} = pq_\nu$ there is a distribution with $\|f\|_{B_\nu^{p,q}} < \infty$ and $\|\mathcal{L}\widehat{f}\|_{L_\nu^{p,q}} = \infty$. We shall present these examples in § 4.4. When $p > 2$ we will construct similar examples, but only for values of $q \geqslant q_{\nu,2} = 2q_\nu$, leaving open the question when $p'q_\nu \leqslant q < \min\{2q_\nu, \widetilde{q}_{\nu,p}\}$. One can conjecture that the theorem also has a positive answer in these cases, with some evidence given by the sharp results for large p presented for light-cones in § 5.

We turn to the second application of our theory, the boundedness of Bergman projectors in $L_\nu^{p,q}$. This is a challenging question which has been open for many years, and which still is not completely solved. The three indices defined above correspond to three steps of difficulty for this question. First, a trivial counterexample (just involving local integrability of the Bergman kernel) shows that P_ν can only be bounded in $L_\nu^{p,q}$ when $\widetilde{q}\,'_{\nu,p} < q < \widetilde{q}_{\nu,p}$. Next, an argument involving Schur's lemma gives $q'_\nu < q < q_\nu$ as the sharp range of boundedness for the *positive operator* P_ν^+ (where the Bergman kernel is replaced by its absolute value $|B_\nu(z,w)|$; see [**1**, **5**]). Finally, it is shown in [**4**] that $q'_{2,p} < q < q_{2,p}$ is the sharp range of boundedness for P_ν in the spaces $L_\nu^{2,q}$ (from which $q_{\nu,p}$ arises by interpolation between q_ν and $q_{\nu,2} = 2q_\nu$). The techniques introduced in [**4**], originally for light-cones, have also been the germ of the Littlewood–Paley decomposition we present here. Our main contribution to this problem, besides the extension to general symmetric cones, is the equivalent formulation in terms of the previous problem, which moreover will lead to new improvements as those presented in § 5. We gather these results in our next two theorems, which by self-adjointness of P_ν we need to state only when $q \geqslant 2$.

THEOREM 1.9. *Let $\nu > n/r - 1$, $1 \leqslant p < \infty$ and $2 \leqslant q < \widetilde{q}_{\nu,p}$. Then, P_ν admits a bounded extension from $L_\nu^{p,q}$ onto $A_\nu^{p,q}$ if and only if*

$$\|\mathcal{L}\widehat{f}\|_{L_\nu^{p,q}} \leqslant C\left[\sum_j \Delta^{-\nu}(\xi_j)\,\|f * \psi_j\|_p^q\right]^{1/q}, \quad \textit{for } f \in \mathcal{S}_\Omega. \tag{1.10}$$

In particular, P_ν is bounded in $L_\nu^{p,q}$ for all $1 \leqslant p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$. Moreover, P_ν does not admit bounded extensions to $L_\nu^{p,q}$ when:

(1) *$1 \leqslant p \leqslant 2$ and $q \geqslant q_{\nu,p}$;*
(2) *$2 < p < \infty$ and $q \geqslant \min\{2q_\nu, \widetilde{q}_{\nu,p}\}$.*

THEOREM 1.11. *For the light-cone Λ_n, with $n \geqslant 3$, and for all $\nu > \frac{1}{2}(n-2)$, there exists a large $p_{n,\nu}$ such that P_ν is bounded in the sharp range $2 \leqslant q < \widetilde{q}_{\nu,p}$ for all $p > p_{n,\nu}$.*

Figure 1.1 illustrates the regions of boundedness and unboundedness for P_ν after the results in Theorems 1.9 and 1.11. Compared with [**4**], the first figure removes the end-points in $1 \leqslant p \leqslant 2$, and the region $q \geqslant q_{\nu,2}$ when $p > 2$, with counter-examples which are new even for light-cones. The second figure improves upon these results for light-cones, reaching for the first time the sharp line $q = \widetilde{q}_{\nu,p}$. We can even improve a bit on this picture when $n = 3$ (see Corollary 5.17 and Figure 5.1 below). At this point it becomes natural to conjecture the boundedness of P_ν in the whole blank region above. We observe that in the particular case of L^p_ν-spaces $(p = q)$ the conjecture becomes $2 \leqslant p < \min\{2q_\nu, q_\nu + n/(n-r)\}$ (together with its dual interval), which in this paper we also settle for light-cones and *sufficiently large* ν (see Remark 5.13 below).

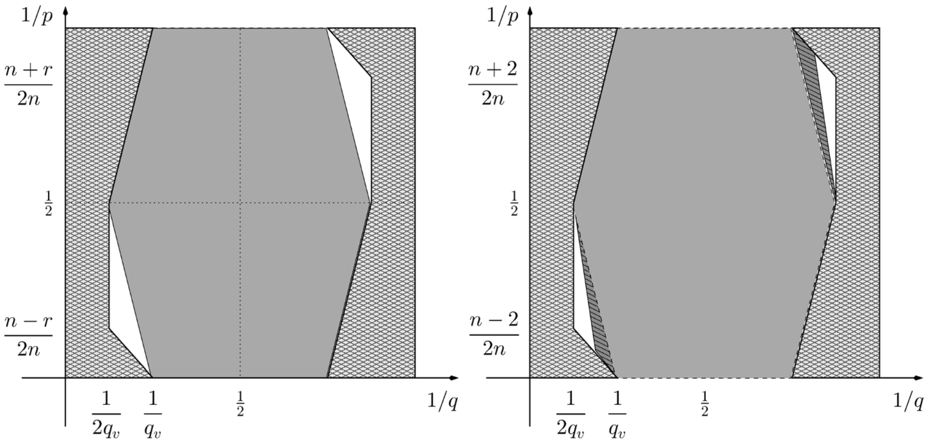

FIGURE 1.1. *Regions of boundedness for P_ν in general cones and light-cones.*

We do not wish to conclude this introductory section without briefly explaining our approach to this problem for light-cones, and its relation with classical multiplier conjectures in Harmonic Analysis. In §§ 4 and 5 we shall reformulate (1.10) in terms of simpler inequalities of the form

$$\|f\|_p \leqslant CN^\alpha \Big[\sum_{\xi_j \in \Omega_N} \|f * \psi_j\|_p^s\Big]^{1/s}, \quad \text{for } f \in \mathcal{S}_{\Omega_N}(\mathbb{R}^n), \tag{1.12}$$

for appropriate combinations of the indices α, p, s, and with the sum restricted to lattice points so that $|\xi_j| \sim 1$ and $\Delta(\xi_j) \sim 1/N$. These inequalities have been studied in light-cones in relation with restriction and cone multiplier problems, where the 'frequency blocks' $B(\xi_j)$ correspond precisely to the usual 'dyadic' partition of a truncated conical shell (as in, for example, [**13**] or [**16**]). When $p = s > 2n/(n-2)$ one can conjecture the validity of (1.12) for all $\alpha > \frac{1}{2}(n-2) - (n-1)/p$, which if true would allow us to push for such p up to the critical line $q = \widetilde{q}_{\nu,p}$ in Figure 1.1. This turns out to coincide with a weak version of the local smoothing conjecture recently posed by T. Wolff [**17**, **12**], and it is precisely from his partial results that we obtain an improvement in our problem. (We thank A. Seeger and A. Vargas for pointing out these references, as well as the implications for our results.) Similarly, when $2 < p \leqslant 2n/(n-2)$ and $s = 2$ the natural conjecture states the validity of (1.12) for all $\alpha > 0$, which would eventually imply an improvement for such p up to the vertical line $q = 2q_\nu$ in Figure 1.1. This is closely related to the stronger local smoothing conjecture, where one replaces the right-hand side of (1.12) with $N^\varepsilon \|(\sum |f * \psi_j|^2)^{1/2}\|_p$ (and restricts to $2 < p \leqslant 2(n-1)/(n-2)$). Again, from the latest progress on this question [**16**, **18**] we shall obtain new positive results for $n = 3$ and $p = 4$ (see Figure 5.1). We finally point out that, regarding our problem, sharp inequalities of the form (1.12) are not only sufficient, but also necessary. In particular, a positive answer for the blank region in Figure 1.1 for all $\nu > 1$ would solve Wolff's conjecture in dimensions 3 and 4 (see Remark 5.14 below).

We conclude by mentioning that a simplified version of our results, specialized to the case when $p = 2$ (for which the use of Plancherel's theorem is available), has been published separately in [**2**]. We also refer to the survey paper [**6**] for complementary information concerning critical indices and their relation with Hardy-type inequalities.

2. *Whitney decompositions on the cone*

In this section we introduce the notation and a list of technical results on symmetric cones, mostly taken from the text [**8**]. We also give a detailed construction of the 'Whitney decomposition' adapted to the analysis of the problems stated above. The main lines and applications of such constructions appear in previous papers: see [**4**] for the light-cone, and [**2**] for general symmetric cones.

2.1. *Background on symmetric cones*

Let Ω be a fixed symmetric cone in a real Euclidean vector space V, endowed with the inner product $(\cdot\,|\,\cdot)$. That is, Ω is a homogeneous open convex cone which is self-adjoint with respect to $(\cdot\,|\,\cdot)$. Let $G(\Omega)$ be the group of linear transformations of the cone, and G its identity component. By definition, $G(\Omega)$ acts transitively on Ω. Further, it is well known that there is a solvable subgroup T of G acting *simply transitively on* Ω. That is, every $y \in \Omega$ can be written uniquely as $y = t\mathbf{e}$, with $t \in T$ and a fixed $\mathbf{e} \in \Omega$. This gives an identification $\Omega \equiv T = G/K$, where K is a maximal compact subgroup of G. Moreover, $K = \{g \in G : g\mathbf{e} = \mathbf{e}\} = G \cap O(V)$. These properties can be found in Chapters I and VI of the text [**8**].

It is well known that for every symmetric cone Ω, its underlying vector space V can be endowed with a multiplication rule which makes it a *Euclidean Jordan algebra* with identity element $\mathbf{e}$. With such multiplication, $\overline{\Omega}$ coincides with the set $\{x^2 : x \in V\}$ of all squares in V. The notions of rank, trace and determinant in Ω are those inherited from the Jordan algebra structure of V (see [**8**, Chapter II]). We say that Ω is *irreducible* when V cannot be decomposed as a direct sum of two lower-dimensional subspaces which contain a pair of symmetric cones whose direct sum equals Ω (alternatively, when V does not contain non-trivial ideals). For irreducible cones we may assume, after multiplying by a positive constant, that the inner product in V is given by $(x|y) = (xy|\mathbf{e}) = \mathrm{tr}(xy)$ [**8**, p. 51]. The reader less familiar with these concepts should keep in mind the example of positive-definite symmetric matrices, which we present in more detail in Example 2.2 below.

Suppose now that the cone is irreducible, has rank r and its underlying space has dimension n. Following [**8**, Chapter IV], we fix a *Jordan frame* $\{c_1, \dots, c_r\}$ in V (that is, a complete system of idempotents), to which we associate a *Peirce decomposition* of the space V, that is, an orthogonal decomposition $V = \bigoplus_{1 \leqslant i \leqslant j \leqslant r} V_{i,j}$, where

$$V_{ii} = \mathbb{R}c_i, \quad \text{and} \quad V_{i,j} = \{x \in V : c_i x = c_j x = \tfrac{1}{2}x\} \text{ if } i < j.$$

Then, by [**8**, Theorem VI.3.6], T may be taken as the corresponding solvable Lie group, which factors as the semidirect product $T = NA = AN$ of a nilpotent subgroup N (of lower triangular matrices), and an abelian subgroup A (of diagonal matrices). The latter takes the explicit form

$$A = \left\{ P(a) : a = \sum_{i=1}^{r} a_i c_i, a_i > 0 \right\},$$

where P is the quadratic representation of V. This also leads to the classical decompositions of the semisimple Lie group $G = NAK$ and $G = KAK$.

Still following [**8**, Chapter VI], we shall denote by $\Delta_1(x), \dots, \Delta_r(x)$ the *principal minors* of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. These are invariant functions under the group N,

$$\Delta_k(nx) = \Delta_k(x), \quad \text{where } n \in N, x \in V, k = 1, \dots, r,$$

and satisfy a homogeneity relation under A,

$$\Delta_k(P(a)x) = a_1^2 \dots a_k^2 \, \Delta_k(x), \quad \text{if } a = a_1 c_1 + \dots + a_r c_r.$$

The determinant function $\Delta(y) = \Delta_r(y)$ is also invariant under K, and moreover, satisfies the formula

$$\Delta(gy) = \Delta(g\mathbf{e})\Delta(y) = \mathrm{Det}(\mathrm{g})^{r/n}\Delta(y). \tag{2.1}$$

It follows from this formula that an invariant measure in Ω is given by $\Delta(y)^{-n/r}\,dy$. Finally, we recall a version of Sylvester's Theorem for symmetric cones, which allows to write these as

$$\Omega = \{x \in V : \Delta_k(x) > 0,\ k = 1, \dots, r\}.$$

EXAMPLE 2.2. *The cone of positive-definite symmetric matrices.* We describe the above concepts for the cone $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$, contained in the vector space $V = \mathrm{Sym}(r, \mathbb{R})$. The *Jordan algebra structure* in V corresponds to the symmetric product $X \circ Y = \frac{1}{2}(XY + YX)$, with the usual identity matrix $\mathbf{e} = I$. A standard *Jordan frame* is the set D_j of diagonal matrices all of whose entries are 0 except for the jth which is equal to 1. The Peirce decomposition in V is just the decomposition of a symmetric matrix in terms of its (i, j) entries.

In this example, the *automorphism group* $G(\Omega)$ can be identified with $\mathrm{Gl}(r, \mathbb{R})$ via the adjoint action

$$g \in \mathrm{Gl}(r, \mathbb{R}), Y \in \mathrm{Sym}(r, \mathbb{R}) \longmapsto g \cdot Y = gYg^* \in \mathrm{Sym}(r, \mathbb{R}). \tag{2.3}$$

Then, the group T consists in the lower triangular matrices in $\mathrm{Gl}(r,\mathbb{R})$, and the factorization $Y = t \cdot I$ is precisely the Gauss decomposition of a positive-definite symmetric matrix. The subgroup N consists of all triangular matrices in $\mathrm{Gl}(r,\mathbb{R})$ with 1s on the diagonal, while A is given by the diagonal matrices $P(a) = \mathrm{diag}\{a_1, \dots, a_r\}$. Finally, the associated *principal minors* are the usual principal minors from linear algebra, that is, the determinants of the $k \times k$ symmetric submatrices obtained by restriction to the first k coordinates. One verifies easily with this example the homogeneity properties with respect to N and A stated above.

2.2. *The invariant metric and the covering lemma*

With the identification $\Omega \equiv G/K$, the cone can be regarded as a Riemannian manifold with the G-invariant metric defined by

$$\langle \xi, \eta \rangle_y := (t^{-1}\xi | t^{-1}\eta)$$

if $y = t\mathbf{e}$ and ξ and η are tangent vectors at $y \in \Omega$. We shall denote by $d(\,\cdot\,,\,\cdot\,)$ the corresponding distance, and by $B_\delta(\xi)$ the ball centered at ξ of radius δ. Note that, for each $g \in G$, the invariance implies that $B_\delta(g\xi) = gB_\delta(\xi)$.

We shall need some weak local invariance properties of the quantities that we have defined on the cone. One consequence is the possibility of obtaining a Whitney-type decomposition for general symmetric cones in terms of invariant balls. Part of this material has already been presented in [**2**].

Lemma 2.4. *Let $\delta > 0$. Then there is a constant $\gamma = \gamma(\delta, \Omega) > 0$ such that*

$$d(\xi, \xi') \leqslant \delta \quad \Longrightarrow \quad \frac{1}{\gamma} \leqslant \frac{\Delta_k(\xi)}{\Delta_k(\xi')} \leqslant \gamma, \quad \text{for } k = 1, \dots, r.$$

Proof. By invariance of the metric and the forms Δ_k under N, we may assume $\xi' = P(a)\mathbf{e}$. Further, since

$$\frac{\Delta_k(\xi)}{\Delta_k(P(a)\mathbf{e})} = \frac{\Delta_k(P(a)^{-1}\xi)}{\Delta_k(\mathbf{e})},$$

we may even assume $\xi' = \mathbf{e}$. Now, the estimates above and below for Δ_k in a ball $\overline{B}_\delta(\mathbf{e})$ follow easily from the continuity of $\xi \mapsto \Delta_k(\xi)$, and a compactness argument. □

The next lemma states the local equivalence between two Riemannian metrics. The proof follows from standard arguments (see, for example, [**14**, 9–22]).

Lemma 2.5. *Let $\delta_0 > 0$ be fixed. Then, there exist two constants $\eta_1 > \eta_0 > 0$, depending only on δ_0 and Ω, so that for every $0 < \delta \leqslant \delta_0$ we have*

$$\{|\xi - \mathbf{e}| < \eta_0\delta\} \subset B_\delta(\mathbf{e}) \subset \{|\xi - \mathbf{e}| < \eta_1\delta\}.$$

We can now estimate the volume of an invariant ball. Recall that the invariant measure in Ω is given by

$$\mathrm{meas}(B) = \int_B \Delta(\xi)^{-n/r}\, d\xi, \quad \text{with } B \subset \Omega \text{ measurable.}$$

Therefore, from the previous results it follows that, for all $y \in \Omega$ and $0 < \delta \leqslant \delta_0$,

$$\operatorname{meas}(B_\delta(y)) = \operatorname{meas}(B_\delta(\mathbf{e})) \sim \operatorname{Vol}(B_\delta(\mathbf{e})) \sim \delta^n,$$

where the equivalences denoted by '$\sim$' are modulo constants depending only on Ω and the fixed number δ_0. Observe, however, that this estimate cannot hold uniformly in $\delta_0 \gg 1$, since the invariant measure is in general not doubling. We can now prove a covering lemma which will be of crucial importance for the rest of the paper.

LEMMA 2.6. Whitney decomposition of the cone. *Let $\delta > 0$ and $R \geqslant 2$. Then, there exist sequences of points $\{\xi_j\}_j$ in Ω such that*

(i) *$\{B_\delta(\xi_j)\}_j$ is a disjoint family in Ω;*

(ii) *$\{B_{R\delta}(\xi_j)\}_j$ is a covering of the cone Ω.*

Moreover, for each such sequence the balls $\{B_{R\delta}(\xi_j)\}_j$ have the finite intersection property. That is, if $\delta, R \leqslant R_0$, then there exists an integer $N = N(R_0, \Omega)$ so that at most N of these balls can intersect an arbitrary set $E \subset \Omega$ with diameter

$$\operatorname{diam}(E) = \sup\{d(\xi, \eta) : \xi, \eta \in E\} \leqslant R_0 \delta. \tag{2.7}$$

Proof. Consider $\{\xi_j\}_j$, a maximal subset of Ω (under inclusion) among those with the property that their elements are distant at least 2δ from one another. Let us denote by B'_j the balls $B_\delta(\xi_j)$. They are pairwise disjoint, while, by maximality, the balls $\{B_j = B_{2\delta}(\xi_j)\}_j$ cover Ω. Note also that, necessarily, the set $\{\xi_j\}_j$ is countable.

For the finite overlapping, let E be a set as in (2.7). Denote by J the set of indices $\{j : B_{R\delta}(\xi_j) \cap E \neq \emptyset\}$, and fix a point $\xi \in B_{R\delta}(\xi_{j_0}) \cap E$ for some $j_0 \in J$. Then, the condition on the diameter gives

$$\bigcup_{j \in J} B_\delta(\xi_j) \subset B_{(2R_0+1)\delta}(\xi).$$

Now, by disjointness and invariance of the measure we have

$$\begin{aligned} |J| \operatorname{meas}(B_\delta(\mathbf{e})) &= \operatorname{meas}\Big(\bigcup_{j \in J} B'_j\Big) \\ &\leqslant \operatorname{meas}(B_{(2R_0+1)\delta}(\xi)) = \operatorname{meas}(B_{(2R_0+1)\delta}(\mathbf{e})). \end{aligned}$$

Thus, the remarks preceding the lemma give us a bound for N depending only on Ω and R_0. □

REMARK 2.8. 1. A sequence of points $\{\xi_j\}_j$ with the above properties will be called a *(δ, R)-lattice of the cone*. Observe that one can always define an associated partition by letting

$$E_1 = B_1, \quad \ldots, \quad E_j = B_j \setminus E_{j-1}, \quad \ldots .$$

We shall call $\{E_j\}_j$ a *Whitney decomposition of* Ω.

2. If $\{\xi_j\}_j$ is a (δ, R)-lattice, then so is $\{\xi_j^{-1}\}_j$. Indeed, this follows from the fact that $y \mapsto y^{-1}$ is an isometry of the cone (see Chapter III of [**8**]). Therefore, $B_\delta(\xi_j^{-1}) = B_\delta(\xi_j)^{-1}$, and the conditions of Lemma 2.6 hold.

3. One can look at the sequences $\{\xi_j\}_j$ and $\{\xi_j^{-1}\}_j$ as a *couple of dual lattices*. In fact, $(\xi_j|\xi_j^{-1}) = r$, while using $\Delta(y^{-1}) = \Delta(y)^{-1}$ we also have $\mathrm{Vol}(B_j) \sim \Delta(\xi_j)^{n/r}$ and $\mathrm{Vol}(B_j^{-1}) \sim \Delta(\xi_j)^{-n/r}$. Moreover, from the next lemma it will follow that actually $(\xi|y) \sim 1$ when $\xi \in B_j$ and $y \in B_j^{-1}$.

LEMMA 2.9. *Let $\delta > 0$. There exists $\gamma = \gamma(\Omega, \delta) > 0$ such that, for $y \in \overline{\Omega}$ and $\xi, \xi' \in \Omega$ with $d(\xi, \xi') \leqslant \delta$, then*

$$\frac{1}{\gamma} \leqslant \frac{(\xi|y)}{(\xi'|y)} \leqslant \gamma. \tag{2.10}$$

In particular, $1/\gamma \leqslant |\xi|/|\xi'| \leqslant \gamma$, when $d(\xi, \xi') \leqslant \delta$.

Proof. By continuity it suffices to show (2.10) for $y \in \Omega$. Using invariance under G (and the fact that $G = G^*$), we may assume that $y = \mathbf{e}$. To show that $(\xi'|\mathbf{e}) \leqslant \gamma(\xi|\mathbf{e})$, let us write $\xi = kP(a)\mathbf{e}$, for $k \in K$ and $a = a_1c_1 + \ldots + a_rc_r$. Then the new vector $\xi'' = P(a)^{-1}k^{-1}\xi'$ belongs to the fixed ball $B_\delta(\mathbf{e})$. Therefore, we have

$$(\xi'|\mathbf{e}) = (P(a)\xi''|\mathbf{e}) \leqslant \sqrt{r}\,\|P(a)\||\xi''| \leqslant \gamma\|P(a)\|,$$

where the last bound appears because $\overline{B}_\delta(\mathbf{e})$ is a compact set. Now $P(a)$ has eigenvalues a_i^2 and a_ia_j, and hence

$$\|P(a)\| = \max\{a_j^2, a_ia_j\} \leqslant \sum_{i=1}^{r} a_i^2 = (P(a)\mathbf{e}|\mathbf{e}) = (\xi|\mathbf{e}). \tag{2.11}$$

Finally, let us remark that $(\xi|\mathbf{e})$ is equivalent to $|\xi|$. Indeed, $(\xi|\mathbf{e}) \leqslant \sqrt{r}\,|\xi|$ by the Schwarz inequality. Conversely, for $\xi = P(a)\mathbf{e}$, we have

$$|\xi| = \left|\sum_{j=1}^{r} a_j^2 c_j\right| \leqslant \sum_{j=1}^{r} a_j^2 = (\xi|\mathbf{e})\,. \qquad \square$$

LEMMA 2.12. *For every $g \in G$ we have*

$$\|g\| \leqslant |g\mathbf{e}| \leqslant \sqrt{r}\,\|g\|.$$

Proof. Write $g = kP(a)h$, for some $h, k \in K$ and $a = a_1c_1 + \ldots + a_rc_r$. Then, as in (2.11),

$$\|P(a)\| \leqslant (a_1^4 + \ldots + a_r^4)^{1/2} = |P(a)\mathbf{e}| = |g\mathbf{e}|.$$

Thus,

$$|g\mathbf{e}|/|\mathbf{e}| \leqslant \|g\| = \|P(a)\| \leqslant |g\mathbf{e}|. \qquad \square$$

2.3. *Integrals on* Ω

To conclude with this preliminary section, we list some basic facts concerning integrals in the cone. Following [**8**], we define the *generalized power function of* $x \in \Omega$ by

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1-s_2}(x)\,\Delta_2^{s_2-s_3}(x)\ldots\Delta_r^{s_r}(x), \quad \text{for } \mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r,$$

where Δ_k are the principal minors with respect to a fixed Jordan frame $\{c_1, \ldots, c_r\}$. In particular, $\Delta_{\mathbf{s}}(x) = a_1^{s_1}\ldots a_r^{s_r}$ when $x = a_1c_1 + \ldots + a_rc_r$. The lemmas from the previous section justify the following discretization of integrals which we shall use often below.

PROPOSITION 2.13. *Let $0 < \delta, R \leqslant R_0$ be fixed, and $\{\xi_j\}_j$ be a (δ, R)-lattice with associated Whitney decomposition $\{E_j\}_j$. Then, for every $\mathbf{s} \in \mathbb{R}^r$ there exists a positive constant C such that, for any $y \in \overline{\Omega}$ and for any non-negative function f on the cone, we have*

$$\frac{1}{C} \sum_j e^{-\gamma(y|\xi_j)} \Delta_{\mathbf{s}}(\xi_j) \int_{E_j} f(\xi) \, \frac{d\xi}{\Delta(\xi)^{n/r}} \leqslant \int_\Omega f(\xi) e^{-(y|\xi)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}}$$
$$\leqslant C \sum_j e^{-(1/\gamma)(y|\xi_j)} \Delta_{\mathbf{s}}(\xi_j) \int_{E_j} f(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}},$$

where $\gamma = \gamma(R_0, \Omega)$ is a constant as in (2.10).

We shall also need the *gamma function in* Ω defined from the generalized powers. That is, given $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, one lets

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-(\xi|\mathbf{e})} \, \Delta_{\mathbf{s}}(\xi) \, \frac{d\xi}{\Delta(\xi)^{n/r}}. \tag{2.14}$$

This integral is known to converge absolutely if and only if

$$\mathfrak{Re}\, s_j > (j-1)\frac{n/r - 1}{r-1}, \quad \text{for all } j = 1, \ldots, r.$$

Moreover, in such a case

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^r \Gamma\left(s_j - (j-1)\frac{n/r - 1}{r-1}\right), \tag{2.15}$$

where Γ is the classical gamma function in $\mathbb{R}_+$ [**8**, Chapter VII]. We shall denote $\Gamma_\Omega(\mathbf{s}) = \Gamma_\Omega(s)$ when $\mathbf{s} = (s, \ldots, s)$. The next formula defines the Laplace transform of a generalized power, and can be found in [**8**, p. 124].

LEMMA 2.16. *For $y \in \Omega$ and $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ with*

$$\mathfrak{Re}\, s_j > (j-1)\frac{n/r - 1}{r-1}, \quad \text{for } j = 1, \ldots, r,$$

we have

$$\int_\Omega e^{-(\xi|y)} \, \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}} = \Gamma_\Omega(\mathbf{s}) \, \Delta_{\mathbf{s}}(y^{-1}).$$

REMARK 2.17. We will sometimes write the above quantity $\Delta_{\mathbf{s}}(y^{-1})$ in terms of the *rotated Jordan frame* $\{c_r, \ldots, c_1\}$. That is, if we denote by Δ_j^*, for $j = 1, \ldots, r$, the principal minors with respect to this new frame, then

$$\Delta_{\mathbf{s}}(y^{-1}) = [\Delta^*_{\mathbf{s}^*}(y)]^{-1}, \quad \text{for all } \mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r,$$

where we have set $\mathbf{s}^* := (s_r, \ldots, s_1)$ (see [**8**, p. 127]).

Our last lemmas are concerned with global and local integrability of generalized powers. The first one is a simple consequence of our last result and the Plancherel formula (see also [**4**]).

Lemma 2.18. *Let* $\alpha \in \mathbb{R}$, *and define*

$$I_\alpha(y) = \int_{\mathbb{R}^n} |\Delta(x+iy)|^{-\alpha}\, dx, \quad \text{for } y \in \Omega.$$

Then, I_α *is finite if and only if* $\alpha > 2n/r - 1$. *In this case,* $I_\alpha(y) = c(\alpha)\,\Delta(y)^{-\alpha+n/r}$.

We next establish the critical index for local integrability at the origin.

Lemma 2.19. *Let* $\alpha \in \mathbb{R}$ *and*

$$g_\alpha(\xi) = \frac{e^{-(\xi|\mathbf{e})}}{\Delta(\xi)(1+|\log \Delta_{(0,\dots,0,1)}(\xi)|)^\alpha}.$$

Then, g_α *is integrable if and only if* $\alpha > 1$.

Proof. This is a simple exercise using Gaussian coordinates (see Chapter VI of [**8**]). Indeed, with the notation in [**8**], the integral of g_α is equal to

$$c_r \int_{(0,\infty)^r} \frac{e^{-\sum u_j^2}}{(1+2|\log u_r|)^\alpha} \left[\prod_{j=1}^r u_j^{(r-j)d-1}\right] du_1 \dots du_r$$
$$= c_r' \int_0^\infty \frac{e^{-u_r^2}}{(1+2|\log u_r|)^\alpha} \frac{du_r}{u_r}. \qquad \square$$

Finally, we conclude with the critical index for integrability at infinity.

Lemma 2.20. *Let* $\alpha, \delta \in \mathbb{R}$, $\beta > -1$ *and*

$$g_{\alpha,\beta,\delta}(y) = \frac{\Delta^\beta(y)}{\Delta^\alpha(y+\mathbf{e})(1+\log \Delta(y+\mathbf{e}))^\delta}.$$

Then, $g_{\alpha,\beta,\delta}$ *is integrable if and only if* $\alpha - \beta > 2n/r - 1$ *or* $\alpha - \beta = 2n/r - 1$ *and* $\delta > 1$.

Proof. This time we use the 'polar coordinates' of the cone

$$y = k(e^{t_1}c_1 + \dots + e^{t_r}c_r), \quad \text{where } t_1 < t_2 < \dots < t_r \text{ and } k \in K$$

(see [**8**, p. 105]). Then, $\Delta(y+\mathbf{e}) = \prod_{j=1}^r (e^{t_j}+1)$, and

$$\int_\Omega g_{\alpha,\beta,\delta}(y)\, dy$$
$$= c \int_{-\infty}^\infty \int_{-\infty}^{t_r} \cdots \int_{-\infty}^{t_2} \frac{e^{(t_1+\dots+t_r)(n/r+\beta)} \prod_{j<k}(\operatorname{sh}(\frac{1}{2}(t_k-t_j)))^d}{\prod_{j=1}^r (e^{t_j}+1)^\alpha\, (1+\sum_{j=1}^r \log(1+e^{t_j}))^\delta}\, dt_1 \dots dt_r,$$

where $d = \dim V_{j,k} = 2(n/r-1)/(r-1)$. For the necessary condition, we can consider only the case when $\alpha - \beta = 2n/r - 1$ and $\delta = 1$. Moreover, we restrict the region of integration so that $\operatorname{sh} t \geqslant ce^t$, and obtain

$$I \geqslant c \int_{2^r}^{\infty} \int_{2^{r-1}}^{2^{r-1}+1} \dots \int_2^3 \frac{e^{(t_1+\ldots+t_r)(1-n/r)}}{1+t_r} \prod_{j<k} e^{d(t_k-t_j)/2}\, dt_1 \dots dt_r$$

$$\geqslant c' \int_{2^r}^{\infty} \frac{e^{t_r(1-n/r)}}{1+t_r}\, e^{d(r-1)t_r/2}\, dt_r = \infty.$$

To estimate from above, we use the bound

$$\prod_{j<k} \operatorname{sh}(\tfrac{1}{2}(t_k - t_j)) \leqslant \prod_{j<k} e^{(t_k-t_j)/2} = \prod_{j=1}^{r} e^{(j-1-(r-1)/2)t_j}.$$

Then the integral I is bounded by the product

$$\prod_{j=1}^{r-1} \int_{-\infty}^{+\infty} \frac{e^{(d(j-1)+\beta+1)t_j}}{(1+e^{t_j})^{\alpha}}\, dt_j \times \int_{-\infty}^{+\infty} \frac{e^{(d(r-1)+\beta+1)t_r}}{(1+e^{t_r})^{\alpha}(1+\log(1+e^{t_r}))^{\delta}}\, dt_r.$$

Each integral is convergent at $-\infty$ since $\beta+1>0$. We use the conditions on α, β and δ to conclude the proof easily for the integrability at $+\infty$. □

3. *Besov spaces with spectrum in* Ω

3.1. *The Littlewood–Paley decomposition*

Through the rest of the paper, $\{\xi_j\}$ will be a fixed (δ, R)-lattice in Ω with $\delta = \frac{1}{2}$ and $R=2$. We can easily construct a smooth partition of the unity associated with the covering $B_j = B_1(\xi_j)$. For this, we choose a real function $\varphi_0 \in C_c^\infty(B_2(\mathbf{e}))$ such that

$$0 \leqslant \varphi_0 \leqslant 1, \quad \text{and} \quad \varphi_0|_{B_1(\mathbf{e})} \equiv 1.$$

We write each point $\xi_j = g_j \mathbf{e}$, for some fixed $g_j \in G$ (which, for simplicity, we take to be self-adjoint). Then, we can define $\varphi_j(\xi) := \varphi_0(g_j^{-1}\xi)$, so that

$$\varphi_j \in C_c^\infty(B_2(\xi_j)), \quad 0 \leqslant \varphi_j \leqslant 1 \quad \text{and} \quad \varphi_j|_{B_j} \equiv 1. \tag{3.1}$$

We assume that $\xi_0 = \mathbf{e}$, so that there is no ambiguity of notation. By the finite intersection property, there exists a constant $c>0$ such that

$$\frac{1}{c} \leqslant \Phi(\xi) := \sum_j \varphi_j(\xi) \leqslant c.$$

Proposition 3.2. *In the conditions above, let* $\widehat{\psi}_j = \varphi_j/\Phi$. *Then*

(1) $\widehat{\psi}_j \in C_c^\infty(B_2(\xi_j))$;
(2) $0 \leqslant \widehat{\psi}_j \leqslant 1$, *and* $\sum_j \widehat{\psi}_j(\xi) = 1$, *for all* $\xi \in \Omega$;
(3) *the* ψ_j *are uniformly bounded in* $L^1(\mathbb{R}^n)$; *in particular, there exists a constant* $C>0$ *such that*

$$\|f * \psi_j\|_p \leqslant C\,\|f\|_p, \quad \text{for all } f \in L^p(\mathbb{R}^n),\ \text{all } j,\ \text{and } 1 \leqslant p \leqslant \infty. \tag{3.3}$$

Proof. The first two statements are clear. For the last one, note first that

$$\|\psi_j\|_{L^1} = \|\mathcal{F}^{-1}(\varphi_0(g_j^{-1}\,\cdot)/\Phi)\|_{L^1} = \|\mathcal{F}^{-1}(\varphi_0/\Phi(g_j\,\cdot))\|_{L^1}.$$

Now, when $\xi \in B_2(\mathbf{e})$ we can write

$$\Phi(g_j\xi) = \sum_{k\in J_j} \varphi_0(g_k^{-1}g_j\xi),$$

where $J_j = \{k : B_2(\xi_k) \cap B_2(\xi_j) \neq \emptyset\}$ is a finite set with at most N elements by the finite intersection property. Further, we claim that the following uniform estimate holds true:

$$\|g_k^{-1}g_j\| \leqslant C, \quad \text{when } k \in J_j, \text{ for all } j. \tag{3.4}$$

Indeed, since $d(g_k^{-1}g_j\mathbf{e}, \mathbf{e}) = d(\xi_j, \xi_k) \leqslant 4$, by Lemmas 2.12 and 2.9,

$$\|g_k^{-1}g_j\| \sim |g_k^{-1}g_j\mathbf{e}| \sim |\mathbf{e}| \sim 1.$$

From (3.4) the proposition follows easily. Indeed, integrating by parts we have

$$\begin{aligned} \mathcal{F}^{-1}(\varphi_0/\Phi(g_j\,\cdot))(x) &= \int_{B_2(\mathbf{e})} e^{i(x|\xi)}\,\frac{\varphi_0(\xi)}{\Phi(g_j\xi)}\,d\xi \\ &= \int_{B_2(\mathbf{e})} e^{i(x|\xi)}\,\frac{D^L(\varphi_0/\Phi(g_j\,\cdot))(\xi)}{(-|x|^2)^L}\,d\xi, \end{aligned} \tag{3.5}$$

where D^L denotes a power of the Laplacian. All functions $D^L(\varphi_0/\Phi(g_j\,\cdot))$ are bounded. Thus, choosing $L=0$ for $|x| \leqslant 1$, and $L > \frac{1}{2}n$ for $|x| > 1$, we can majorize $\mathcal{F}^{-1}(\varphi_0/\Phi(g_j\,\cdot))$ uniformly in j by an integrable function, and this establishes the result. □

In this paper we shall mainly be concerned with Besov-type seminorms derived from the couple $\{\xi_j, \psi_j\}$ as in (1.3). That is, for $\nu \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$, and $f \in \mathcal{S}'(\mathbb{R}^n)$ we let

$$\|f\|_{B_\nu^{p,q}} := \begin{cases} \left[\sum_j \Delta^{-\nu}(\xi_j)\,\|f * \psi_j\|_p^q\right]^{1/q} & \text{if } q < \infty, \\ \sup_j \Delta^{-\nu}(\xi_j)\,\|f * \psi_j\|_p & \text{if } q = \infty. \end{cases} \tag{3.6}$$

We shall make use of the fact that these seminorms do not actually depend on the choice of the lattice $\{\xi_j\}$ or the test functions ψ_j. Moreover, they can also be defined with test functions which are not normalized as in the previous proposition. That is, we may replace $\{\psi_j\}$ by any family

$$\widehat{\chi}_j(\xi) := \widehat{\chi}(g_j^{-1}\xi), \tag{3.7}$$

defined from an arbitrary $\widehat{\chi} \in C_c^\infty(B_4(\mathbf{e}))$ so that $0 \leqslant \widehat{\chi} \leqslant 1$ and $\widehat{\chi}$ is identically 1 in $B_2(\mathbf{e})$. These and other elementary equivalences are stated and proved in the following lemma.

LEMMA 3.8. *Let $\{\xi_j, \psi_j\}$ be as at the beginning of this section, and fix $\nu \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$. Then, for any other (δ, R)-lattice $\{\widetilde{\xi}_j\}$ with associated Littlewood–Paley functions $\{\widetilde{\psi}_j\}$, and for any family $\{\chi_j\}$ as in (3.7), we have the* *equivalences*

$$\Big[\sum_j \Delta^{-\nu}(\xi_j)\,\|f*\psi_j\|_p^q\Big]^{1/q} \sim \Big[\sum_j \Delta^{-\nu}(\widetilde{\xi}_j)\,\|f*\widetilde{\psi}_j\|_p^q\Big]^{1/q},$$

$$\Big[\sum_j \Delta^{-\nu}(\xi_j)\,\|f*\psi_j\|_p^q\Big]^{1/q} \sim \Big[\sum_j \Delta^{-\nu}(\xi_j)\,\|f*\chi_j\|_p^q\Big]^{1/q},$$

for all $f\in\mathcal{S}'(\mathbb{R}^n)$. Moreover, when $g\in G$ and $q<\infty$, the following equivalence holds:

$$\sum_j \Delta^{-\nu}(\xi_j)\,\|(f\circ g)*\psi_j\|_p^q \sim \Delta(g\mathbf{e})^{-(n/r)(q/p)-\nu}\sum_j \Delta^{-\nu}(\xi_j)\,\|f*\psi_j\|_p^q.$$

Proof. We just consider the case $q<\infty$, the modifications for $q=\infty$ being trivial. For the first part, we can write, for each j,

$$\widehat{\psi}_j=\sum_k \widehat{\psi}_j\widehat{\widetilde{\psi}}_k,$$

where the index k runs through a set $J_j=\{k: B_{R\delta}(\widetilde{\xi}_k)\cap B_2(\xi_j)\neq\emptyset\}$ of at most $N=N(\delta,R,\Omega)$ elements. Then, using (3.3) and Lemma 2.4 we have

$$\begin{aligned}\sum_j \Delta^{-\nu}(\xi_j)\,\|f*\psi_j\|_p^q &\leqslant C\sum_j\sum_{k\in J_j}\Delta^{-\nu}(\xi_j)\,\|f*\widetilde{\psi}_k\|_p^q\\ &\leqslant C'\sum_k \Delta^{-\nu}(\widetilde{\xi}_k)\,\|f*\widetilde{\psi}_k\|_p^q.\end{aligned}$$

The converse inequality follows similarly. For the second equivalence in the lemma, the fact that $\widehat{\chi}_j\widehat{\psi}_j=\widehat{\psi}_j$ immediately implies the left inequality. A similar use of the finite intersection property as we did above gives the right-hand side.

Finally, for our last statement, it is sufficient to prove an inequality of the form '$\lesssim$', the converse inequality '$\gtrsim$' following after replacing g by its inverse. Now, using a first change of variables, and the fact that the determinant of the transformation g in $\mathbb{R}^n$ is equal to $\Delta(g\mathbf{e})^{n/r}$, we are linked to consider the L^p-norms of the functions

$$\Delta(g\mathbf{e})^{-(1+1/p)n/r}\,f*(\psi_j\circ g^{-1})=\Delta(g\mathbf{e})^{-(1+1/p)n/r}\sum_k f*\psi_k*(\psi_j\circ g^{-1}).$$

For each fixed j this last sum has at most N terms, since the Fourier transform of $\psi_k*(\psi_j\circ g^{-1})$ is non-zero only if $d(\xi_k,g^*\xi_j)<4$. So

$$\|f*(\psi_j\circ g^{-1})\|_p\leqslant C\Delta(g\mathbf{e})^{n/r}\sum_{k;d(g^*\xi_k,\xi_j)<4}\|f*\psi_k\|_p,$$

the factor $\Delta(g\mathbf{e})^{n/r}$ appearing as the determinant of the transformation g in the computation of the L^1-norm of $\psi_j\circ g^{-1}$. Now, when $d(g^*\xi_k,\xi_j)<4$, then $\Delta(\xi_j)$ is equivalent to $\Delta(g^*\xi_k)=\Delta(g\mathbf{e})\Delta(\xi_k)$. Thus, we conclude that

$$\sum_j \Delta^{-\nu}(\xi_j)\,\|f*(\psi_j\circ g^{-1})\|_p^q\leqslant C\Delta(g\mathbf{e})^{nq/r-\nu}\sum_j\sum_{k;d(g^*\xi_k,\xi_j)<4}\Delta^{-\nu}(\xi_k)\|f*\psi_k\|_p^q.$$

We get the required inequality by multiplying by $\Delta(g\mathbf{e})^{-q(1+1/p)n/r}$ and summing in the j indices first. □

Recall now that $\mathcal{S}_\Omega$ denotes the space of Schwartz functions f on $\mathbb{R}^n$ with $\operatorname{Supp}\widehat{f} \subset \overline{\Omega}$. The next proposition gives the Littlewood–Paley decomposition $f = \sum_j f * \psi_j$ for functions in $f \in \mathcal{S}_\Omega$, and relates it with the Besov space norm.

Proposition 3.9. *Every $f \in \mathcal{S}_\Omega$ admits a Littlewood–Paley decomposition $f = \sum_j f * \psi_j$ with convergence in $\mathcal{S}(\mathbb{R}^n)$. Further, for every $\nu \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$ there are a constant $C = C(p, q, \nu) > 0$ and an integer $\ell = \ell(p, q, \nu) \geqslant 0$ so that*

$$\begin{aligned} \|f\|_{B^{p,q}_\nu} &= \Big[\sum_j \Delta^{-\nu}(\xi_j)\, \|f * \psi_j\|_p^q\Big]^{1/q} \\ &\leqslant C p_\ell(\widehat{f}) < \infty, \quad \text{for all } f \in \mathcal{S}_\Omega, \end{aligned} \tag{3.10}$$

where $p_\ell(\varphi) = \sup_{|\alpha| \leqslant \ell} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^\ell\, |\partial^\alpha \varphi(\xi)|$ denotes a Schwartz seminorm.

The proof depends on a lemma which gives appropriate estimates for test functions in $\mathcal{S}_\Omega$.

Lemma 3.11. *Let $N, M \geqslant 0$. Then, there are a constant $C = C(N, M) > 0$ and an integer $\ell = \ell(N, M) \geqslant 0$ such that, for every $f \in \mathcal{S}_\Omega$,*

(1) *$|\widehat{f}(\xi)| \leqslant C p_\ell(\widehat{f})\, \Delta^M(\xi)\, (1 + |\xi|)^{-N}$, for all $\xi \in \Omega$;*

(2) *if $1 \leqslant p \leqslant \infty$, then*

$$\|f * \psi_j\|_p \leqslant C p_\ell(\widehat{f})\, \Delta(\xi_j)^{M + (n/r)(1/p')}\, (1 + |\xi_j|)^{-N}, \quad \textit{for all } j.$$

Proof of Lemma 3.11. For the first statement, it suffices to show that for every $f \in \mathcal{S}_\Omega$ and $M \geqslant 1$ there is $M' \geqslant 1$ so that

$$|\widehat{f}(\xi)| \leqslant C p_{M'}(\widehat{f})\, \Delta^M(\xi), \quad \text{whenever } \Delta(\xi) \leqslant 1, \xi \in \Omega. \tag{3.12}$$

Indeed, we then write (3.12) for $D^N f$ to get the full statement. So, let us prove (3.12). Let $\xi \in \Omega$ be fixed, and choose $\xi_0 \in \partial\Omega$ so that $\operatorname{dist}(\xi, \partial\Omega) = |\xi - \xi_0|$. Since $\operatorname{Supp}\widehat{f} \subset \overline{\Omega}$, we have $\partial^\alpha \widehat{f}(\xi_0) = 0$, for every multi-index α. Thus, given $M \geqslant 1$ there is a constant $C = C(M)$ such that $|\widehat{f}(\xi)| \leqslant C p_M(\widehat{f})\, |\xi - \xi_0|^M$. We claim that $|\xi - \xi_0| \leqslant \Delta(\xi)^{1/r}$, which will clearly establish (3.12).

To show our claim, we may assume that $\xi = P(a)\mathbf{e}$, where $a = a_1 c_1 + \ldots + a_r c_r$. Suppose also that $a_1 = \min\{a_1, \ldots, a_r\}$. Then

$$\begin{aligned} |\xi - \xi_0| = \operatorname{dist}(\xi, \partial\Omega) &\leqslant |\xi - (a_2^2 c_2 + \ldots + a_r^2 c_r)| \\ &= a_1^2 \leqslant (a_1^2 \ldots a_r^2)^{1/r} = \Delta(\xi)^{1/r}. \end{aligned}$$

Let us now prove the second statement in Lemma 3.11. It is sufficient to prove the same inequality, with the system χ_j instead of ψ_j. Given $f \in \mathcal{S}_\Omega$, we proceed as in (3.5):

$$f * \chi_j(x) = \int_\Omega e^{i(x|\xi)}\, \widehat{f}(\xi)\widehat{\chi}(g_j^{-1}\xi)\, d\xi \tag{3.13}$$
$$= \Delta^{n/r}(\xi_j) \int_\Omega e^{i(g_j x|\xi)}\, \widehat{f}(g_j\xi)\widehat{\chi}(\xi)\, d\xi$$
$$= \Delta^{n/r}(\xi_j) \int_{B_2(\mathbf{e})} e^{i(g_j x|\xi)}\, \frac{D^L(\widehat{f}(g_j\xi)\widehat{\chi}(\xi))}{(-|g_j x|^2)^L}\, d\xi.$$

The estimates in the first part, together with Lemmas 2.4, 2.9 and 2.12, imply that, on the invariant ball $B_2(\mathbf{e})$,

$$|D^L(\widehat{f}(g_j\xi))| \leqslant C(1+\|g_j\|)^{2L} \sum_{|\alpha|\leqslant 2L} |(\partial^\alpha \widehat{f})(g_j\xi)|$$
$$\leqslant C' p_\ell(\widehat{f})\, \frac{\Delta^M(\xi_j)}{(1+|\xi_j|)^N},$$

for some integer $\ell = \ell(M,N,L)$. Therefore

$$|f * \chi_j(x)| \leqslant C p_\ell(\widehat{f})\, \Delta^{n/r}(\xi_j)\, \frac{\Delta^M(\xi_j)}{(1+|\xi_j|)^N}\, \frac{1}{(1+|g_j x|^2)^L}, \quad \text{for } x \in \mathbb{R}^n.$$

Taking L^p-norms and changing variables, we conclude with

$$\|f * \chi_j\|_p \leqslant C p_\ell(\widehat{f})\, \frac{\Delta(\xi_j)^{M+(n/r)(1/p')}}{(1+|\xi_j|)^N}. \tag{3.14}$$

□

Proof of Proposition 3.9. Once we show the convergence of the series $\sum_j f * \psi_j$, the fact that the sum equals f is immediate from $\operatorname{Supp} \widehat{f} \subset \overline{\Omega}$ and $\sum_j \widehat{\psi}_j = \chi_\Omega$. Now, from the previous lemma and Proposition 2.13, we have

$$\sum_j \|\widehat{f}\,\widehat{\psi}_j\|_\infty = \sum_j \Big\|\widehat{f}(g_j\,\cdot)\,\frac{\varphi_0}{\Phi(g_j\,\cdot)}\Big\|_\infty$$
$$\leqslant C_f \sum_j \Delta^M(\xi_j)\,(1+|\xi_j|)^{-N} \leqslant C_f' \int_\Omega \frac{\Delta^M(\xi)}{(1+|\xi|)^N}\, \frac{d\xi}{\Delta(\xi)^{n/r}},$$

where the last integral is finite for N and M large enough. A similar argument applies to $\|(1+|\xi|)^L \partial^\alpha(\widehat{f}\,\widehat{\psi}_j)\|_\infty$, establishing our claim.

For the second assertion in the proposition, we use the second estimate in Lemma 3.11. Assuming $q < \infty$ (otherwise the estimate is trivial) we have

$$\|f\|^q_{B^{p,q}_\nu} \leqslant C p_\ell(\widehat{f})^q \sum_j \Delta^{-\nu}(\xi_j)\, \frac{\Delta(\xi_j)^{Mq+(n/r)(q/p')}}{(1+|\xi_j|)^{Nq}} \tag{3.15}$$
$$\leqslant C' p_\ell(\widehat{f})^q \int_\Omega \frac{\Delta(\xi)^{Mq+(n/r)(q/p')-\nu}}{(1+|\xi|)^{Nq}}\, \frac{d\xi}{\Delta(\xi)^{n/r}},$$

which is finite for a sufficiently large choice of N and M. □

We observe that we have strongly relied on the assumption on the support of $\widehat{f}$. The next proposition gives the sharp region of the indices ν, p and q for which general Schwartz functions have finite $B_\nu^{p,q}$-seminorms. We state this fact separately since such conditions will appear in the sequel in relation with the index $\widetilde{q}_{\nu,p}$.

Proposition 3.16. *Let $\nu \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$ be such that*

$$\frac{q}{p'}\frac{n}{r} > \nu + \frac{n}{r} - 1 \tag{3.17}$$

(or $(1/p')(n/r) \geqslant \nu$ when $q = \infty$). Then, there exist $C = C(p,q,\nu) > 0$ and an integer $\ell = \ell(p,q,\nu) \geqslant 0$ so that

$$\|f\|_{B_\nu^{p,q}} \leqslant C p_\ell(\widehat{f}) < \infty, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n). \tag{3.18}$$

Moreover, this property can hold for all $f \in \mathcal{S}(\mathbb{R}^n)$ only if ν, p and q satisfy (3.17).

Proof. As before we assume that $q < \infty$, with obvious modifications when $q = \infty$. First one observes that, when $f \in \mathcal{S}(\mathbb{R}^n)$, the conclusion of Lemma 3.11 is still valid with $M = 0$. Thus, similar reasoning as in (3.15) gives

$$\|f\|_{B_\nu^{p,q}} \leqslant C p_\ell(\widehat{f}) \left[\int_\Omega \frac{\Delta^{(q/p')(n/r)-\nu}(\xi)}{(1+|\xi|)^{Nq}} \frac{d\xi}{\Delta(\xi)^{n/r}} \right]^{1/q}, \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n),$$

and this integral is finite under the condition $(q/p')(n/r) > \nu + n/r - 1$. To show that this condition is critical, take any $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{f}$ is identically 1 in the Euclidean ball centered at 0 of radius 1. Then, for such an f, one has the bound from below

$$\|f\|_{B_\nu^{p,q}}^q \geqslant c \sum_{j; |\xi_j| < c} \Delta(\xi_j)^{-\nu} \|\chi_j\|_p^q \geqslant c'' \int_{\xi \in \Omega; |\xi| < c'} \Delta^{(q/p')(n/r)-\nu}(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}},$$

and this last integral is infinity unless $(q/p')(n/r) > \nu + n/r - 1$. □

3.2. *Properties of Besov spaces*

Given a closed set $F \subset \mathbb{R}^n$, we shall denote by $S'_F = S'_F(\mathbb{R}^n)$ the space of tempered distributions with Fourier transform supported in F. Recall the expression of the 'seminorm' $\|f\|_{B_\nu^{p,q}}$ in (1.3), and observe that a distribution $f \in \mathcal{S}'_{\overline{\Omega}}$ satisfies $\|f\|_{B_\nu^{p,q}} = 0$ if and only if $f \in \mathcal{S}'_{\partial\Omega}$. This leads to the following natural definition of Besov spaces with spectrum in Ω.

Definition 3.19. Given $\nu \in \mathbb{R}$ and $1 \leqslant p, q < \infty$, we define $B_\nu^{p,q}$ as the space of equivalence classes of tempered distributions

$$B_\nu^{p,q} = \{f \in \mathcal{S}'_{\overline{\Omega}} : \|f\|_{B_\nu^{p,q}} < \infty\} / \mathcal{S}'_{\partial\Omega}.$$

It follows right away from Lemma 3.8 that $B_\nu^{p,q}$ does not depend on the choice of $\{\psi_j\}$ or the lattice $\{\xi_j\}$. Moreover, $B_\nu^{p,q}$ is invariant under the action of G, and

$$\|f \circ g\|_{B_\nu^{p,q}} \sim \Delta(g\mathbf{e})^{-(n/rp)-(\nu/q)} \|f\|_{B_\nu^{p,q}}. \tag{3.20}$$

Before collecting in the next proposition other basic properties of the spaces $B^{p,q}_\nu$, we make a minor comment on the notation used below.

NOTATION 3.21. Throughout this paper, the standard action of tempered distributions over Schwartz functions will be denoted by

$$(f,\varphi)=\int f\,\varphi, \quad \text{for } f\in\mathcal{S}'(\mathbb{R}^n),\ \varphi\in\mathcal{S}(\mathbb{R}^n).$$

For convenience, we shall often use the *anti-linear pairing*:

$$\langle f,\varphi\rangle:=(f,\overline{\varphi})=\int_{\mathbb{R}^n} f\,\overline{\varphi}, \quad \text{for } f\in\mathcal{S}'(\mathbb{R}^n),\ \varphi\in\mathcal{S}(\mathbb{R}^n).$$

This has the notational advantage of a simple Plancherel identity: $\langle f,\varphi\rangle=\langle\widehat{f},\widehat{\varphi}\rangle$, leading to a natural pairing between $f\in\mathcal{S}'_{\overline{\Omega}}(\mathbb{R}^n)$ and $\varphi\in\mathcal{S}_\Omega$ (rather than using $(f,\varphi)=(\widehat{f},\widehat{\varphi}(-\,\cdot\,))$, which requires one to deal with $\varphi\in\mathcal{S}_{-\Omega}$).

With the above considerations we have the following lemma.

LEMMA 3.22. *Let* $\nu\in\mathbb{R}$, $1\leqslant p\leqslant\infty$ *and* $1<q<\infty$. *Then, there exists* $\ell=\ell(\nu,p,q)\geqslant 0$ *so that, for every distribution* $f\in\mathcal{S}'(\mathbb{R}^n)$ *with* $\|f\|_{B^{p,q}_\nu}<\infty$, *we have*

$$|\langle f,\varphi\rangle|\leqslant C\,\|f\|_{B^{p,q}_\nu}\,\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}\leqslant C'\,\|f\|_{B^{p,q}_\nu}\,p_\ell(\widehat{\varphi}), \quad \text{for all } \varphi\in\mathcal{S}_\Omega. \tag{3.23}$$

When $q=1$ *or* $q=\infty$, *the same holds with* $\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}$ *replaced by* $\|\varphi\|_{B^{p',q'}_{-\nu}}$.

Proof. Remember that $\widehat{\psi}_j=\widehat{\psi}_j\widehat{\chi}_j$ for all j. Therefore, using Proposition 3.9 and Lemma 3.8 we have

$$\begin{aligned}|\langle f,\varphi\rangle|&\leqslant\sum_j|\langle f*\psi_j,\varphi*\chi_j\rangle|\leqslant\sum_j\|f*\psi_j\|_p\|\varphi*\chi_j\|_{p'}\\&\leqslant C\,\|f\|_{B^{p,q}_\nu}\,\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}.\end{aligned} \tag{3.24}$$

Finally, observe that $\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}=\|\varphi\|_{B^{p',q'}_{\nu(1-q')}}$, which, for $\varphi\in\mathcal{S}_\Omega$, is bounded by a Schwartz seminorm by Proposition 3.9. The modifications needed for the cases $q=1,\infty$ are obvious. □

PROPOSITION 3.25. *Let* $1\leqslant p,q<\infty$ *and* $\nu\in\mathbb{R}$. *Then*

(1) $B^{p,q}_\nu$ *is a Banach space;*
(2) *the space* $\mathcal{D}_\Omega:=\{f\in\mathcal{S}(\mathbb{R}^n):\operatorname{Supp}\widehat{f}$ *is compact in* $\Omega\}$ *is dense in* $B^{p,q}_\nu$; *moreover, for every class* $f+\mathcal{S}'_{\partial\Omega}$ *in* $B^{p,q}_\nu$, *the series* $\sum_j f*\psi_j$ *converges to* (*the class of*) f *in the space* $B^{p,q}_\nu$.

Proof. Suppose $\{f_m\}_m$ is a Cauchy sequence of distributions in $\mathcal{S}'_{\overline{\Omega}}$ for the $B^{p,q}_\nu$-seminorm. Then, from the previous lemma it follows that $\langle f_m,\varphi\rangle$ converges for every $\varphi\in\mathcal{S}_\Omega$, and moreover, it defines a continuous (anti-linear) functional in $\mathcal{S}_\Omega$. We can extend it to $\mathcal{S}_\Omega\oplus\mathcal{S}_{\Omega^c}$ by letting it be identically zero in the second summand, and finally extend it to the whole Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by the Hahn–Banach theorem. This gives a tempered distribution $f\in\mathcal{S}'_{\overline{\Omega}}$ which in particular satisfies

$$f * \psi_j(x) = \lim_{m\to\infty} f_m * \psi_j(x), \quad \text{for all } x \in \mathbb{R}^n \text{ and all } j.$$

Therefore, by Fatou's lemma

$$\|f\|_{B_\nu^{p,q}} \leqslant \varliminf_{m\to\infty} \|f_m\|_{B_\nu^{p,q}} < \infty,$$

and in a similar fashion $\lim_{m\to\infty} \|f - f_m\|_{B_\nu^{p,q}} = 0$. This shows that $B_\nu^{p,q}$ is a Banach space.

For the density, let f be a fixed distribution in $\mathcal{S}'_{\overline{\Omega}}$ with $\|f\|_{B_\nu^{p,q}} < \infty$. We shall show that f is the $B_\nu^{p,q}$-limit of the partial sums of the series $\sum_j f * \psi_j$. Remark that each finite sum belongs to $L^p(\mathbb{R}^n)$, and therefore can be approached by a Schwartz function with Fourier transform supported in a compact set of Ω, justifying the density of $\mathcal{D}_\Omega$. Now, it is easily seen that partial sums (for any order) constitute a Cauchy sequence in $B_\nu^{p,q}$. Since $B_\nu^{p,q}$ is a Banach space, they converge to a distribution $u \in \mathcal{S}'_{\overline{\Omega}}$. It remains to show that u and f belong to the same equivalence class in $\mathcal{S}'_{\overline{\Omega}}/\mathcal{S}'_{\partial\Omega}$, which is an immediate consequence of the fact that $f * \psi_j = u * \psi_j$. □

For the duality of the spaces $B_\nu^{p,q}$, recall the Hölder type inequality in (3.23):

$$|(\overline{f}, g)| = |\langle f, g\rangle| \leqslant C \, \|f\|_{B^{p',q'}_{-\nu q'/q}} \, \|g\|_{B_\nu^{p,q}}, \quad \text{for } g \in \mathcal{D}_\Omega,$$

valid for every $f \in \mathcal{S}'_{\overline{\Omega}}$ with $\|f\|_{B^{p',q'}_{-\nu q'/q}} < \infty$. Observe that $g \in \mathcal{D}_\Omega \mapsto (\overline{f}, g)$ is a linear functional in $\mathcal{D}_\Omega$ which depends only on the equivalence class $f + \mathcal{S}'_{\partial\Omega}$. Thus, by the above inequality and the density of $\mathcal{D}_\Omega$, it defines a continuous linear functional Φ_f in $B_\nu^{p,q}$. Further, if $\Phi_f = 0$, then $(\overline{f}, g) = 0$, for all $g \in \mathcal{D}_\Omega$, and necessarily $f \in \mathcal{S}'_{\partial\Omega}$. Thus, the correspondence

$$f + \mathcal{S}'_{\overline{\Omega}} \in B^{p',q'}_{-\nu q'/q} \longrightarrow \Phi_f \in (B_\nu^{p,q})^* \tag{3.26}$$

is well defined and injective.

Proposition 3.27. *Let $\nu \in \mathbb{R}$ and $1 \leqslant p, q < \infty$. Then, the mapping in (3.26) is an anti-linear isomorphism of Banach spaces.*

Proof. By the previous comments it suffices to show that for every $\Phi \in (B_\nu^{p,q})^*$, there exists a distribution $f \in \mathcal{S}'_{\overline{\Omega}}$ such that

$$\Phi(g) = (\overline{f}, g), \quad \text{for all } g \in \mathcal{D}_\Omega \quad \text{and} \quad \|f\|_{B^{p',q'}_{-\nu q'/q}} \leqslant C\,\|\Phi\|. \tag{3.28}$$

Now, since $\mathcal{D}_\Omega \subset \mathcal{S}_\Omega \hookrightarrow B_\nu^{p,q}$, by the Hahn–Banach theorem we can extend Φ continuously to $\mathcal{S}(\mathbb{R}^n)$, and find a tempered distribution $f \in \mathcal{S}'_{\overline{\Omega}}$ such that $\Phi(g) = (\overline{f}, g)$, for all $g \in \mathcal{D}_\Omega$.

We now claim that each $f * \psi_j$, which *a priori* is only a smooth function with polynomial growth, does belong to $L^{p'}(\mathbb{R}^n)$, and moreover, the sequence of their $L^{p'}$-norms belongs to the suitable space of sequences. Indeed, for every finite sequence $g_j \in \mathcal{S}(\mathbb{R}^n)$ with $\sum_j \Delta(\xi_j)^{-\nu}\|g_j\|_p^q \leqslant 1$ we have

$$\Big|\sum_j \langle f * \psi_j, g_j\rangle\Big| = \Big|\Phi\Big(\sum_j g_j * \psi_j\Big)\Big|$$
$$\leqslant \|\Phi\| \,\Big\|\sum_j g_j * \psi_j\Big\|_{B^{p,q}_\nu} \leqslant C\,\|\Phi\|.$$

The constant C depends only on the number N in the finite intersection property, the constant γ related to the variation of the function Δ inside an invariant ball of radius 2, and the L^1-norm of the ψ_j. Since the constant is independent of the finite set of indices, (3.28) follows. We do not give the details of the proof, since it is completely analogous to that of Lemma 3.8. □

Let us remark that, for two classes of tempered distributions $f+\mathcal{S}'_{\partial\Omega}$ in $B^{p,q}_\nu$ and $g+\mathcal{S}'_{\partial\Omega}$ in $B^{p',q'}_{-\nu q'/q}$, the duality pairing can also be expressed as

$$\Phi_f(g+\mathcal{S}'_{\partial\Omega}) = \sum_j \langle f*\psi_j, g*\chi_j\rangle, \tag{3.29}$$

where the series converges absolutely by (3.23). This representation is sometimes convenient, and of course, independent on the choice of $\{\psi_j,\chi_j\}$.

3.3. *The □ operator and Besov multipliers*

Next we describe some analytic properties of the spaces $B^{p,q}_\nu$. The first one concerns the role of the generalized wave operator □ (introduced in (1.2)) as a natural isomorphism between these spaces. Below we shall be interested in fractional and negative powers of □, which can be defined by the rule

$$\Box^\beta f = \mathcal{F}^{-1}(\Delta^\beta \widehat{f}), \tag{3.30}$$

at least for distributions $f \in \mathcal{S}'(\Omega)$ so that $\operatorname{Supp}\widehat{f}$ is compact in Ω. Our next result is a more general version than (2) in Theorem 1.4.

Proposition 3.31. *Let $\nu,\beta\in\mathbb{R}$ and $1\leqslant p,q<\infty$. Then there is a constant $C>0$ such that, for every distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ with* $\operatorname{Supp}\widehat{f}$ *compact in Ω,*

$$C^{-1}\,\|f\|_{B^{p,q}_\nu} \leqslant \|\Box^\beta f\|_{B^{p,q}_{\nu+\beta q}} \leqslant C\|f\|_{B^{p,q}_\nu}. \tag{3.32}$$

In particular, $\Box^\beta$ extends to an isomorphism $\widetilde{\Box}^\beta\colon B^{p,q}_\nu \to B^{p,q}_{\nu+\beta q}$.

Proof. Indeed, given $f\in\mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{Supp}\widehat{f}$ compact in Ω we have

$$\|\Box^\beta f\|^q_{B^{p,q}_{\nu+\beta q}} = \sum_j \Delta(\xi_j)^{-(\nu+\beta q)}\,\|\mathcal{F}^{-1}(\widehat{f}\widehat{\psi}_j\Delta^\beta)\|_p^q$$
$$\leqslant \sum_j \Delta(\xi_j)^{-(\nu+\beta q)}\|f*\psi_j\|_p^q\,\|\mathcal{F}^{-1}(\widehat{\chi}_j\Delta^\beta)\|_1^q.$$

Using $\Delta(g\xi)=\Delta(g\mathbf{e})\Delta(\xi)$, for $g\in G$, we have

$$\|\mathcal{F}^{-1}(\widehat{\chi}_j\Delta^\beta)\|_1 = \Delta^\beta(\xi_j)\,\|\mathcal{F}^{-1}(\widehat{\chi}\Delta^\beta)\|_1 = c_\beta\Delta^\beta(\xi_j),$$

from which (3.32) follows easily. To extend $\Box^\beta$ to the space $B^{p,q}_\nu$ one proceeds by density. More precisely, given any $f \in \mathcal{S}'_{\overline{\Omega}}$ with $\|f\|_{B^{p,q}_\nu} < \infty$, we denote by $\widetilde{\Box}^\beta f$ a representative from the equivalence class of $\sum_j \Box^\beta(f * \psi_j)$, which by (3.32) (and Proposition 3.25) is a Cauchy series in the $B^{p,q}_{\nu+\beta q}$-seminorm. Observe that $\widetilde{\Box}^\beta \mathcal{S}'_{\partial\Omega} = 0$ (or its equivalence class), while by uniqueness of the extension, $\widetilde{\Box}^\beta$ does not depend on the Littlewood–Paley functions $\{\psi_j\}$. □

A further step in the previous idea leads to a *functional calculus in* $B^{p,q}_\nu$ based on the operator $\Box$. Let $m \in C^\infty(0,\infty)$ be a Mihlin-type multiplier in one dimension. That is, there is a constant $C = C(m)$ so that

$$\sup_{\xi>0} |\xi|^k \, |m^{(k)}(\xi)| \leqslant C, \quad \text{for all } k = 0, 1, \dots . \tag{3.33}$$

Then, it makes sense to define the operator $m(\Box)$ by

$$m(\Box)(f) = \mathcal{F}^{-1}(m(\Delta)\widehat{f}),$$

at least for $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{Supp} \widehat{f}$ compact in Ω. Observe that a typical example is given by the imaginary powers $m(\xi) = \xi^{i\gamma}$ for $\gamma \in \mathbb{R}$. Then we have the following result.

Proposition 3.34. *Let* $\nu \in \mathbb{R}$, $1 \leqslant p, q < \infty$ *and* $m \in C^\infty(0,\infty)$ *satisfying* (3.33). *Then, there is a constant* $C = C(m)$ *such that*

$$\|m(\Box)(f)\|_{B^{p,q}_\nu} \leqslant C_m \, \|f\|_{B^{p,q}_\nu}, \quad \textit{for } f \in \mathcal{D}_\Omega.$$

In particular, $m(\Box)$ *extends to a bounded operator in* $B^{p,q}_\nu$.

Proof. For the proof it suffices to estimate

$$\|m(\Box)(f) * \psi_j\|_p \leqslant \|f * \psi_j\|_p \, \|\mathcal{F}^{-1}(m(\Delta)\widehat{\chi}_j)\|_1, \quad \text{for } f \in \mathcal{D}_\Omega. \tag{3.35}$$

Now,

$$\begin{aligned} \|\mathcal{F}^{-1}(m(\Delta)\widehat{\chi}_j)\|_1 &= \|\mathcal{F}^{-1}(m(\Delta(\xi_j)\Delta)\widehat{\chi})\|_1 \\ &\leqslant C p_\ell(m(\Delta(\xi_j)\Delta)\widehat{\chi}) \\ &\leqslant C' \sup_{\xi \in B(\mathbf{e},4)} \sum_{s=0}^{\ell} \Delta(\xi_j)^s \, |m^{(s)}(\Delta(\xi_j)\Delta(\xi))| \leqslant C''. \end{aligned} \tag{3.36}$$

Thus, raising (3.35) to the qth power and summing we easily complete the proof. □

Remark 3.37. 1. The previous proposition also holds under the milder hypothesis $m \in C^{n+1}(0,\infty)$ and (3.33) for $k = 0, 1, \dots, n+1$. This follows from the fact that the key inequality (3.36) is actually valid with $\ell = n+1$.

2. A similar result can be proved with higher-dimensional multipliers. More precisely, given $m \in C^\infty((0,\infty)^r)$ satisfying

$$\sup_{\xi \in (0,\infty)^r} \left| \xi_1^{\alpha_1} \dots \xi_r^{\alpha_r} \, \frac{\partial^\alpha m}{\partial \xi_1^{\alpha_1} \dots \partial \xi_r^{\alpha_r}}(\xi) \right| \leqslant C_\alpha, \quad \text{for all } \alpha = (\alpha_1, \dots, \alpha_r) \geqslant 0,$$

we can define the operator

$$T_m f = \mathcal{F}^{-1}(m(\Delta_1, \dots, \Delta_r)\widehat{f}) \quad \text{if } f \in \mathcal{D}_\Omega.$$

Then, an analogous proof gives $\|T_m f\|_{B^{p,q}_\nu} \leqslant C \|f\|_{B^{p,q}_\nu}$.

3. In the previous examples m is a Fourier multiplier belonging to M_p for all $1<p<\infty$. More general examples of multipliers for $B^{p,q}_\nu$ can be constructed as follows. Let $\{m_j\}_j$ be a uniformly bounded family of multipliers in $M_p(\mathbb{R}^n)$ with $\operatorname{Supp} m_j$ contained in a fixed compact set of Ω. Let $m(\xi)=\sum_j m_j(g_j^{-1}\xi)$ and $T_m f=\mathcal{F}^{-1}(m\widehat{f})$. Then $\|T_m f\|_{B^{p,q}_\nu}\leqslant C\|f\|_{B^{p,q}_\nu}$. In particular, we may take $m_j=\varepsilon_j\widehat{\psi}$, where $\varepsilon_j=\pm 1$, and conclude that $m_\varepsilon=\sum_j\varepsilon_j\psi_j$ is a multiplier for $B^{p,q}_\nu$. Observe however, that if we let $\varepsilon_j=1$, the function $m_\varepsilon=\chi_\Omega\notin M_p(\mathbb{R}^n)$ for any $p\neq 2$.

3.4. *Fourier–Laplace extensions*

It is well known that to every distribution supported in a closed cone $\overline{\Omega}$ we can associate an analytic function in the tube domain T_Ω via the *Fourier–Laplace integral*. More precisely, this is given by

$$\mathcal{L}g(z)=(g,e^{i(z|\xi)})=\int_\Omega e^{i(z|\xi)}\,g(\xi)\,d\xi,\quad \text{for } z\in T_\Omega,$$

which makes sense for compactly supported distributions g in Ω, and can also be given a meaning for all $g\in\mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{Supp} g\subset\overline{\Omega}$ (see [**11**, Chapter VII]).

In this section we wish to describe the analytic functions associated with (classes of) distributions in our Besov spaces $B^{p,q}_\nu$. To avoid dealing with equivalence classes, it is convenient to restrict the indices ν, p and q so that $B^{p,q}_\nu$ can be embedded in the usual space of tempered distributions.

LEMMA 3.38. *Let $\nu>0$, $1\leqslant p<\infty$ and $1\leqslant q<\widetilde{q}_{\nu,p}$. Then, for every $f\in\mathcal{S}'_{\overline{\Omega}}$ with $\|f\|_{B^{p,q}_\nu}<\infty$, the series $\sum_j f*\psi_j$ converges in the space $\mathcal{S}'(\mathbb{R}^n)$. Moreover, the correspondence*

$$\begin{aligned} B^{p,q}_\nu &\longrightarrow \mathcal{S}'(\mathbb{R}^n),\\ f+\mathcal{S}'_{\partial\Omega} &\longmapsto f^\sharp=\sum_j f*\psi_j \end{aligned}$$

is continuous, injective, and does not depend on the Littlewood–Paley functions $\{\psi_j\}$.

Proof. The proof of the convergence of the series is completely analogous to that of Lemma 3.22. In fact, using the Hölder-type inequality in (3.23) we can write

$$\sum_j|\langle f*\psi_j,\varphi\rangle|\leqslant C\|f\|_{B^{p,q}_\nu}\|\varphi\|_{B^{p',q'}_{-\nu q'/q}},\quad \text{for } \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{3.39}$$

Now, from $q<\widetilde{q}_{\nu,p}$ and Proposition 3.16 we obtain

$$\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}\leqslant Cp_\ell(\widehat{\varphi})<\infty,\quad \text{for all } \varphi\in\mathcal{S}(\mathbb{R}). \tag{3.40}$$

Using the previous two formulas and the density of $\mathcal{D}_\Omega$ we are able to verify the last statement of the lemma. □

REMARK 3.41. From now on, whenever we restrict to the indices $\nu>0$, $1\leqslant p<\infty$ and $1\leqslant q<\widetilde{q}_{\nu,p}$ as in the previous lemma, we shall identify $B^{p,q}_\nu$ with the corresponding space $(B^{p,q}_\nu)^\sharp$ of tempered distributions. Observe that all these distributions have finite order, are supported in $\overline{\Omega}$ and satisfy the Littlewood–Paley decomposition $f = \sum_j f * \psi_j$.

For the purposes of this paper, we shall prefer to speak of *Fourier–Laplace extensions* for distributions f with spectrum in $\overline{\Omega}$. That is, we define

$$\mathcal{E}f(z) = \mathcal{L}\widehat{f}(z) = (\widehat{f}, e^{i(z|\xi)}) = \int_\Omega e^{i(z|\xi)}\widehat{f}(\xi)\,d\xi, \quad \text{for } z \in T_\Omega,$$

which we shall call the *Fourier–Laplace extension* of f, and which defines a holomorphic function in the tube domain T_Ω. For distributions $f \in B_\nu^{p,q}$ this takes the form

$$\widetilde{\mathcal{E}}f(z) = \mathcal{E}\Big(\sum_j f * \psi_j\Big)(z) = \sum_j \mathcal{L}(\widehat{f}\widehat{\psi}_j)(z), \quad \text{for } z \in T_\Omega, \tag{3.42}$$

where as we shall see below, the last series converges uniformly in compact sets of T_Ω. We stick to the notation $\widetilde{\mathcal{E}}f$ to recall that we are choosing the special representative $f^\sharp = \sum_j f * \psi_j$ from the equivalence class $f + \mathcal{S}'_{\partial\Omega}$. Observe that in general $\widetilde{\mathcal{E}}(\mathcal{S}'_{\partial\Omega}) = 0$, while $\mathcal{E}(\mathcal{S}'_{\partial\Omega})$ is not zero. The main result about Fourier–Laplace extensions is the following proposition.

PROPOSITION 3.43. *Let $\nu > 0$, $1 \leqslant p < \infty$ and $1 \leqslant q < \widetilde{q}_{\nu,p}$. Then for every distribution $f = \sum_j f * \psi_j \in B_\nu^{p,q}$ the series in (3.42) converges uniformly on compact sets to a holomorphic function in T_Ω, and moreover*

$$|\widetilde{\mathcal{E}}f(x+iy)| \leqslant C\Delta(y)^{-(n/r)(1/p)-\nu/q}\,\|f\|_{B_\nu^{p,q}}, \quad \text{for } x+iy \in T_\Omega. \tag{3.44}$$

In addition, for each $y \in \Omega$, the distributions $\widetilde{\mathcal{E}}f(\cdot + iy)$ satisfy

$$\widetilde{\mathcal{E}}(f)(\cdot + iy) = \sum_j \widetilde{\mathcal{E}}(f)(\cdot + iy) * \psi_j, \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

and

$$\|\widetilde{\mathcal{E}}f(\cdot + iy)\|_{B_\nu^{p,q}} \leqslant C\,\|f\|_{B_\nu^{p,q}}, \qquad \lim_{\substack{y\to 0\\ y\in\Omega}} \|\widetilde{\mathcal{E}}f(\cdot + iy) - f\|_{B_\nu^{p,q}} = 0. \tag{3.45}$$

Proof. In the first part we shall only prove the pointwise convergence and (3.44). The proof can easily be adapted to obtain uniform convergence on compact sets. Since $B_\nu^{p,q}$ is invariant under the action of G as well as under translations in the x variable, we can reduce to the case $y = i\mathbf{e}$ and $x = 0$, using (3.20) to prove (3.44) in the general case. Now, by definition of $\mathcal{E}$ and following the same steps as in (3.23), we can write

$$\begin{aligned}\sum_j |\mathcal{E}(f * \psi_j)(i\mathbf{e})| &= \sum_j |\langle f * \psi_j, \mathcal{F}^{-1}(\widehat{\chi}_j e^{(-\mathbf{e}|\cdot)})\rangle| \\ &\leqslant C\,\|f\|_{B_\nu^{p,q}}\,\|\mathcal{F}^{-1}(\chi_\Omega e^{(-\mathbf{e}|\cdot)})\|_{B^{p',q'}_{-\nu q'/q}}.\end{aligned}$$

So, it suffices to compute the norm of

$$h = \mathcal{F}^{-1}(\chi_\Omega e^{-(\mathbf{e}|\cdot)}) = \Gamma_\Omega(n/r)\,\Delta^{-n/r}((\cdot + i\mathbf{e})/i).$$

Now, proceeding as in (3.13) with the function h we obtain the estimate

$$|h * \psi_j(x)| \leqslant C\Delta^{n/r}(\xi_j)\, \frac{(1+|\xi_j|)^{2n}}{(1+|g_j x|^2)^n}\, e^{-(\mathbf{e}|\xi_j)/\gamma} \tag{3.46}$$
$$\leqslant C'\Delta^{n/r}(\xi_j)\, \frac{e^{-(\mathbf{e}|\xi_j)/2\gamma}}{(1+|g_j x|^2)^n}.$$

Taking $L^{p'}$-norms and summing, we are led to

$$\|h\|_{B^{p',q'}_{-\nu q'/q}} \leqslant C\left[\int_\Omega \Delta(\xi)^{\nu q'/q+(n/r)(q'/p)}\, e^{-c(\mathbf{e}|\xi)}\, \frac{d\xi}{\Delta(\xi)^{n/r}}\right]^{1/q'}.$$

By Lemma 2.16 this integral is finite provided $q < \widetilde{q}_{\nu,p}$.

For the second part, fix $y \in \Omega$ and j, and use the Dominated Convergence Theorem with $\sum_k |\mathcal{E}(f * \psi_k)(x+iy)| \leqslant C_y$ (from the first part) to write

$$(\widetilde{\mathcal{E}}(f)(\cdot+iy) * \psi_j)(x) = \sum_k (\mathcal{E}(f*\psi_k)(\cdot+iy)*\psi_j)(x)$$
$$= \sum_k \int_\Omega \widehat{f}(\xi)\widehat{\psi}_k(\xi)e^{-(y|\xi)}\, \widehat{\psi}_j(\xi)e^{i(x|\xi)}\, d\xi$$
$$= \int_\Omega e^{i(x+iy|\xi)}\, \widehat{f}(\xi)\widehat{\psi}_j(\xi)\, d\xi = \mathcal{E}(f*\psi_j)(x+iy).$$

Summing in j and using the previous step again we see that

$$\widetilde{\mathcal{E}}(f)(x+iy) = \sum_j (\widetilde{\mathcal{E}}(f)(\cdot+iy)*\psi_j)(x),$$

converging uniformly and absolutely in x, and hence also in $\mathcal{S}'(\mathbb{R}^n)$.

Let us finally prove the statements in (3.45). First of all,

$$\|\widetilde{\mathcal{E}}f(\cdot+iy)\|^q_{B^{p,q}_\nu} = \sum_j \Delta^{-\nu}(\xi_j)\, \|\mathcal{F}^{-1}(\widehat{f}\widehat{\psi}_j e^{-(y|\cdot)})\|^q_p$$
$$\leqslant \sum_j \Delta^{-\nu}(\xi_j)\, \|f*\psi_j\|^q_p\, \|\mathcal{F}^{-1}(\widehat{\chi}_j e^{-(y|\cdot)})\|^q_1 \leqslant C\, \|f\|^q_{B^{p,q}_\nu},$$

where in the last step we have estimated the L^1-norm by a Schwartz seminorm

$$\|\mathcal{F}^{-1}(\widehat{\chi}_j e^{-(y|\cdot)})\|_1 = \|\mathcal{F}^{-1}(\widehat{\chi} e^{-(g_j y|\cdot)})\|_1 \tag{3.47}$$
$$\leqslant C p_\ell(\widehat{\chi} e^{-(g_j y|\cdot)})$$
$$\leqslant C'(1+|g_j y|)^\ell\, e^{-(g_j y|\mathbf{e})/\gamma}$$
$$\leqslant C'', \quad \text{for all } j \text{ and all } y \in \Omega.$$

Finally, for the convergence, use the density to write $f = g + h$, with $g \in \mathcal{D}_\Omega$ and h with a small $B^{p,q}_\nu$-norm. It is well known that $\mathcal{E}g(z)$ is smooth up to the boundary with convergence of $\mathcal{E}g(\cdot+iy)$ to g in the $\mathcal{S}(\mathbb{R}^n)$-topology (and by Proposition 3.16 also in the $B^{p,q}_\nu$-topology). The convergence for f then follows from a standard $\frac{1}{3}\varepsilon$ argument. □

Remark 3.48. Let us finally remark that the index $\widetilde{q}_{\nu,p}$ in the previous proposition is optimal. Indeed, from the duality theorem, the continuity of the linear form $f \mapsto \widetilde{\mathcal{E}} f(i\mathbf{e})$ will imply that $\mathcal{F}^{-1}(\chi_\Omega e^{-(\mathbf{e}|\cdot)})$ belongs to $B^{p',q'}_{-\nu q'/q}$. This continues to be the case after convolution with $\mathcal{F}^{-1}(e^{(\mathbf{e}|\cdot)}\widehat{\varphi})$, for any $\widehat{\varphi} \in C^\infty_c(\mathbb{R}^n)$ identically 1 in a neighborhood of 0. Thus, it follows from Proposition 3.16 that we must have $q < \widetilde{q}_{\nu,p}$.

4. *Bergman spaces and projectors*

In this section we shall show Theorems 1.7 and 1.8, as well as the boundedness of the Bergman projector announced in the introduction. Heuristically, the correspondence between holomorphic functions F in the Bergman space $A^{p,q}_\nu$ and distributions f in $B^{p,q}_\nu$ is given by the Fourier–Laplace formula

$$F(z) = \mathcal{E}f(z) = \mathcal{L}\widehat{f}(z) = \int_\Omega e^{i\,(z|\xi)}\widehat{f}(\xi)\,d\xi, \quad \text{for } z \in T_\Omega.$$

The distribution f plays the role of a Shilov boundary value for the holomorphic function F. The main result in this section is the equivalence of norms $\|F\|_{A^{p,q}_\nu} \sim \|f\|_{B^{p,q}_\nu}$, which follows from a suitable discretization of the integral above using the Whitney decomposition in § 2. Several technical estimates will appear in this process, involving gamma integrals in Ω and Littlewood–Paley inequalities as in (1.12), forcing us at some point to assume further restrictions in the indices ν, p and q. The sharp range of parameters for the equivalence of these two norms is still an open question, which depends on the conjecture associated with (1.12) (see § 5 for a further discussion on these matters).

For the proof of the theorems we shall need three preliminary results. Recall that a holomorphic function $F \in \mathcal{H}(T_\Omega)$ belongs to the *Hardy space* $H^2(T_\Omega)$ when

$$\|F\|_{H^2} = \sup_{y\in\Omega} \|F(\cdot + iy)\|_{L^2(\mathbb{R}^n)} < \infty.$$

The first result is known as the *Paley–Wiener Theorem* for Hardy spaces (see Chapter III of [**15**]).

Proposition 4.1. *A function $F \in H^2(T_\Omega)$ if and only if $F = \mathcal{L}\widehat{f}$ for some $f \in L^2(\mathbb{R}^n)$ with* $\operatorname{Supp}\widehat{f} \subset \overline{\Omega}$. *In this case,* $\|F\|_{H^2} = \|f\|_{L^2(\mathbb{R}^n)}$.

Next we need a density result that is a simple modification of the one presented in [**4**] (see also [**10**]).

Proposition 4.2. *Let $\nu > n/r - 1$ and $1 \leqslant p, q < \infty$. Then, the norms of the spaces $A^{p,q}_\nu$ are complete. Moreover, the intersection $H^2(T_\Omega) \cap A^{p,q}_\nu$ is dense in $A^{p,q}_\nu$.*

Finally we establish an elementary discretization of $A^{p,q}_\nu$-norms.

Proposition 4.3. *Let $\nu > n/r - 1$ and $1 \leqslant p, q < \infty$. Then, for every lattice $\{y_j\}$ in Ω there exists $c > 0$ such that*

$$\frac{1}{c}\,\|F\|_{A^{p,q}_\nu} \leqslant \Big(\sum_j \Delta^\nu(y_j)\|F(\cdot + iy_j)\|_p^q \Big)^{1/q} \leqslant c\,\|F\|_{A^{p,q}_\nu}, \quad \text{for all } F \in A^{p,q}_\nu(T_\Omega). \tag{4.4}$$

Proof. The proof relies on the mean value property, via the following lemma. □

LEMMA 4.5. *Let $1\leqslant p,q<\infty$. Then, for every $F\in\mathcal{H}(T_\Omega)$ and $y_0\in\Omega$ we have*

$$\|F(\cdot+iy_0)\|_p\leqslant C\left[\int_{B(y_0,1)}\|F(\cdot+iy)\|_p^q\,\frac{dy}{\Delta^{n/r}(y)}\right]^{1/q}, \tag{4.6}$$

where the constant depends only on p and q.

Proof of Lemma 4.5. By homogeneity, we may assume $y_0=\mathbf{e}$. Let us consider first the case $p\leqslant q$. Then, by the mean value property for subharmonic functions,

$$|F(x+i\mathbf{e})|^p\leqslant c\int_{B(\mathbf{e},1)}\int_{|x'|\leqslant 1}|F(x+x'+iy)|^p\,dx'\,dy.$$

Thus, integrating in x and using Hölder's inequality we obtain

$$\|F(\cdot+i\mathbf{e})\|_p^p\leqslant c\int_{B(\mathbf{e},1)}\|F(\cdot+iy)\|_p^p\,dy\leqslant c\left(\int_{B(\mathbf{e},1)}\|F(\cdot+iy)\|_p^q\,dy\right)^{p/q}.$$

Suppose instead that $q\leqslant p$. Then, the mean value property gives

$$|F(x+i\mathbf{e})|^p\leqslant c\left(\int_{B(\mathbf{e},1)}\int_{|x'|\leqslant 1}|F(x+x'+iy)|^q\,dx'\,dy\right)^{p/q}.$$

A new integration in x and Minkowski's inequality give the same result. □

Continuing with the proposition, we assume for simplicity that $\{y_j\}$ is a $(\frac{1}{2},2)$-lattice, that is, $\{B_1(y_j)\}_j$ covers Ω with the finite intersection property. The right-hand side of (4.4) follows from a discretization of the integral on Ω defining $\|F\|^q_{A^{p,q}_\nu}$ as in Proposition 2.13. We then use the previous lemma to conclude the required result. For the left-hand side we just need a kind of converse for (4.6), where as before we can assume $y_0=\mathbf{e}$. But this follows from the fact that, when $F\in A^{p,q}_\nu$, the function $y\to\|F(\cdot+iy)\|_p$ is monotonic in Ω (see, for example, [**4**, **10**]), and therefore

$$\left[\int_{B(\mathbf{e},1)}\|F(\cdot+iy)\|_p^q\,\frac{dy}{\Delta^{n/r}(y)}\right]^{1/q}\leqslant C\|F(\cdot+ic\mathbf{e})\|_p,$$

for some constant $c=c(\Omega)>0$. This establishes the proposition. □

4.1. *The proof of Theorem* 1.7

We wish to show that every $F\in A^{p,q}_\nu$ can be written as $F=\widetilde{\mathcal{E}}f$ for some distribution $f\in B^{p,q}_\nu$. Suppose first that F belongs to the dense set $H^2(T_\Omega)\cap A^{p,q}_\nu$, so that, by Proposition 4.1, $F=\mathcal{L}\widehat{f}$ for some function $f\in L^2(\mathbb{R}^n)$ with $\operatorname{Supp}\widehat{f}\subset\overline{\Omega}$. We shall show the inequality $\|f\|_{B^{p,q}_\nu}\leqslant C\,\|F\|_{A^{p,q}_\nu}$. Observe that, since $f=\sum_j f*\psi_j$ in L^2 (and hence in $\mathcal{S}'(\mathbb{R}^n)$), it will follow from this and the definition of $\widetilde{\mathcal{E}}$ that $\widetilde{\mathcal{E}}f=\mathcal{L}\widehat{f}$.

To prove the inequality of norms, let us first choose $y_j = \xi_j^{-1}$ the dual lattice of $\{\xi_j\}$. Then, Young's inequality gives

$$\begin{aligned}\|f * \chi_j\|_p &= \|\mathcal{F}^{-1}(\widehat{f}(\xi)e^{-(y_j|\xi)}\widehat{\chi}_j(\xi)e^{(y_j|\xi)})\|_p \\ &\leqslant \|\mathcal{F}^{-1}(\widehat{f}e^{-(y_j|\cdot)})\|_p \, \|\mathcal{F}^{-1}(\widehat{\chi}_j e^{(y_j|\cdot)})\|_1.\end{aligned}$$

Since $\xi_j^{-1} = g_j^{-1}\mathbf{e}$ and g_j is self-adjoint, we observe that the last factor is actually constant,

$$\|\mathcal{F}^{-1}(\widehat{\chi}_j e^{(y_j|\cdot)})\|_1 = \|\mathcal{F}^{-1}(\widehat{\chi} e^{(\mathbf{e}|\cdot)})\|_1 = c_1 < \infty.$$

This leads to the estimate

$$\begin{aligned}\|f\|^q_{B^{p,q}_\nu} &\leqslant c \sum_j \Delta^{-\nu}(\xi_j)\, \|f * \chi_j\|_p^q \\ &\leqslant c' \sum_j \Delta^{\nu}(y_j)\, \|\mathcal{F}^{-1}(\widehat{f}e^{-(y_j|\cdot)})\|_p^q \\ &= c' \sum_j \Delta^{\nu}(y_j)\, \|F(\cdot + iy_j)\|_p^q \leqslant c'' \|F\|^q_{A^{p,q}_\nu},\end{aligned} \tag{4.7}$$

where in the last inequality we have used Proposition 4.3.

For general $F \in A^{p,q}_\nu$ one proceeds by density. Approximating with $\{F_m\}$ in $H^2(T_\Omega) \cap A^{p,q}_\nu$, we obtain a corresponding sequence of functions $f_m \in L^2(\mathbb{R}^n)$, which by (4.7) is a Cauchy sequence in $B^{p,q}_\nu$. Then, by completeness of this space and Lemma 3.38, f_m converges (in $B^{p,q}_\nu$ and in $\mathcal{S}'$) to a distribution $f \in B^{p,q}_\nu$ such that $f = \sum_j f * \psi_j$. Moreover, $\|f\|_{B^{p,q}_\nu} \leqslant C\, \|F\|_{A^{p,q}_\nu}$. It remains to prove that $F = \widetilde{\mathcal{E}}(f)$, for which we can use the continuity of the pointwise evaluation functional $f \mapsto \widetilde{\mathcal{E}}f(z)$ in (3.44). Indeed, for each $z \in T_\Omega$ we have

$$\widetilde{\mathcal{E}}f(z) = \lim_{m\to\infty} \widetilde{\mathcal{E}}f_m(z) = \lim_{m\to\infty} \mathcal{L}\widehat{f}_m(z) = \lim_{m\to\infty} F_m(z) = F(z).$$

Finally, the convergence

$$\lim_{\substack{y\to 0\\ y\in\Omega}} F(\cdot + iy) = f, \quad \text{in } B^{p,q}_\nu \text{ and } \mathcal{S}'(\mathbb{R}^n),$$

is just a consequence of Proposition 3.43 and Lemma 3.38. This completes the proof of the theorem. □

4.2. *The proof of Theorem* 1.8

We start with a preliminary result which gives the trivial range of indices for which an inequality similar to (1.12) in the introduction holds.

LEMMA 4.8. *Let $1 \leqslant p < \infty$ and let $1 \leqslant s \leqslant p_\sharp = \min\{p, p'\}$. Then there exists a constant C such that, for every sequence of functions $f_j \in L^p(\mathbb{R}^n)$ satisfying* $\operatorname{Supp} \widehat{f}_j \subset B_2(\xi_j)$, *we have the inequality*

$$\Big\| \sum_j f_j \Big\|_p \leqslant C \Big(\sum_j \|f_j\|_p^s \Big)^{1/s}. \tag{4.9}$$

Proof. It is sufficient to prove the stronger inequality

$$\Big\| \sum_j f_j * \chi_j \Big\|_p \leqslant \Big(\sum_j \|f_j\|_p^s \Big)^{1/s},$$

valid for all sequences of functions in L^p. The χ_j are chosen as in § 3.1 with their L^1-norm uniformly bounded, and their Fourier transform supported in $B_4(\xi_j)$ and identically 1 on the ball $B_2(\xi_j)$. For this last inequality, the proof is immediate when $s=1$ by Minkowski's inequality, as well as for $s=p=2$ by the finite intersection property of the balls. We interpolate between these two cases to conclude. □

REMARK 4.10. Our proof for Theorem 1.8 depends directly on (4.9), in which unfortunately the best exponent s for each fixed p seems not to be known (ideally, $s=2$ would be the best possible). Having in mind future improvements, we restate our theorem below with a more general version of the previous inequality. We shall find later some equivalent expressions closer to (1.12), and discuss in § 5 the validity of such inequalities for light-cones.

THEOREM 4.11. *Let $\nu > n/r-1$ and $1 \leqslant p, s < \infty$. Assume that there exist numbers $\mu, \delta \geqslant 0$ and a constant $C = C(\mu,\delta) > 0$ such that*

$$\Big\| \sum_j f_j \Big\|_p \leqslant C \Big[\sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|\mathbf{e})} \|f_j\|_p^s \Big]^{1/s} \tag{4.12}$$

holds for every finite sequence $\{f_j\} \subset L^p(\mathbb{R}^n)$ with $\operatorname{Supp} \widehat{f}_j \subset B_2(\xi_j)$. *Then, for every index*

$$q < \min\Big\{ s\frac{q_\nu}{q_\mu}\,,\ s\,\frac{\nu-(n/r-1)}{\mu}\,,\ \widetilde{q}_{\nu,p} \Big\}, \tag{4.13}$$

and for every distribution f with $\|f\|_{B_\nu^{p,q}} < \infty$, the function $F = \mathcal{E}(\sum_j f * \psi_j)$ belongs to $A_\nu^{p,q}$, and moreover,

$$\|F\|_{A_\nu^{p,q}} \lesssim \|f\|_{B_\nu^{p,q}}.$$

Observe that Theorem 1.8 follows from Lemma 4.8, which establishes the validity of (4.12) for $\mu=\delta=0$ and $s = p_\sharp = \min\{p,\, p'\}$. In this case $q_\mu = 1$, and the condition on q simplifies to $q < \min\{sq_\nu, \widetilde{q}_{\nu,p}\} = p_\sharp q_\nu$, as stated in Theorem 1.8.

Turning to the proof of Theorem 4.11, we need to show that, for every $f \in \mathcal{D}_\Omega$, the function $F(z) := \mathcal{L}\widehat{f}(z)$ belongs to $A_\nu^{p,q}(T_\Omega)$ with $\|F\|_{A_\nu^{p,q}} \leqslant C\,\|f\|_{B_\nu^{p,q}}$. This will be enough to conclude, since in the general case one can proceed by density. We shall use the following consequence of (4.12), which as we shall see later is actually equivalent to it.

LEMMA 4.14. *Let $1 \leqslant p, s < \infty$, and assume that* (4.12) *holds for some $\mu, \delta \geqslant 0$. Then, for every $f \in \mathcal{D}_\Omega$ and $y \in \Omega$, the function $F(\cdot + iy) = \mathcal{F}^{-1}(\widehat{f} e^{-(y|\cdot)})$ belongs* *to* $L^p(\mathbb{R}^n)$. *Moreover,*

$$\|F(\cdot + iy)\|_p \lesssim \Delta^{-\mu/s}(y)\|f\|_{B^{p,s}_\mu} \tag{4.15}$$

with constants independent of f *or* $y \in \Omega$.

Proof. By homogeneity (see Lemma 3.8), it is sufficient to prove (4.15) when $y = \eta\mathbf{e}$, for some fixed $\eta > 0$ to be chosen below. Let us denote $\widehat{g} = \widehat{f}e^{-\eta(\mathbf{e}|\cdot)}$, so that $g = \sum_j g * \psi_j \in \mathcal{D}_\Omega$. Applying (4.12) to g we obtain

$$\begin{aligned}\|g\|_p &= \|F(\cdot + i\eta\mathbf{e})\|_p \\ &\lesssim \Big(\sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|\mathbf{e})} \|\mathcal{F}^{-1}(\widehat{f}\widehat{\psi}_j e^{-\eta(\mathbf{e}|\cdot)})\|_p^s, \Big)^{1/s} \\ &\lesssim \Big(\sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|\mathbf{e})} \|f * \psi_j\|_p^s \, \|\mathcal{F}^{-1}(e^{-\eta(\mathbf{e}|\cdot)}\widehat{\chi}_j)\|_1^s \Big)^{1/s}.\end{aligned}$$

Now, $\|\mathcal{F}^{-1}(e^{-\eta(\mathbf{e}|\cdot)}\widehat{\chi}_j)\|_1$ is bounded by a constant times $e^{-\gamma\eta(\xi_j|\mathbf{e})}$ (by (3.47)), and thus we only have to choose η larger than δ/γ. □

We now conclude the proof of our theorem. Given $f \in \mathcal{D}_\Omega$, and $F(z) := \mathcal{L}\widehat{f}(z)$, the previous lemma applied to $\mathcal{F}^{-1}(\widehat{f}e^{-(y|\mathbf{e})})$ gives us

$$\begin{aligned}\|F(\cdot + i2y)\|_p &\lesssim \Delta^{-\mu/s}(y) \Big[\sum_j \Delta^{-\mu}(\xi_j) \, \|\mathcal{F}^{-1}(\widehat{f}\widehat{\psi}_j e^{-(y|\cdot)})\|_p^s \Big]^{1/s} \\ &\lesssim \Delta^{-\mu/s}(y) \Big[\sum_j \Delta^{-\mu}(\xi_j) \, e^{-\gamma(y|\xi_j)} \, \|f * \psi_j\|_p^s \Big]^{1/s},\end{aligned}$$

where once again we have used (3.47). Thus,

$$\begin{aligned}I &:= \int_\Omega \|F(\cdot + iy)\|_p^q \, \Delta^{\nu - n/r}(y) \, dy \\ &\lesssim \int_\Omega \Delta^{-\mu q/s}(y) \Big(\sum_j \Delta^{-\mu}(\xi_j) e^{-\gamma(y|\xi_j)} \|f * \psi_j\|_p^s \Big)^{q/s} \Delta^{\nu - n/r}(y) \, dy.\end{aligned}$$

When $q \leqslant s$ we directly conclude that

$$I \lesssim \sum_j \Delta^{-\mu q/s}(\xi_j) \|f * \psi_j\|_p^q \int_\Omega \Delta^{-\mu q/s}(y) \, e^{-\gamma'(y|\xi_j)} \, \Delta^{\nu - n/r}(y) \, dy,$$

where the gamma integral equals a multiple of $\Delta(\xi_j)^{\mu q/s - \nu}$, whenever $\nu - \mu q/s > n/r - 1$. This leads to one of the conditions stated in (4.13).

Suppose now that $q > s$. Then we multiply and divide the summands by $\Delta_{\mathbf{t}}(\xi_j)$, for some multi-index $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{R}^r$ to be chosen below. After applying Hölder's inequality, we obtain

$$I \lesssim \int_\Omega \Delta^{-\mu q/s}(y)\Big[\sum_j \Delta^{-\mu q/s}(\xi_j)e^{-\gamma(y|\xi_j)}\|f*\psi_j\|_p^q\,\Delta_{-\mathbf{t}q/s}(\xi_j)\Big] \times\Big[\sum_j \Delta_{\mathbf{t}(q/s)'}(\xi_j)e^{-\gamma(y|\xi_j)}\Big]^{(q/s)/(q/s)'}\Delta(y)^{\nu-n/r}\,dy.$$

According to Proposition 2.13, the last bracket can be transformed into a gamma integral, which in order to be finite requires the following condition in the indices:

$$t_j(q/s)' > (j-1)\frac{n/r-1}{r-1}, \quad \text{for all } j=1,\dots,r.$$

Thus, replacing the expression inside the brackets by a multiple of $\Delta^*_{-\mathbf{t}^*(q/s)'}(y)$, we have

$$I \lesssim \sum_j \Delta^{-\mu q/s}(\xi_j)\,\|f*\psi_j\|_p^q\,\Delta_{-\mathbf{t}q/s}(\xi_j) \times\int_\Omega \Delta^*_{-(\mathbf{t}^*+\mu)q/s}(y)e^{-\gamma(y|\xi_j)}\Delta(y)^{\nu-n/r}\,dy.$$

Computing the gamma integral again, we obtain a finite multiple of $\Delta_{(\mathbf{t}+\mu)q/s-\nu}(\xi_j)$ if we impose the following condition in the indices:

$$-\frac{q}{s}(t_{r-(j-1)}+\mu)+\nu > (j-1)\frac{n/r-1}{r-1}, \quad \text{for all } j=1,\dots,r.$$

Therefore, we will conclude with $I \lesssim \|f\|^q_{B^{p,q}_\nu}$ if we can choose real numbers t_j so that

$$\frac{1}{(q/s)'}\frac{j-1}{r-1}\left(\frac{n}{r}-1\right) < t_j < \frac{1}{q/s}\left(\nu-\frac{r-j}{r-1}\left(\frac{n}{r}-1\right)\right)-\mu, \quad \text{for } j=1,\dots,r.$$

Using the fact that

$$\frac{1}{(q/s)'} = 1-\frac{1}{q/s},$$

we see that this is only possible when

$$\frac{j-1}{r-1}\left(\frac{n}{r}-1\right)+\mu < \frac{1}{q/s}\left(\nu-\left(\frac{n}{r}-1\right)+2\,\frac{j-1}{r-1}\left(\frac{n}{r}-1\right)\right), \quad \text{for } j=1,\dots,r.$$

Solving for q/s, this forces us to have

$$\frac{q}{s} < \min_{1\leqslant j\leqslant r}\frac{\nu-(n/r-1)+2\,((j-1)/(r-1))(n/r-1)}{\mu+((j-1)/(r-1))(n/r-1)} = \begin{cases} q_\nu/q_\mu & \text{if } \nu > 2\mu+n/r-1,\\ \dfrac{\nu-(n/r-1)}{\mu} & \text{if } \nu\leqslant 2\mu+n/r-1.\end{cases}$$

This is precisely the range of ν and q assumed in (4.13) so the theorem is completely proved. □

To conclude this section we state some equivalent but simpler expressions of the sufficient condition (4.12) above.

Proposition 4.16. *Let* $1 \leqslant p, s < \infty$ *and* $\mu > 0$. *Then the following properties are equivalent.*

1. *There exist* $\delta > 0$ *and a constant* $C_\delta > 0$ *such that*

$$\Big\| \sum_j f_j \Big\|_p \leqslant C_\delta \Big[\sum_j \Delta^{-\mu}(\xi_j)\, e^{\delta(\xi_j|\mathbf{e})}\, \|f_j\|_p^s \Big]^{1/s}, \tag{4.17}$$

for every finite sequence $\{f_j\}$ *in* $L^p(\mathbb{R}^n)$ *satisfying* $\operatorname{Supp} \widehat{f}_j \subset B_2(\xi_j)$.

2. *There exists a constant* $C > 0$ *such that*

$$\Big\| \sum_j f_j \Big\|_p \leqslant C \Big[\sum_j \Delta^{-\mu}(\xi_j) \|f_j\|_p^s \Big]^{1/s}, \tag{4.18}$$

for every finite sequence $\{f_j\}$ *in* $L^p(\mathbb{R}^n)$ *satisfying* $\operatorname{Supp} \widehat{f}_j \subset B_2(\xi_j) \cap H_0$, *where* H_0 *is the band in between two hyperplanes* $H_0 = \{\frac{1}{2} < (\mathbf{e}|\xi) < 2\}$.

3. *There exists a constant* $C > 0$ *such that*

$$\|\mathcal{F}^{-1}(\widehat{f} e^{-(y|\cdot)})\|_p \leqslant C \Delta^{-\mu/s}(y)\, \|f\|_{B_\mu^{p,s}}, \quad \text{for all } f \in \mathcal{D}_\Omega \text{ and all } y \in \Omega. \tag{4.19}$$

Proof. We have already proved in Lemma 4.14 that (4.17) implies (4.19). To prove that (4.19) implies (4.18), take a corresponding sequence $\{f_j\}$ as in (4.18). Define the functions $\widehat{g}_j = e^{(\mathbf{e}|\cdot)} \widehat{f}_j$, and apply the inequality (4.19) for $y = \mathbf{e}$ to the function $g = \sum_j g_j \in \mathcal{D}_\Omega$. Then,

$$\Big\| \sum_j f_j \Big\|_p = \|\mathcal{F}^{-1}(\widehat{g} e^{-(\mathbf{e}|\cdot)})\|_p \leqslant C \Big[\sum_k \Delta^{-\mu}(\xi_k)\, \|\mathcal{F}^{-1}(\widehat{g}\widehat{\psi}_k)\|_p^s \Big]^{1/s}.$$

Using the finite intersection property, and the fact that the L^p-norms of f_j and g_j are comparable, we easily obtain the right-hand side of (4.18).

It remains to prove that (4.18) implies (4.17). To do this, we are going to slice the cone with hyperplanes, and then apply a scaled version of (4.18) to the restrictions of $\sum_j f_j$ to the bands

$$H_k = \{2^{k-1} < (\xi|\mathbf{e}) < 2^{k+1}\}, \quad \text{where } k \in \mathbb{Z}.$$

To make this argument precise, we select a sequence of smooth 1-variable functions $\{\rho_k\}$ so that $\operatorname{Supp} \rho_k \subset (2^{k-1}, 2^{k+1})$ and $\sum_{k \in \mathbb{Z}} \rho_k \equiv 1$ in $(0, \infty)$. We let $\widehat{f}_{j,k}(\xi) = \rho_k((\xi|\mathbf{e}))\widehat{f}_j(\xi)$, so that

$$\operatorname{Supp} \widehat{f}_{j,k} \subset B_2(\xi_j) \cap H_k \quad \text{and} \quad \|f_{j,k}\|_p \leqslant C\, \|f_j\|_p. \tag{4.20}$$

By Minkowski's inequality we can write

$$I = \Big\| \sum_j f_j \Big\|_p \leqslant \sum_{k \in \mathbb{Z}} \Big\| \sum_{j \in J_k} f_{j,k} \Big\|_p =: \sum_{k \in \mathbb{Z}} \|F_k\|_p,$$

where the sets of indices J_k are defined so that $B_2(\xi_j) \cap H_k \neq \emptyset$. In order to estimate the norm of F_k, we must first perform a dilation by $\delta = 2^{-k}$ so that, replacing F_k with $F_k^{(\delta)} = \delta^{n/p}\, F_k(\delta\, \cdot)$, we do not change the L^p-norms and the

Fourier transform is now supported in H_0. Thus, we are able to apply (4.18):

$$\|F_k\|_p = \Big\| \sum_\ell F_k^{(\delta)} * \psi_\ell \Big\|_p \leqslant C \Big[\sum_\ell \Delta^{-\mu}(\xi_\ell)\, \|F_k^{(\delta)} * \psi_\ell\|_p^s \Big]^{1/s}.$$

Now, observe that each $f_{j,k}^{(\delta)}$ has Fourier transform supported in $B_2(\delta\xi_j)$, and $\{\delta\xi_j\}$ is still a $(\frac{1}{2}, 2)$-lattice in the cone. Thus, by the finite intersection property, the set of indices j for which $B_2(\delta\xi_j)$ intersects a fixed set $B_2(\xi_\ell)$ has at most $N = N(\Omega)$ elements, independently of ℓ and δ. Thus,

$$\|F_k\|_p^s \lesssim \sum_\ell \Delta^{-\mu}(\xi_\ell) \sum_j \|f_{j,k}^{(\delta)} * \psi_\ell\|_p^s.$$

Now, changing the order of sums, restricting ℓ to the bounded set of indices

$$\widetilde{J}_j = \{\ell : B_2(\xi_\ell) \cap B_2(\delta\xi_j) \neq \emptyset\},$$

and using the fact that $\Delta(\xi_\ell) \sim \Delta(\delta\xi_j)$ for such indices, we obtain

$$\begin{aligned}\|F_k\|_p^s &\lesssim \sum_j \Delta^{-\mu}(\delta\xi_j) \sum_{\ell\in\widetilde{J}_j} \|f_{j,k}^{(\delta)}\|_p^s \|\psi_\ell\|_1^s \\ &\lesssim \delta^{-\mu r} \sum_j \Delta^{-\mu}(\xi_j)\|f_{j,k}\|_p^s \lesssim 2^{k\mu r} \sum_{j\in J_k} \Delta^{-\mu}(\xi_j)\|f_j\|_p^s,\end{aligned}$$

where in the last step we have also used (4.20). Thus, raising to the power $1/s$ and summing in k, we have shown that

$$I \lesssim \sum_{k\in\mathbb{Z}} \Big[\sum_{j\in J_k} \Delta^{-\mu}(\xi_j)\|f_j\|_p^s \Big]^{1/s} 2^{k\mu r/s}.$$

Multiplying and dividing by e^{2^k}, and applying Hölder's inequality we obtain

$$I \lesssim \Big[\sum_{k\in\mathbb{Z}} \sum_{j\in J_k} \Delta^{-\mu}(\xi_j)\, e^{s2^k}\, \|f_j\|_p^s \Big]^{1/s} \Big[\sum_{k\in\mathbb{Z}} 2^{k\mu r s'/s}\, e^{-s'2^k} \Big]^{1/s'}.$$

The last term is a finite constant when $\mu > 0$, while in the first factor we can replace e^{s2^k} by $e^{\eta(\xi_j|\mathbf{e})}$, for a sufficiently large η. To conclude it remains only to show that, for each fixed j, the set of all $k \in \mathbb{Z}$ such that H_k intersects $B_2(\xi_j)$ contains at most $N = N(\Omega)$ elements. To see this observe, from Lemma 2.9, that each such k must satisfy

$$2^{k-1}/\gamma < (\xi_j|\mathbf{e}) < \gamma 2^{k+1},$$

or equivalently $(1/2\gamma)\,(\xi_j|\mathbf{e}) < 2^k < 2\gamma(\xi_j|\mathbf{e})$. Taking logarithms we see that this is only possible for a constant number of such values of k. The proof of Proposition 4.16 is then complete. □

4.3. *Boundedness of Bergman projectors in* $L^{p,q}_\nu$

In this section we shall prove the equivalence between the identification of $A^{p,q}_\nu$ and $B^{p,q}_\nu$ and the boundedness of the Bergman projector P_ν in $L^{p,q}_\nu$.

Recall that the Bergman projector P_ν is defined for functions $F \in L^2_\nu(T_\Omega)$ as

$$P_\nu F(x+iy) = \int_{\mathbb{R}^n}\int_\Omega B_\nu(x-u+i(y+v))F(u+iv)\,\Delta(v)^{\nu-n/r}\,dv\,du, \tag{4.21}$$

with the Bergman kernel given by

$$\begin{aligned} B_\nu(z-\overline{w}) &= d(\nu)\,\Delta^{-(\nu+n/r)}((z-\overline{w})/i) \\ &= c_\nu \int_\Omega e^{i(z-\overline{w}|\xi)}\,\Delta(\xi)^\nu\,d\xi, \quad \text{for } z, w \in T_\Omega, \end{aligned} \tag{4.22}$$

for some positive constants c_ν and $d(\nu)$ (see, for example, Chapter XIII of [**8**]). It is clear that $P_\nu F(z)$ defines a holomorphic function in T_Ω whenever the integral in (4.21) converges absolutely. The following lemma shows that this is the case exactly when $F \in L^{p,q}_\nu$ and $q < \widetilde{q}_{\nu,p}$. This elementary fact also gives us a trivial range of unboundedness for P_ν (see [**5**]).

LEMMA 4.23. *Let $\nu > n/r - 1$ and $1 \leqslant p < \infty$. Then*

$$B_\nu(z+i\mathbf{e}) \in L^{p',q'}_\nu(T_\Omega) \iff q < \widetilde{q}_{\nu,p} := \frac{\nu + n/r - 1}{((n/r)(1/p') - 1)_+}. \tag{4.24}$$

Moreover, if $q \leqslant \widetilde{q}'_{\nu,p}$ or $q \geqslant \widetilde{q}_{\nu,p}$ then P_ν does not admit bounded extensions into $L^{p,q}_\nu$.

Proof. The first statement is an elementary application of Lemma 2.18 to the formula in (4.22). For the second statement test with $F(z) = \Delta^{-\nu+n/r}(\Im z)\chi_{Q(i\mathbf{e})}(z)$, where $Q(i\mathbf{e})$ is a closed polydisk in T_Ω centered at $i\mathbf{e}$. Then

$$P_\nu F(z) = c_n\,B_\nu(z+i\mathbf{e}), \quad \text{for } z \in T_\Omega,$$

by the mean value property for (anti)-holomorphic functions. Therefore, if $q \geqslant \widetilde{q}_{\nu,p}$, then P_ν cannot be bounded into $L^{p',q'}_\nu$, and by self-adjointness not into $L^{p,q}_\nu$ either. □

We pass now to the study of boundedness of P_ν in $L^{p,q}_\nu$ when $\widetilde{q}'_{\nu,p} < q < \widetilde{q}_{\nu,p}$. This is a difficult open question for which only partial results are known (see [**4**] for the light-cone, and [**2**] for the simpler case $L^{2,q}_\nu$). We prove here the following equivalence between this problem and the kind of estimates for Fourier–Laplace integrals that we have considered before.

THEOREM 4.25. *Let $\nu > n/r - 1$ and $2 \leqslant q < \widetilde{q}_{\nu,p}$. Then, the Bergman projector P_ν is bounded in $L^{p,q}_\nu$ if and only if there exists a constant C such that*

$$\|\mathcal{L}\widehat{f}\|_{L^{p,q}_\nu} \leqslant C\|f\|_{B^{p,q}_\nu}, \quad \textit{for } f \in \mathcal{D}_\Omega. \tag{4.26}$$

Proof of the necessity condition. Let $2 \leqslant q < \widetilde{q}_{\nu,p}$ and assume that the projector P_ν is bounded in $L^{p,q}_\nu$. We want to compute $\|\mathcal{L}\widehat{f}\|_{L^{p,q}_\nu}$ for $f \in \mathcal{D}_\Omega$. It is sufficient to test it on functions in $L^{p',q'}_\nu \cap L^2_\nu$. Moreover, since $\mathcal{L}\widehat{f} \in A^{p,q}_\nu$ and the projection is self-adjoint (and hence, bounded in $L^{p',q'}_\nu$), we can also test it on functions which are in $A^{p',q'}_\nu \cap A^2_\nu$. Such functions may be written as $\widetilde{\mathcal{E}}g = \mathcal{L}\widehat{g}$, with $g = \sum_j g * \psi_j \in B^{p',q'}_\nu$, and we know from Theorem 1.8 that, for this range of exponents, the norm of $\mathcal{L}\widehat{g}$ in $A_\nu^{p',q'}$ is equivalent to the norm of g in $B_\nu^{p',q'}$. So it is sufficient to prove that

$$\left|\int_{\mathbb{R}^n}\int_\Omega \mathcal{L}\widehat{f}(x+iy)\overline{\mathcal{L}\widehat{g}(x+iy)}\Delta(y)^{\nu-n/r}dy\,dx\right| \leqslant C\|f\|_{B_\nu^{p,q}}\|g\|_{B_\nu^{p',q'}},$$

for some constant C which does not depend on f and g. Using the Paley–Wiener Theorem for A_ν^2 (see, for example, [**8**, p. 260]), we know that the left-hand side is equal to

$$\left|\int_\Omega \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\frac{d\xi}{\Delta(\xi)^\nu}\right|.$$

Then, Plancherel's Theorem and the definition of the fractional powers of $\Box$ tell us that this is the usual duality pairing between f and $\Box^{-\nu}g$. As a consequence, it is bounded by

$$C\|f\|_{B_\nu^{p,q}}\|\Box^{-\nu}g\|_{B^{p',q'}_{-\nu q'/q}}.$$

To conclude, we use Proposition 3.31, which gives the equivalence of the norm $\|\Box^{-\nu}g\|_{B^{p',q'}_{-\nu q'/q}}$ with the norm $\|g\|_{B_\nu^{p',q'}}$. This finishes the proof of this direction. $\Box$

For the other direction, we will prove a little more. We will show that P_ν is always bounded from $L_\nu^{p,q}$ into a new holomorphic function space $\mathcal{B}_\nu^{p,q} := \widetilde{\mathcal{E}}(B_\nu^{p,q})$, consisting of Fourier–Laplace extensions of distributions in $B_\nu^{p,q}$.

Definition 4.27. Given $\nu>0$, $1\leqslant p<\infty$ and $1\leqslant q<\widetilde{q}_{\nu,p}$, we define the *holomorphic function space*

$$\mathcal{B}_\nu^{p,q}(T_\Omega) := \left\{F=\widetilde{\mathcal{E}}f=\sum_j \mathcal{L}(\widehat{f}\widehat{\psi}_j) : f\in\mathcal{S}'_\Omega \text{ with } \|f\|_{B_\nu^{p,q}}<\infty\right\},$$

endowed with the norm $\|F\|_{\mathcal{B}_\nu^{p,q}} = \|f\|_{B_\nu^{p,q}}$.

By Proposition 3.43, $\mathcal{B}_\nu^{p,q}$ is continuously embedded into $\mathcal{H}(T_\Omega)$ and its functions satisfy the inequality

$$|F(x+iy)|\leqslant C\,\Delta(y)^{-(n/r)(1/p)-(\nu/q)}\,\|f\|_{B_\nu^{p,q}}, \quad \text{for } x+iy\in T_\Omega.$$

Moreover, Theorems 1.7 and 1.8 tell us that

$$A_\nu^{p,q}\subset\mathcal{B}_\nu^{p,q} \text{ when } 1\leqslant q<\widetilde{q}_{\nu,p}, \quad \text{and} \quad A_\nu^{p,q}=\mathcal{B}_\nu^{p,q} \text{ when } 1\leqslant q<q_{\nu,p}.$$

Observe also that $A_\nu^{p,q}$ is a dense subspace of $\mathcal{B}_\nu^{p,q}$ (since $\mathcal{E}(\mathcal{D}_\Omega)\subset A_\nu^{p,q}$), but in general is not closed. In fact, examples in the next subsection show that the inclusion is strict whenever $q\geqslant\min\{2,p\}q_\nu$. We now prove the announced statement, which allows us to complete the proof of Theorem 4.25.

Proposition 4.28. *Let $\nu>n/r-1$, $1\leqslant p<\infty$ and $2\leqslant q<\widetilde{q}_{\nu,p}$. Then, with the previous notation, P_ν extends as a bounded operator from $L_\nu^{p,q}$ into $\mathcal{B}_\nu^{p,q}$. That is, for every $F\in L_\nu^{p,q}$ there exists $g\in B_\nu^{p,q}$ such that $P_\nu F=\widetilde{\mathcal{E}}g$ with*

$$\|P_\nu F\|_{\mathcal{B}_\nu^{p,q}} = \|g\|_{B_\nu^{p,q}}\leqslant C\,\|F\|_{L_\nu^{p,q}}, \quad \text{for } F\in L_\nu^{p,q}(T_\Omega). \tag{4.29}$$

Remark 4.30. Observe that for $F \in A^2_\nu \cap A^{p,q}_\nu$, $P_\nu F = F = \widetilde{\mathcal{E}} f$, and therefore, (4.29) actually generalizes the inequality $\|f\|_{B^{p,q}_\nu} \leqslant C\,\|F\|_{A^{p,q}_\nu}$ in Theorem 1.7.

Proof. It is enough to show (4.29) for functions F in the dense set $L^2_\nu \cap L^{p,q}_\nu$, proceeding otherwise as in the last part of § 4.1. Since P_ν is a projector, for such functions we will have $P_\nu F \in A^2_\nu$, and therefore, by the Paley–Wiener Theorem for A^2_ν there exists a unique function $\widehat{g} \in L^2(\Omega; \Delta^{-\nu}(\xi)\,d\xi)$ such that

$$P_\nu F(z) = \mathcal{L}\widehat{g}(z) = \int_\Omega e^{i(x+iy|\xi)}\widehat{g}(\xi)\,d\xi, \quad \text{for } z = x+iy \in T_\Omega. \tag{4.31}$$

Observe that $g = \sum_j g * \psi_j$ in $\mathcal{S}'(\mathbb{R}^n)$ (since $\widehat{g} = \sum_j \widehat{g}\widehat{\psi}_j$ in $L^2(\Omega; \Delta^{-\nu}(\xi)\,d\xi)$), and therefore $P_\nu F = \mathcal{L}\widehat{g} = \widetilde{\mathcal{E}} g$. Thus, to prove $P_\nu F \in \mathcal{B}^{p,q}_\nu$ we just need to bound $\|g\|_{B^{p,q}_\nu}$. From the duality of Besov spaces (Proposition 3.27), it follows that it is sufficient to prove that

$$|\langle g, \varphi\rangle| \leqslant C \|F\|_{L^{p,q}_\nu} \|\varphi\|_{B^{p',q'}_{-\nu q'/q}}, \quad \text{for } \varphi \in \mathcal{D}_\Omega.$$

As in the previous proof, we use the fact that

$$\begin{aligned}\langle g, \varphi\rangle =& \int_{\mathbb{R}^n}\int_\Omega \overline{\mathcal{L}\widehat{g}(x+iy)}\,\mathcal{L}\widehat{h}(x+iy)\Delta(y)^{\nu-n/r}\,dy\,dx\\ & \int_{\mathbb{R}^n}\int_\Omega \overline{F(x+iy)}\,\mathcal{L}\widehat{h}(x+iy)\Delta(y)^{\nu-n/r}\,dy\,dx,\end{aligned}$$

with $h = \Box^\nu \varphi$. So,

$$|\langle g, \varphi\rangle| \leqslant C \|F\|_{L^{p,q}_\nu} \|\mathcal{E}h\|_{A^{p',q'}_\nu}.$$

To conclude, we use Theorem 1.8 applied to h, and Proposition 3.31 as before to have the equivalence of the norm $\|\varphi\|_{B^{p',q'}_{-\nu q'/q}}$ with the norm $\|h\|_{B^{p',q'}_\nu}$. □

As a corollary, we can use the previous section to extend the range of exponents for which the Bergman projector is bounded.

Corollary 4.32. *If $\nu > n/r - 1$, $1 \leqslant p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$, then P_ν admits a bounded extension to $L^{p,q}_\nu$. That is, there exists a constant $C > 0$ such that*

$$\|P_\nu F\|_{L^{p,q}_\nu} \leqslant C\,\|F\|_{L^{p,q}_\nu}, \quad \textit{for all } F \in L^{p,q}_\nu.$$

4.4. *Necessary conditions*

In this section we shall construct counter-examples to Theorem 1.8, for some values of ν, p and q above the critical indices. That is, we shall show that the space $\mathcal{B}^{p,q}_\nu$, of Fourier–Laplace extensions of $B^{p,q}_\nu$, cannot be embedded into $A^{p,q}_\nu$. Our examples are actually stronger and show that $\mathcal{B}^{p,q}_\nu$ cannot even be embedded into $A^{p,\infty}_\nu$, that is, there *does not* exist a constant C such that, for all $f \in B^{p,q}_\nu$, one has the inequality

$$\int_{\mathbb{R}^n} |F(x+i\mathbf{e})|^p\,dx \leqslant C\|f\|^p_{B^{p,q}_\nu}, \quad \text{for } F = \widetilde{\mathcal{E}}(f). \tag{4.33}$$

Observe that this indeed contradicts Theorem 1.8 since, by Lemma 4.5, the integral on the left-hand side is always smaller than $\|F\|_{A^{p,q}_\nu}$. Our results are stated in the following proposition.

PROPOSITION 4.34. *Let $\nu > n/r - 1$, $1 \leqslant p < \infty$ and $1 \leqslant q < \widetilde{q}_{\nu,p}$. Then, there cannot exist a constant C such that the inequality (4.33) is valid for all $f \in B^{p,q}_\nu$ in the following two cases:*
(a) *$1 \leqslant p \leqslant 2$ and $q \geqslant q_{\nu,p}$;*
(b) *$2 < p < \infty$ and $q \geqslant \min\{2q_\nu, \widetilde{q}_{\nu,p}\}$.*

Proof. We shall use a different method for (a) and (b). The first one is based on an explicit holomorphic function, and the second on a Rademacher argument with Littlewood–Paley inequalities. For the first part, we shall find $F \in \mathcal{H}(T_\Omega)$ such that

$$\int_{\mathbb{R}^n} |F(x+i\mathbf{e})|^p \, dx = \infty \quad \text{and} \quad \|\Box F\|_{L^{p,q}_{\nu+q}} < \infty,$$

for all $q \geqslant pq_\nu$ (with $q < \widetilde{q}_{\nu,p}$). This gives a contradiction with (4.33), since if that held, we would conclude that

$$\begin{aligned}\|F(\cdot + i\mathbf{e})\|_p &\lesssim \|f\|_{B^{p,q}_\nu} \sim \|\Box f\|_{B^{p,q}_{\nu+q}} \\ &\leqslant C\|\widetilde{\mathcal{E}}(\Box f)\|_{L^{p,q}_{\nu+q}} = C\|\Box F\|_{L^{p,q}_{\nu+q}} < \infty.\end{aligned}$$

At this point, an example involving the $\Box$ operator may seem a bit cumbersome, but it is actually quite natural since the boundedness of the Bergman projection turns out to be equivalent to the existence of *generalized Hardy inequalities*. For more on this direction we refer to [**4**] (in the case of the light-cone), and to the survey paper [**6**]. Our specific example will be the holomorphic function

$$F(z) = \Delta((z+i\mathbf{e})/i)^{-\alpha} \, (1 + \log \Delta((z+i\mathbf{e})/i))^{-1/p}, \quad \text{for } z \in T_\Omega,$$

where we choose $\alpha = (2n/r - 1)/p$. We are using the standard convention

$$\log \Delta(z/i) = \sum_{j=1}^r \log\left[\frac{\Delta_j}{\Delta_{j-1}}(z/i)\right],$$

since in this case $\Re e[(\Delta_j/\Delta_{j-1})(z/i)] > 0$, for each $z \in T_\Omega$ and $j = 1, \dots, r$ (see, for example, the discussion in [**10**, § 7]). We also remark that, since $|\Delta((z+i\mathbf{e})/i)| \geqslant \Delta(y+\mathbf{e}) > 1$, the expression under the power $1/p$ has positive real part, and defines a holomorphic function.

To compute the first integral we estimate the denominator of $F(z)$ using the elementary facts

$$|1 + \log \Delta((x+i\mathbf{e})/i)| \leqslant \tfrac{1}{2} r\pi + 1 + \log|\Delta(x+i\mathbf{e})| \quad \text{and} \quad |\Delta(x+i\mathbf{e})| \leqslant \Delta(x+\mathbf{e}),$$

when $x \in \Omega$. This leads to the expression

$$\int_{\mathbb{R}^n} |F(x+i\mathbf{e})|^p \, dx \geqslant C \int_\Omega \frac{dx}{|\Delta(x+\mathbf{e})|^{2n/r-1} \, (1 + \log \Delta(x+\mathbf{e}))} = \infty,$$

by Lemma 2.20.

For the second integral we first calculate $\Box F(z)$. Observe that around each $z_0 \in T_\Omega$ there is a neighborhood U so that

$$F(z) = g_{z_0}(\Delta((z+i\mathbf{e})/i)), \quad \text{for } z \in U,$$

where $g_{z_0}(w) = w^{-\alpha}(1+\log w)^{-1/p}$ is a function of one complex variable with a determination of the log depending on z_0. We remark that functions corresponding to two points z_0 and z_1 will only differ by constants which are irrelevant for our estimates below, and for this reason we shall drop the subindex in g.

We can now compute $\Box F(z)$ using the formula

$$\Box[g(\Delta(z/i))] = \frac{(Bg)(\Delta(z/i))}{\Delta(z/i)}, \quad \text{for } z \in T_\Omega, \tag{4.35}$$

where $B = b(w\frac{d}{dw})$ is the 1-variable differential operator of degree r given by the Bernstein polynomial

$$b(\lambda) = (-1)^r \lambda\Big(\lambda + \frac{d}{2}\Big)\dots\Big(\lambda + (r-1)\frac{d}{2}\Big).$$

One can verify the equality (4.35) directly, using the Taylor series of g and $\Box[\Delta^n(z/i)] = b(n)\Delta^{n-1}(z/i)$ (see, for example, [**8**, p. 142]). Thus, an easy computation of the derivatives of $g(w)$ leads to the expression

$$|\Box F(z)| \leqslant C\,|\Delta((z+i\mathbf{e})/i)|^{-(\alpha+1)}\,(1+\log\Delta(y+\mathbf{e}))^{-1/p}, \quad \text{for } z = x+iy \in T_\Omega,$$

where we have also used $|\Delta(u+iv)| \geqslant \Delta(v)$, for $v \in \Omega$. Now, we can apply Lemmas 2.18 and 2.20 to estimate the integral

$$\begin{aligned}\int_\Omega \Big(\int_{\mathbb{R}^n} &|\Box F(x+iy)|^p\,dx\Big)^{q/p} \Delta^{\nu+q-n/r}(y)\,dy \\ &\leqslant C\int_\Omega\Big(\int_{\mathbb{R}^n}\frac{dx}{|\Delta(x+i(y+\mathbf{e}))|^{(\alpha+1)p}}\Big)^{q/p}\frac{\Delta^{\nu+q-n/r}(y)\,dy}{(1+\log\Delta(y+\mathbf{e}))^{q/p}} \\ &\leqslant C'\int_\Omega\frac{\Delta^{\nu+q-n/r}(y)}{\Delta(y+\mathbf{e})^{((\alpha+1)p-n/r)q/p}}\,\frac{dy}{(1+\log\Delta(y+\mathbf{e}))^{q/p}},\end{aligned}$$

and observe that the last quantity is finite for every $q \geqslant p(1+\nu/(n/r-1))$.

Let us now pass to the second type of counter-examples, and obtain case (b) in the proposition. We may assume $q > 2$. We know from Proposition 4.16 that the inequality (4.33) implies the existence of a constant C such that

$$\Big\|\sum f_j\Big\|_p^q \leqslant C\sum \Delta(\xi_j)^{-\nu}\|f_j\|_p^q, \tag{4.36}$$

for any finite sequence $\{f_j\}$ of Schwartz functions satisfying $\operatorname{Supp}\widehat{f_j} \subset B_{1/2}(\xi_j)$ and restricted to indices j so that $|\xi_j| < 1$. Let us prove that this implies very easily the necessity of $q < 2q_\nu$ (which has already been announced in [**4**] for light-cones). Indeed, let us take $f_j = \varepsilon_j a_j e^{i(\xi_j|\cdot)} f$, with $\{\varepsilon_j\}$ a sequence of Rademacher functions, and the support of $\widehat{f}$ a small neighborhood of 0. Taking the mean over all the ε_j and using Khintchine inequalities, we find that

$$\left[\sum_j |a_j|^2 \right]^{1/2} \leqslant C \left[\sum_j \Delta(\xi_j)^{-\nu} |a_j|^q \right]^{1/q},$$

with perhaps a different constant C, independent of the sequence $\{a_j\}$. Now, choosing $a_j = \Delta(\xi_j)^{\nu/(q-2)}$ and using $q > 2$, we see this implies that

$$\sum_{j;|\xi_j|<1} \Delta(\xi_j)^{2\nu/(q-2)} \leqslant C < \infty.$$

Using Proposition 2.13, we find that this is equivalent to the fact that

$$\int_{|\xi|<1} \Delta(\xi)^{2\nu/(q-2)} \frac{d\xi}{\Delta(\xi)^{n/r}} < \infty,$$

which, in turn, is equivalent to the condition $q < 2q_\nu$. □

5. *Sharp results for light-cones*

In this section we particularize the previous problems to the light-cone Λ_n. As we shall see, the sufficient conditions given in Proposition 4.16 are related to inequalities which appear in study of the so-called 'cone multiplier problem'. In particular, we establish a link between our problem and Wolff's conjecture for $\|\cdot\|_{p,mic}$-norms (see [**17**, **12**]). Moreover, our sufficient conditions are 'essentially necessary', so whatever progress appears in each of these problems will have consequences in the other.

Figure 1.1 shows the regions of boundedness for the Bergman projector P_ν in $L^{p,q}_\nu$-spaces for a general symmetric cone. In the 'blank region', our main contribution up to now is Theorem 4.25, which gives the *equivalence* between boundedness of P_ν in $L^{p,q}_\nu$ and the inequality

$$\|\mathcal{L}\widehat{f}\|_{L^{p,q}_\nu} \lesssim \|f\|_{B^{p,q}_\nu}, \quad \text{for } f \in \mathcal{D}_\Omega. \tag{5.1}$$

Recall that the natural range of indices for this question is

$$\nu > \tfrac{1}{2}n - 1, \quad 1 \leqslant p < \infty, \quad 2 \leqslant q < \widetilde{q}_{\nu,p},$$

while the 'blank region' corresponds to

$$p' q_\nu \leqslant q < \min\{2q_\nu, \widetilde{q}_{\nu,p}\}, \quad \text{for } p > 2.$$

We shall apply results by Łaba and Wolff and by Tao and Vargas to conclude positively in this region when p is sufficiently large, and obtain a small interval around the left-hand point for all $p > 2$.

5.1. *A Whitney covering for the light-cone*

From now on, we let $\Omega = \Lambda_n$ denote the light-cone in $\mathbb{R}^n$ with $n \geqslant 3$. We first describe explicitly a Whitney decomposition for Λ_n.

A lattice in the light-cone is constructed as follows. For every $j \geqslant 1$, take a maximal 2^{-j}-separated sequence $\{\omega_k^{(j)}\}_{k=1}^{k_j}$ in the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$, with respect to the Euclidean distance (so that $k_j \sim 2^{j(n-2)}$). Then, define the following grid of points in Λ_n:

$$\xi^{\ell}_{j,k} = (2^{\ell}, 2^{\ell}\sqrt{1-2^{-2j}}\,\omega^{(j)}_k), \quad \text{for } \ell \in \mathbb{Z},\ j \geqslant 1,\ k = 1, \dots, k_j, \tag{5.2}$$

and the corresponding sets

$$E^{\ell}_{j,k} = \{(\tau, \xi') \in \Lambda_n : 2^{\ell-1} < \tau < 2^{\ell+1},\ 2^{-2j-2} < 1 - |\xi'|^2/\tau^2 < 2^{-2j+2}, \\ \text{and } |\xi'/|\xi'| - \omega^{(j)}_k| \leqslant \delta 2^{-j}\},$$

where the constant δ is chosen in such a way that the regions cover the cone. The geometric picture in $\mathbb{R}^3$ is as follows: the sets $E^{\ell}_{j,k}$ are truncated conical shells of height $\sim 2^{\ell}$, of thickness $\sim 2^{\ell-2j}$, and further decomposed into $k_j \sim 2^j$ sectors, all of equal arc-length $\sim 2^{\ell-j}$. This is the usual decomposition of Λ_n in the study of cone multipliers (see, for example, [**13**, **16**, **17**]). The next proposition shows that it is also a Whitney covering of the cone.

PROPOSITION 5.3. *With the notation above, the grid $\{\xi^{\ell}_{j,k}\}$ is a lattice in Λ_n. Moreover, there exist $0 < \eta_1 < \eta_2$ such that the corresponding family of invariant balls satisfies*

(a) *$\{B_{\eta_1}(\xi^{\ell}_{j,k})\}$ is disjoint in Ω;*
(b) *$\{B_{\eta_2}(\xi^{\ell}_{j,k})\}$ is a covering of Ω;*
(c) *$B_{\eta_1}(\xi^{\ell}_{j,k}) \subset E^{\ell}_{j,k} \subset B_{\eta_2}(\xi^{\ell}_{j,k})$.*

Proof. In view of the definition in § 2.2, it suffices to find two fixed (invariant) balls $B \subset \widetilde{B}$, centered at $\mathbf{e}$ and such that

$$g^{\ell}_{j,k}(B) \subset E^{\ell}_{j,k} \subset g^{\ell}_{j,k}(\widetilde{B}), \tag{5.4}$$

for some fixed automorphisms of the cone $g^{\ell}_{j,k}$ mapping $\mathbf{e}$ into $\xi^{\ell}_{j,k}$. Using dilations as well as rotations with axis $\mathbf{e} = (1, 0, \dots, 0)$, it is sufficient to prove this for the points $(1, \sqrt{1-2^{-2j}}, 0) = g_j\mathbf{e}$. Then, an elementary exercise shows that the corresponding set E_j is such that

$$E_j(c) \subset E_j \subset E_j(\widetilde{c}),$$

for constants c and $\widetilde{c}$ independent of j, and where $E_j(c)$ is the set of all $\xi \in \Lambda_n$ for which

$$c^{-1} < \frac{(\xi|\mathbf{e})}{(g_j\mathbf{e}|\mathbf{e})} < c, \quad c^{-1} < \frac{\Delta(\xi)}{\Delta(g_j\mathbf{e})} < c, \quad c^{-1} < \frac{\Delta_1(\xi)}{\Delta_1(g_j\mathbf{e})} < c$$

(recall that for light-cones $\Delta_1(\xi) = \xi_1 - \xi_2$). Let us finally prove that $g_j^{-1}(E_j(c))$ is contained in a ball and contains a ball, centered at $\mathbf{e}$ and with radii independent of j. From the invariance properties of the quantities involved, this last set consists of elements ξ for which

$$c^{-1} < (g_j\xi|\mathbf{e}) < c, \quad c^{-1} < \Delta(\xi) < c, \quad c^{-1} < \Delta_1(\xi) < c.$$

Using the explicit value $(g_j\xi|\mathbf{e}) = \xi_1 + \sqrt{1-2^{-2j}}\,\xi_2$, one finds that it is an elementary exercise to find two such balls with radii independent of j. □

With this covering, the necessary and sufficient conditions for (5.1) take the following explicit form. To simplify notation, we write $E_{j,k} = E^0_{j,k}$.

Proposition 5.5. Necessary and sufficient condition in Λ_n. *Let $1 \leqslant p, s < \infty$. Suppose that for some $\mu > 0$ there exists C_μ such that*

$$\Big\| \sum_{k=1}^{k_j} f_k \Big\|_p \leqslant C_\mu \, 2^{2j\mu/s} \Big[\sum_{k=1}^{k_j} \|f_k\|_p^s \Big]^{1/s}, \quad \text{for all } j \geqslant 1, \tag{5.6}$$

for every sequence $\{f_k\}$ satisfying $\operatorname{Supp} \widehat{f}_k \subset E_{j,k}$. *Then P_ν is bounded in $L^{p,q}_\nu$ for all q and ν such that*

$$q/s < \min\{q_\nu/q_\mu, (\nu - (\tfrac{1}{2}n - 1))/\mu\}. \tag{5.7}$$

Conversely, if P_ν is bounded in $L^{p,q}_\nu$ for some $\nu > n/r - 1$ and $2 < p, q < \infty$, then (5.6) *holds with $\mu = \nu$ and $s = q$.*

Proof. For the first part, by Theorem 4.11 and the second equivalence in Proposition 4.16, we only need to show that (5.6) implies the inequality

$$\Big\| \sum_j \sum_{k=1}^{k_j} f_{j,k} \Big\|_p \leqslant C_{\mu'} \Big[\sum_j 2^{2j\mu'/s} \sum_{k=1}^{k_j} \|f_{j,k}\|_p^s \Big]^{1/s},$$

with μ' perhaps larger than, but arbitrarily close to, μ. Here $f_{j,k}$ are arbitrary functions with Fourier transforms in the regions $E_{j,k}$. To prove the previous inequality, start using Minkowski's inequality for the sum in j, apply (5.6) in each block for fixed j, and conclude with Hölder's inequality. For the converse, we know from (4.33) that the condition

$$\|\mathcal{F}^{-1}(\widehat{f} e^{-(\mathbf{e}|\cdot)})\|_p \leqslant C \, \|f\|_{B^{p,s}_\mu}, \quad \text{for } f \in \mathcal{D}_\Omega,$$

is necessary for the boundedness of P_μ in $L^{p,s}_\mu(T_\Omega)$, which by Proposition 4.16 we can write as (5.6) when $f = \sum_k f_k$ and $\widehat{f}_k$ is supported in $E_{j,k}$. □

5.2. *Positive results for light-cones*

There are two situations where we can obtain new regions of boundedness from the sufficient condition in (5.6). These correspond to two fine inequalities related with the so-called 'cone multiplier problem' [**13**, **7**, **16**, **17**].

The first one is a consequence of the following results of T. Wolff and I. Łaba, which we state below adapted to our notation.

Theorem 5.8 (see [**17**, **12**]). *If $n \geqslant 6$ and $p > p_n := 2n/(n-4)$, the inequality*

$$\Big\| \sum_k f_k \Big\|_p \leqslant C_\varepsilon \, 2^{2j((n-2)/2-(n-1)/p+\varepsilon)} \Big[\sum_k \|f_k\|_p^p \Big]^{1/p}, \tag{5.9}$$

for all $\varepsilon > 0$, $j = 1, 2, \ldots,$ holds for all smooth functions f_k with spectrum in $E_{j,k}$. Moreover, (5.9) *also holds when $n = 3$ for all $p > p_3 = 74$, when $n = 4$ for all $p > p_4 = 18$, and when $n = 5$ for all $p > p_5 = \frac{42}{5}$.*

Remark 5.10. As stated in [**17**, **12**], the natural conjecture says that (5.9) must hold for all $p > 2n/(n-2)$. The exponent $\frac{1}{2}(n-2) - (n-1)/p$ is the best possible for each such p.

The following corollary is a personal communication by A. Seeger.

Corollary 5.11. *If $n \geqslant 3$ and $\nu > \frac{1}{2}(n-2)$, the Bergman projector P_ν is bounded in $L^{p,q}_\nu(T_{\Lambda_n})$ for all $q < \widetilde{q}_{\nu,p}$, whenever $p > p_{n,\nu}$, where*

$$p_{n,\nu} := p_n + \frac{(\nu_n - \nu)_+}{\nu - \frac{1}{2}(n-2)} \quad \text{and} \quad \nu_n := \tfrac{1}{2}(n-2) + (n-2)\left(p_n - \frac{2(n-1)}{n-2}\right).$$

Proof. If we first assume $\nu > \nu_n$, then (5.9) implies (5.6) with $s = p > p_n$ and all $\mu > p(\frac{1}{2}(n-2) - (n-1)/p)$. Computing the numbers in (5.7) one easily sees that $q/p < q_\nu/q_\mu$ (for all such μ) implies $q < \widetilde{q}_{\nu,p}$. On the other hand, the second condition does not play any role when $q_\nu/q_\mu \leqslant (\nu - \frac{1}{2}(n-2))/\mu$, or equivalently $\nu \geqslant 2\mu + \frac{1}{2}(n-2)$, from which one easily covers all cases $\nu > \nu_n$.

When $\nu \leqslant \nu_n$ one must use the inequality

$$\left\| \sum_k f_k \right\|_p \leqslant C_\varepsilon \, 2^{2j((n-2)/2s' - n/2p + \varepsilon)} \left[\sum_k \|f_k\|_p^s \right]^{1/s}, \quad \text{for all } \varepsilon > 0, \tag{5.12}$$

valid by interpolation when $(1/p, 1/s)$ lies in the segment

$$S_\rho = \left\{ \left(\frac{\theta}{\rho}, \frac{\theta}{\rho} + \frac{1-\theta}{1} \right) : 0 \leqslant \theta \leqslant 1 \right\}$$

and for all $\rho > p_n$. Computing again in (5.7) for all $\mu > s((n-2)/2s' - n/2p)$ we see that $q/s < q_\nu/q_\mu$ leads to $q < \widetilde{q}_{\nu,p}$, while the second condition is irrelevant if

$$\nu > \nu(p,s) := (s-1)(n-2) - \frac{sn}{p} + \tfrac{1}{2}(n-2).$$

Now, if $\nu \leqslant \nu_n$ we can choose $(1/p_\theta, 1/s_\theta) \in S_{p_n}$ so that $\nu = \nu(p_\theta, s_\theta)$. Then P_ν is bounded in $L^{p,q}_\nu$ for all $q < \widetilde{q}_{\nu,p}$ and $p > p_\theta$. An elementary calculation shows that $p_\theta = p_n + (\nu_n - \nu)/(\nu - \frac{1}{2}(n-2))$. □

Remark 5.13. In particular, if we study boundedness of P_ν in L^p_ν (that is, $p = q$), then by interpolation we obtain a positive answer in the full conjectured range $2 < p < q_\nu + n/(n-2)$ (and its dual interval) whenever

$$\nu \geqslant \tfrac{1}{2}(n-2) + \tfrac{1}{2}(n-2)\,(p_n - 2(n-1)/(n-2)).$$

If Wolff's conjecture (5.9) held for the best possible p_n (that is, $2n/(n-2)$), this would imply boundedness of the unweighted Bergman projector $P_{n/2}$ in $L^p(\Lambda_n)$, for all $2 < p < (3n-2)/(n-2)$, which was the original problem stated in [**1**].

Remark 5.14. Conversely, solving the Bergman projection problem in $L^p_\nu(\Lambda_n)$ would have implications on Wolff's conjecture, at least in low dimensions. More precisely, if we assume that P_ν is bounded in $L^{q,q}_\nu(\Lambda_n)$ for all $q \in (2, q_\nu + n/(n-2))$ and some fixed ν, then the inequality (5.9) would hold as well with $p = q_\nu + n/(n-2)$ (by Proposition 5.5). In particular, if $n = 3$ or 4, and we are assuming the above assertion for all $\nu > 1$, then (5.9) would follow for all $p > 2n/(n-2)$. When $n \geqslant 5$ this implication is weaker due to the restriction $\nu > \frac{1}{2}n - 1$, which eventually would imply (5.9) only in the (non-critical) range $p > (3n-4)/(n-2)$.

A second but less direct approach also gives a slight improvement for the 3-dimensional cone Λ_3. This is based on the *square function estimate*

$$\Big\| \sum_{k=1}^{k_j} f_k \Big\|_p \leqslant C_\mu \, 2^{j\mu} \Big\| \Big(\sum_{k=1}^{k_j} |f_k|^2 \Big)^{1/2} \Big\|_p . \tag{5.15}$$

Observe that (5.15) and Minkowski's inequality imply (5.6) with $s = 2$. The natural conjecture states that (5.15) must hold for all $\mu > 0$ when $p = 2(n-1)/(n-2)$. There are partial non-trivial results when $n = 3$ and $p = 4$. The first one, due to Mockenhaupt, is a geometric argument leading to $\mu = \frac{1}{4}$ (see [**13**]), but this index does not produce new results on the boundedness of Bergman projectors. There are however several improved estimates using *bilinear restriction on the cone*, which appear in works of Bourgain [**7**], Tao and Vargas [**16**], and Wolff [**18**]. These lead to the following theorem, which seems to contain the current best-known exponent.

Theorem 5.16 (see [**16**, **18**]). *If $n = 3$, the inequality (5.15) holds with $p = 4$ for all $\mu > \frac{1}{4} - \frac{1}{44}$.*

Proof. Combine Theorem 5.1 in [**16**], with the sharp bilinear restriction index for $n = 3$ in [**18**]. □

From this discussion and a straightforward computation of the numerology we conclude the following result.

Corollary 5.17. *When $n = 3$ and $p = 4$, the Bergman projector P_ν is bounded in $L^{4,q}_\nu(\Lambda_3)$ for all $2 \leqslant q < (\frac{4}{3} + \frac{1}{24})\, q_\nu$ and all $\nu > \frac{1}{2} + \frac{5}{11}$. (See Figure 5.1).*

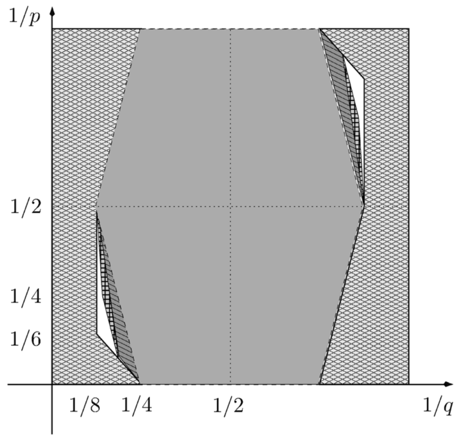

Figure 5.1. *Region of boundedness for P_ν in Λ_3.*

Remark 5.18. When $p = 4$, this result slightly improves the estimate which by interpolation can be obtained from Corollary 5.11. In fact, with $p_3 = 74$, the latter leads to boundedness of P_ν in $L^{4,q}_\nu(\Lambda_3)$ for all $2 \leqslant q < (\frac{4}{3} + \frac{4}{159})\, q_\nu$ and $\nu > \frac{1}{2} + 70$.

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