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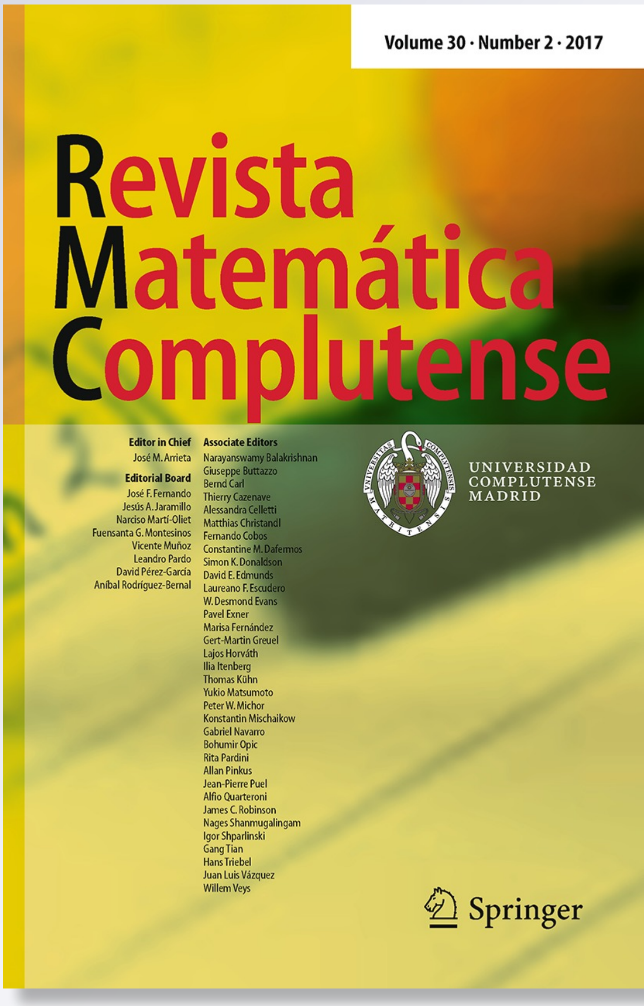

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Lebesgue inequalities for the greedy algorithm in general bases

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Abstract We present various estimates for the Lebesgue type inequalities associated with the thresholding greedy algorithm, in the case of general bases in Banach spaces. We show the optimality of the involved constants in some situations. Our results recover and slightly improve various estimates appearing earlier in the literature.

Keywords Thresholding greedy algorithm · Quasi-greedy basis · Conditional basis

Mathematics Subject Classification 41A65 · 41A46 · 46B15

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1 Introduction

Let $\mathbb{X}$ be a Banach space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty}$ a biorthogonal system such that $\mathscr{B} = \{\mathbf{e}_n\}$ has dense span in $\mathbb{X}$ and $0 < \kappa_1 \leq \|\mathbf{e}_n\|, \|\mathbf{e}_n^*\| \leq \kappa_2 < \infty$. Examples include (semi-normalized) Schauder bases $\mathscr{B}$, as well as more general structures (such as Markushevich bases [11]). As suggested in [24,25], greedy algorithms can be considered in this generality, by formally associating with every $x \in \mathbb{X}$ the series $x \sim \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n$. Note that $\lim_{n\to\infty} \mathbf{e}_n^*(x) = 0$, so one may speak of decreasing rearrangements of $\{\mathbf{e}_n^*(x)\}$.

We recall a few standard notions about greedy algorithms; see e.g. [21,22] for a detailed presentation and background. We say that a finite set $\Gamma \subset \mathbb{N}$ is a greedy set for $x \in \mathbb{X}$, denoted $\Gamma \in \mathscr{G}(x)$, if

$$\min_{n\in\Gamma} |\mathbf{e}_n^*(x)| \;\geq\; \max_{n\in\Gamma^c} |\mathbf{e}_n^*(x)|,$$

and write $\Gamma \in \mathscr{G}(x, N)$ if in addition $|\Gamma| = N$. A *greedy operator of order* N is a mapping $G : \mathbb{X} \to \mathbb{X}$ such that

$$Gx = \sum_{n\in\Gamma_x} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad \text{for some } \Gamma_x \in \mathscr{G}(x, N).$$

We write $\mathscr{G}_N$ for the set of all greedy operators of order N, and $\mathscr{G} = \cup_{N\geq 1}\mathscr{G}_N$. By convention, we set $\mathscr{G}_0 = \{0\}$. Given $G, G' \in \mathscr{G}$ we shall write $G' < G$ whenever $G \in \mathscr{G}_N$ and $G' \in \mathscr{G}_M$ with $0 \leq M < N$ and $\Gamma'_x \subset \Gamma_x$ for all x.

For every *finite* set $A \subset \mathbb{N}$ we also consider the projection operator

$$P_A x = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

and the "complement" projection $P_{A^c} = I - P_A$.

Greedy operators are frequently used for N-term approximation. As usual, we let $\Sigma_N = \left\{ \sum_{n\in A} a_n \mathbf{e}_n \;:\; |A| \leq N,\; a_n \in \mathbb{K} \right\}$ and $\sigma_N(x) = \operatorname{dist}(x, \Sigma_N)$. To quantify the efficiency of greedy approximation one defines, for each $N = 1, 2, \ldots$, the smallest number $\mathbf{L}_N$ such that

$$\|x - Gx\| \leq \mathbf{L}_N\, \sigma_N(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, G \in \mathscr{G}_N. \tag{1.1}$$

This is sometimes called a Lebesgue-type inequality for the greedy algorithm [22], and $\mathbf{L}_N$ is its associated Lebesgue-type constant. Likewise, one may consider "expansional" N-term approximations using $\widetilde{\sigma}_N(x) = \inf\{\|x - P_A x\| \;:\; |A| \leq N\}$, and define the smallest $\widetilde{\mathbf{L}}_N$ such that

$$\|x - Gx\| \leq \widetilde{\mathbf{L}}_N\, \widetilde{\sigma}_N(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, G \in \mathscr{G}_N. \tag{1.2}$$

A celebrated result of Konyagin and Temlyakov [14] establishes that $\mathbf{L}_N = O(1)$ if and only if $\mathscr{B}$ is unconditional and democratic. Explicit estimates for $\mathbf{L}_N$ have been obtained in various contexts for greedy bases [2,5,25], quasi-greedy bases [1,6,7,9, 23], and a few examples of non quasi-greedy bases [17,19,20]. The goal of this paper is to present these inequalities in a more general setting, and improve them as much as possible so that they actually become optimal in certain Banach spaces. This of course depends on the quantities used for the bounds, which we list next. We shall use the following notation

$$\mathbf{1}_A = \sum_{n\in A} \mathbf{e}_n \quad \text{and} \quad \mathbf{1}_{\boldsymbol{\varepsilon} A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n, \quad \text{if } \boldsymbol{\varepsilon} = \{\varepsilon_n\},$$

and we say that $\boldsymbol{\varepsilon} = \{\varepsilon_n\} \in \Upsilon$ if $|\varepsilon_n| = 1$ for all n (where ε_n could be real or complex). We also set $|x|_\infty = \sup_n |\mathbf{e}_n^*(x)|$ and $\operatorname{supp} x = \{n : \mathbf{e}_n^*(x) \neq 0\}$, and we write $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ to mean that A, B and $\operatorname{supp} x$ are pairwise disjoint.

- Unconditionality constants:

$$k_N = \sup_{|A|\le N} \|P_A\| \quad \text{and} \quad k_N^c = \sup_{|A|\le N} \|I - P_A\|.$$

- Quasi-greedy constants[1]:

$$g_N = \sup_{G\in\cup_{k\le N}\mathscr{G}_k} \|G\| \quad \text{and} \quad g_N^c = \sup_{G\in\cup_{k\le N}\mathscr{G}_k} \|I - G\|.$$

  We shall also use the following variants

$$\hat{g}_N = \min\{g_N, g_N^c\} \quad \text{and} \quad \tilde{g}_N = \sup_{\substack{G\in\cup_{k\le N}\mathscr{G}_k \\ G'<G}} \|G - G'\|\ .$$

- Democracy (and superdemocracy) constants:

$$\mu_N = \sup_{|A|=|B|\le N} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \quad \text{and} \quad \tilde{\mu}_N = \sup_{\substack{|A|=|B|\le N \\ \boldsymbol{\varepsilon},\boldsymbol{\eta}\in\Upsilon}} \frac{\|\mathbf{1}_{\boldsymbol{\varepsilon} A}\|}{\|\mathbf{1}_{\boldsymbol{\eta} B}\|},$$

  and their counterparts for disjoint sets, $A \cap B = \emptyset$, denoted μ_N^d and $\tilde{\mu}_N^d$.
- A-property constants:

$$\nu_N = \sup\left\{ \frac{\|\mathbf{1}_{\boldsymbol{\varepsilon} A} + x\|}{\|\mathbf{1}_{\boldsymbol{\eta} B} + x\|} \ :\ |A| = |B| \le N,\ \boldsymbol{\varepsilon},\boldsymbol{\eta} \in \Upsilon,\ |x|_\infty \le 1,\ A \mathbin{\dot\cup} B \mathbin{\dot\cup} x \right\}.$$

All these are natural quantities in the greedy literature, and quite often it is not hard to compute them explicitly; see Sect. 5 below for some examples. Elementary inequalities for the less frequent $\tilde{g}_N$ and ν_N are also given in Sect. 2.1 below. These sequences of constants produce natural *lower* bounds for the Lebesgue inequalities.

[1] We use the notation $\|G\| = \sup_{x\neq 0} \|Gx\|/\|x\|$, even if $G : \mathbb{X} \to \mathbb{X}$ may be a non-linear map.

Proposition 1.1 *For all $N \geq 1$ we have*

$$\mathbf{L}_N \geq \max\left\{k_N^c\,,\ \widetilde{\mathbf{L}}_N\right\}, \quad \textit{and} \quad \widetilde{\mathbf{L}}_N \geq \max\left\{g_N^c\,,\ \nu_N\right\}. \tag{1.3}$$

Moreover,

$$\nu_N \geq \max\left\{\mu_N\,,\ \tilde{\mu}_N^d\,,\ \tfrac{1}{2\kappa}\tilde{\mu}_N\right\}, \tag{1.4}$$

where $\kappa = 1$ or 2, if $\mathbb{X}$ is real or complex, respectively.

We shall present two results concerning upper bounds.

Theorem 1.2 *For all $N \geq 1$ we have*

$$\mathbf{L}_N \leq k_{2N}^c\,\nu_N \quad \textit{and} \quad \widetilde{\mathbf{L}}_N \leq g_N^c\,\nu_N. \tag{1.5}$$

Moreover, there exists $(\mathbb{X}, \mathscr{B})$ for which both equalities are attained.

Theorem 1.3 *For all $N \geq 1$ we have*

$$\mathbf{L}_N \leq k_{2N}^c + \tilde{g}_N\,\tilde{\mu}_N \quad \textit{and} \quad \widetilde{\mathbf{L}}_N \leq g_N^c + \tilde{g}_N\,\tilde{\mu}_N. \tag{1.6}$$

Moreover, there exists $(\mathbb{X}, \mathscr{B})$ for which both equalities are attained.

We discuss a bit these theorems and their relation with earlier estimates in the literature. Theorem 1.2 is a variant of a result of Albiac and Ansorena [1], which for $\mathscr{B}$ quasi-greedy and democratic showed that

$$\widetilde{\mathbf{L}}_N \leq g^c\nu, \quad \text{where} \quad g^c = \sup_{N\geq 1} g_N^c \quad \text{and} \quad \nu = \sup_{N\geq 1} \nu_N;$$

see [1, Theorem 3.3.ii]. In the unconditional case, they announced as well the bound $\mathbf{L}_N \leq k^c\nu$ with $k^c = \sup k_N^c$ (see [1, Remark 3.8]), which itself improves the earlier bound $\mathbf{L}_N \leq (k^c)^2\nu$ by Dilworth et al. [5, Theorem 2]. Our (modest) contribution here is the explicit dependence on N of the involved constants, together with a slightly shorter and more direct proof. As discussed in [1], an interesting special case occurs when $\mathscr{B}$ is an unconditional basis with $k_N^c \equiv 1$. Actually, (1.3), (1.5) and the trivial estimate

$$\widetilde{\mathbf{L}}_N \leq \mathbf{L}_N \leq k_N^c\,\widetilde{\mathbf{L}}_N$$

(see [9, (1.7)]), give

Corollary 1.4 *If for some N_0 we have $k_{N_0}^c = 1$, then $k_N^c \equiv 1$ and*

$$\mathbf{L}_N = \widetilde{\mathbf{L}}_N = \nu_N\,, \quad \forall\, N \geq 1.$$

In particular, the optimality asserted in the last sentence of Theorem 1.2 is attained for any 1-suppression unconditional basis. Optimality also holds in the following case.

Corollary 1.5 *If for some N_0 we have $\nu_{N_0} = 1$, then $\nu_N \equiv 1$ and*

$$\mathbf{L}_N = k_N^c, \quad \textit{and} \quad \widetilde{\mathbf{L}}_N = g_N^c = 1\,, \quad \forall\, N \geq 1.$$

This result is essentially proved in [1]. It is an open question whether in this case it could happen that $k_N^c \to \infty$; see [1, Problem 4.4]. As we show in Example 5.5 below, if one merely assumes $\sup_N \nu_N < \infty$, then it may actually happen that $k_N^c \geq g_N^c \to \infty$. This is based on an example appearing earlier in [3, Example 4.8].

Theorem 1.2, however, has some drawbacks. The first one concerns ν_N, which in practice may be much harder to compute explicitly than the standard democracy constants μ_N and $\tilde{\mu}_N$. A second drawback comes from the multiplicative bound $k_{2N}^c \nu_N$, which may be far from optimal when both k_N^c and ν_N grow to ∞. This already occurs with simple examples of quasi-greedy bases (see e.g. [9, (6.9)]).

Theorem 1.3 intends to cover some of these drawbacks, with an estimate which is asymptotically optimal at least for quasi-greedy bases. In fact, if we set

$$\mathbf{q} := \sup_N \hat{g}_N \;=\; \min\Big\{ \sup_{G\in\mathscr{G}} \|G\|, \;\sup_{G\in\mathscr{G}} \|I-G\| \Big\} \tag{1.7}$$

then, in Sect. 3.6 we shall show that

Corollary 1.6 *If $\mathscr{B}$ is a quasi-greedy basis then*

$$\max\{k_N^c, \mu_N\} \;\leq\; \mathbf{L}_N \;\leq\; k_{2N}^c \;+\; 8\,\kappa^2\,\mathbf{q}^2\,\mu_N \tag{1.8}$$

and

$$\max\{g_N^c, \mu_N\} \;\leq\; \widetilde{\mathbf{L}}_N \;\leq\; g_N^c \;+\; 8\,\kappa^2\,\mathbf{q}^2\,\mu_N, \tag{1.9}$$

where $\kappa = 1$ if $\mathbb{K} = \mathbb{R}$, and $\kappa = 2$ if $\mathbb{K} = \mathbb{C}$.

The fact that $\mathbf{L}_N \approx k_N + \mu_N$ for quasi-greedy bases is already known [9]. Our contribution here is an improvement of the implicit constants in the second summand, compared to $O(\mathbf{q}^4)$ in [9], and $8\mathbf{q}^3$ in [6]. Similarly, for $\widetilde{\mathbf{L}}_N$ the earlier estimates in [23, Theorem 2] only gave $8\mathbf{q}^4$ for the involved constants in the second summand.

Another consequence of Theorem 1.3 is the following asymptotic equivalence

Corollary 1.7 *If $\mathcal{B}$ is superdemocratic (that is $\sup_N \tilde{\mu}_N < \infty$), then*

$$\mathbf{L}_N \approx k_N \quad \textit{and} \quad \widetilde{\mathbf{L}}_N \approx g_N. \tag{1.10}$$

Example 5.5 below provides a non-trivial application of this result. We do not know whether (1.10) continues to hold for all democratic bases.

Finally, we should say that the estimates in (1.6), being multiplicative, suffer from a similar drawback as (1.5), namely they may be far from efficient when both $\tilde{\mu}_N$ and $\tilde{g}_N$ grow to infinity. For such cases one always has the following trivial upper bounds

Theorem 1.8 *If $K = \sup_{m,n} \|\mathbf{e}_m\|\|\mathbf{e}_n^*\|$, then for all $N \geq 1$ we have*

$$\mathbf{L}_N \leq 1 + 3K\,N \quad \textit{and} \quad \nu_N \leq \widetilde{\mathbf{L}}_N \leq 1 + 2K\,N. \tag{1.11}$$

Moreover, there exists an example of $(\mathbb{X}, \mathscr{B})$ *for which all the equalities hold.*

The optimality for $\mathbf{L}_N$ in Theorem 1.8 was first proved by Oswald [17]. We give a different and simpler construction in Example 5.1 below.

The outline of the paper is the following. We start in Sect. 2 with a few elementary lemmas. In Sect. 3 we give the details proofs of Theorems 1.2, 1.3, 1.8, and their corollaries. In Sect. 4 we prove the lower bounds asserted in Proposition 1.1. Finally, Sect. 5 is devoted to the computations of explicit examples.

2 Some elementary Lemmas

2.1 Elementary bounds for $\tilde{g}_N$ and ν_N

Lemma 2.1 *For each* $N \in \mathbb{N}$ *we have*

$$g_N \leq \tilde{g}_N \leq \min\{2\hat{g}_N,\ g_N g_N^c,\ k_N\}. \tag{2.1}$$

In particular, $\tilde{g}_N = g_N$ *when* $g_N^c = 1$.

Proof $g_N \leq \tilde{g}_N \leq k_N$ is obvious by definition and $\tilde{g}_N \leq 2\hat{g}_N$ follows easily from the triangle inequality. Finally, for each $G \in \cup_{k\leq N}\mathscr{G}_k$ and $G' < G$ we can write $Gx - G'x = \sum_{n\in\Gamma_x\setminus\Gamma'_x} e_n^*(x)e_n$ with $\Gamma_x\setminus\Gamma'_x \in \cup_{k\leq N}\mathscr{G}(x - G'x, k)$; hence

$$\|Gx - G'x\| \leq g_N\|x - G'x\| \leq g_N g_N^c\|x\|.$$

□

Lemma 2.2 *For each* $N \in \mathbb{N}$ *we have*

$$\max\{\tilde{\mu}_N^d, \mu_N\} \leq \nu_N \leq g_N^c + g_N\tilde{\mu}_N^d. \tag{2.2}$$

Proof $\tilde{\mu}_N^d \leq \nu_N$ and $\mu_N \leq \nu_N$ follow selecting $x = 0$ and $x = \mathbf{1}_{A\cap B}$, respectively, in the definition of ν_N. On the other hand, for each $|A| = |B| \leq N$, $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \Upsilon$, $|x|_\infty \leq 1$, $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ we have $\|x\| \leq g_N^c\|\mathbf{1}_{\boldsymbol{\eta} B}+x\|$ and $\|\mathbf{1}_{\boldsymbol{\varepsilon} A}\| \leq \tilde{\mu}_N^d\|\mathbf{1}_{\boldsymbol{\eta} B}\| \leq \tilde{\mu}_N^d g_N\|\mathbf{1}_{\boldsymbol{\eta} B}+x\|$. Hence the inequality $\nu_N \leq g_N^c + g_N\tilde{\mu}_N^d$ is easily obtained. □

2.2 Truncation operators

For each $\alpha > 0$, we define the α-truncation of $z \in \mathbb{C}$ by

$$T_\alpha(z) = \alpha\ \text{sign}(z)\ \text{ if } |z| \geq \alpha, \quad \text{and} \quad T_\alpha(z) = z\ \text{ if } |z| \leq \alpha.$$

We extend T_α to an operator in $\mathbb{X}$ by

$$T_\alpha(x) = \sum_n T_\alpha(\mathbf{e}_n^*(x))\mathbf{e}_n = \sum_{n\in\Lambda_\alpha} \alpha\frac{\mathbf{e}_n^*(x)}{|\mathbf{e}_n^*(x)|}\mathbf{e}_n + \sum_{n\notin\Lambda_\alpha} \mathbf{e}_n^*(x)\mathbf{e}_n, \tag{2.3}$$

where $\Lambda_\alpha = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$. Since Λ_α is a finite set, the last summand can be expressed as $(I - P_{\Lambda_\alpha})x$, so the operator is well-defined for all $x \in \mathbb{X}$.

Lemma 2.3 *If $x \in \mathbb{X}$ and $\boldsymbol{\varepsilon} = \{\text{sign}\, \mathbf{e}_n^*(x)\}$, then*

$$\min_{\Lambda} |\mathbf{e}_n^*(x)| \left\| \mathbf{1}_{\boldsymbol{\varepsilon}\Lambda} \right\| \leq \tilde{g}_N \|x\|, \quad \forall\, \Lambda \in \mathscr{G}(x, N). \tag{2.4}$$

Proof Set $\alpha = \min_\Lambda |\mathbf{e}_n^*(x)|$. Notice first that

$$T_\alpha x = \int_0^1 \Big[\sum_n \chi_{[0, \frac{\alpha}{|\mathbf{e}_n^*(x)|}]}(s)\, \mathbf{e}_n^*(x)\mathbf{e}_n \Big]\, ds = \int_0^1 (I - P_{\Lambda_{\alpha,s}})x\, ds, \tag{2.5}$$

where we have set $\Lambda_{\alpha,s} = \{n : |\mathbf{e}_n^*(x)| > \frac{\alpha}{s}\}$ for each $s \in (0, 1]$. Hence

$$\alpha \mathbf{1}_{\boldsymbol{\varepsilon}\Lambda} = T_\alpha x - P_{\Lambda^c} x = \int_0^1 (P_\Lambda x - P_{\Lambda_{\alpha,s}} x)\, ds.$$

Note that $\Lambda_{\alpha,s} \in \mathscr{G}(x, k_s)$ with $k_s = |\Lambda_{\alpha,s}|$ and $\Lambda_{\alpha,s} \subseteq \Lambda_\alpha \subset \Lambda$. Hence

$$\|P_\Lambda x - P_{\Lambda_{\alpha,s}} x\| \leq \tilde{g}_N \|x\|, \quad 0 < s \leq 1.$$

The result now follows. □

Remark 2.4 The slightly weaker inequality

$$\min_{\Lambda} |\mathbf{e}_n^*(x)| \left\| \mathbf{1}_{\boldsymbol{\varepsilon}\Lambda} \right\| \leq 2 \min\{g_N, g_N^c\} \|x\|. \tag{2.6}$$

was proved in [4, Lemma 2.2] with an elementary Abel summation argument.

The next lemma is a slight improvement over [3, Proposition 3.1].

Lemma 2.5 *For all $\alpha > 0$ and $x \in \mathbb{X}$ we have*

$$\|T_\alpha x\| \leq g^c_{|\Lambda_\alpha|} \|x\|, \quad \|(I - T_\alpha)x\| \leq g_{|\Lambda_\alpha|} \|x\|, \tag{2.7}$$

where $\Lambda_\alpha = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$. Moreover, for every $|A| < \infty$

$$\|T_\alpha (I - P_A)x\| \leq k^c_{|A \cup \Lambda_\alpha|} \|x\|. \tag{2.8}$$

Proof The result follows from Minkowski's integral inequality applied to (2.5), and to the following two formulae derived from it

$$(I - T_\alpha)x = \int_0^1 P_{\Lambda_{\alpha,s}} x\, ds$$

and

$$T_\alpha(I-P_A)x = \int_0^1 (I-P_{\Lambda_{\alpha,s}})(I-P_A)x\,ds, = \int_0^1 (I-P_{A\cup\Lambda_{\alpha,s}})x\,ds.$$

□

Remark 2.6 Observe that, together with (2.8), one has the trivial estimate

$$\|T_\alpha(I-P_A)x\| \le g^c_{|\Lambda_\alpha|}\, k^c_{|A|}\, \|x\|. \tag{2.9}$$

Being multiplicative, (2.9) is typically worse than (2.8) (if say both k^c_N and g^c_N grow fast as $N\to\infty$). However in some cases it may better (e.g. when $g^c_{|\Lambda_\alpha|}=1$).

2.3 Convex extensions

We shall use an elementary convexity lemma. As usual, the convex envelop of a set S is defined by co $S=\{\sum_{j=1}^n \lambda_j x_j \;:\; x_j\in S,\ 0\le\lambda_j\le 1,\ \sum_{j=1}^n\lambda_j=1,\ n\in\mathbb{N}\}$.

Lemma 2.7 *For every finite $A\subset\mathbb{N}$, we have*

$$\text{co}\Big\{\mathbf{1}_{\varepsilon A} \;:\; \varepsilon\in\Upsilon\Big\} = \Big\{\sum_{n\in A} z_n\mathbf{e}_n \;:\; |z_n|\le 1\Big\}.$$

Proof We sketch the proof in the complex case, where it may be less obvious. The inclusion "$\subseteq$" is clear, since each $\mathbf{1}_{\varepsilon A}$ belongs to the set R on the right hand side, and R is a convex set. To show "$\supseteq$" one proceeds by induction in $N=|A|$. It is clear for $N=1$, so we show the case N from the case $N-1$. We may assume that $A=\{1,\dots,N\}$. Pick any $z=\sum_{n=1}^N z_n\mathbf{e}_n\in R$, that is $|z_n|\le 1$. Write $z_N=re^{i\theta}$, and by the induction hypothesis

$$z' = \sum_{n=1}^{N-1} z_n\mathbf{e}_n = \sum_{\varepsilon}\lambda_\varepsilon\,(\varepsilon_1\mathbf{e}_1+\ldots+\varepsilon_{N-1}\mathbf{e}_{N-1}),$$

for suitable numbers $0\le\lambda_\varepsilon\le 1$ such that $\sum_\varepsilon\lambda_\varepsilon=1$. Then we have

$$\begin{aligned} z &= \tfrac{1+r}{2}\left[z'+e^{i\theta}\mathbf{e}_N\right] + \tfrac{1-r}{2}\left[z'-e^{i\theta}\mathbf{e}_N\right] \\ &= \sum_{\varepsilon,\pm}\tfrac{1\pm r}{2}\lambda_\varepsilon\,(\varepsilon_1\mathbf{e}_1+\ldots+\varepsilon_{N-1}\mathbf{e}_{N-1}\pm e^{i\theta}\mathbf{e}_N), \end{aligned}$$

which belongs to the set on the left hand side. □

The next lemma is a straightforward extension of the inequality defining ν_N.

Lemma 2.8 *Let $x \in \mathbb{X}$ and $\alpha \geq \max |\mathbf{e}_n^*(x)|$. Then*

$$\|x + z\| \leq \nu_N \|x + \alpha \mathbf{1}_{\eta B}\|, \quad \forall\, \eta \in \Upsilon$$

and for all B and z such that $|\operatorname{supp} z| \leq |B| \leq N$, $B \mathbin{\dot\cup} x \mathbin{\dot\cup} z$ and $|z|_\infty \leq \alpha$.

Proof We may assume that $\alpha = 1$. By definition of ν_N, the result is true when $z = \mathbf{1}_{\varepsilon A}$, for any $\varepsilon \in \Upsilon$ and any set A with $|A| = |B|$ and $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$. By convexity of the norm, it continues to be true for any $z \in \operatorname{co}\{\mathbf{1}_{\varepsilon A} \;:\; \varepsilon \in \Upsilon\}$. Then the general case follows from Lemma 2.7. □

In a similar fashion one shows

Lemma 2.9 *Let $z \in \mathbb{X}$ and $B \subset \mathbb{N}$ such that $|\operatorname{supp} z| \leq |B| \leq N$. Then*

$$\|z\| \leq \tilde{\mu}_N \max |\mathbf{e}_n^*(z)| \, \|\mathbf{1}_{\eta B}\|, \quad \forall\, \eta \in \Upsilon.$$

3 Proof of the Theorems

The general outline for proving estimates of $\mathbf{L}_N$ and $\widetilde{\mathbf{L}}_N$ goes back to the work of Konyagin and Temlyakov [14], with the improvements coming from refinements in certain steps. In Theorem 1.2 we use the ideas developed by Albiac and Ansorena [1], slightly simplified according to our previous lemmas.

3.1 Proof of Theorem 1.2

Let $x \in \mathbb{X}$ and $\Gamma \in \mathscr{G}(x, N)$, and call $\alpha = \min_\Gamma |\mathbf{e}_n^*(x)|$. Pick any $z \in \Sigma_N$ and $A \supset \operatorname{supp} z$ with $|A| = |\Gamma| = N$. Then we can write

$$x - P_\Gamma x = (I - P_{A\cup\Gamma})x + P_{A\setminus\Gamma} x =: X + Z. \tag{3.1}$$

Since $|X|_\infty, |Z|_\infty \leq \alpha$ and $|\operatorname{supp} Z| \leq |A \setminus \Gamma| = |\Gamma \setminus A|$, we can apply Lemma 2.8 with $\eta = \{\operatorname{sign} \mathbf{e}_n^*(x)\}$ to obtain

$$\begin{aligned} \|x - P_\Gamma x\| &\leq \nu_N \left\|\alpha \mathbf{1}_{\eta(\Gamma\setminus A)} + P_{(A\cup\Gamma)^c} x\right\| \\ &= \nu_N \left\|T_\alpha\big[(I - P_A)x\big]\right\| = \nu_N \left\|T_\alpha\big[(I - P_A)(x - z)\big]\right\| \\ &\leq \nu_N\, k^c_{|A\cup\Gamma|} \|x - z\| \leq \nu_N\, k^c_{2N} \|x - z\|, \end{aligned} \tag{3.2}$$

using Lemma 2.5 in the second to last inequality. Thus, taking the infimum over all $z \in \Sigma_N$ we conclude that

$$\mathbf{L}_N \leq \nu_N\, k^c_{2N}.$$

The estimate for $\widetilde{\mathbf{L}}_N$ is similar: for any set A with $|A| = |\Gamma| = N$ we have

$$\|x - P_\Gamma x\| \leq \nu_N \left\|T_\alpha\big[(I - P_A)x\big]\right\| \leq \nu_N\, g^c_N \|x - P_A x\|,$$

using again Lemma 2.5 (and $|\Lambda_\alpha| \le |\Gamma| = N$). By a standard perturbation argument as in [1, Lemma 3.4], this inequality continues to hold for all $|A| \le N$. This implies that $\widetilde{\mathbf{L}}_N \le \nu_N g_N^c$, and establishes the theorem. □

Remark 3.1 Notice that we could use in (3.2) the estimate in Remark 2.6, leading to the slightly smaller bound

$$\mathbf{L}_N \le \min\{k_{2N}^c, k_N^c g_N^c\}\, \nu_N.$$

For instance, if we assume $g_N^c = 1$ for some N (or equivalently, for all N), since $k_N^c \le k_{2N}^c$, we obtain

$$\mathbf{L}_N \le k_N^c\, \nu_N,$$

which we shall use in Corollary 1.5.

3.2 Proof of Theorem 1.3

With the same notation as in (3.1), it is clear that

$$\|(I - P_{A\cup\Gamma})x\| = \|(I - P_{A\cup\Gamma})(x - z)\| \le k_{2N}^c\, \|x - z\|. \tag{3.3}$$

So we only need to estimate the term $\|P_{A\setminus\Gamma}x\|$. We pick any set $\widetilde{\Gamma} \in \mathscr{G}(x-z, |A\setminus\Gamma|)$, and use the elementary observation

$$\max_{A\setminus\Gamma} |\mathbf{e}_n^*(x)| \le \min_{\widetilde{\Gamma}} |\mathbf{e}_n^*(x - z)|; \tag{3.4}$$

see e.g. [9, p. 453]. Then, Lemma 2.9 with $\boldsymbol{\eta} = \{\mathrm{sign}\, \mathbf{e}_n^*(x - z)\}$, followed by (3.4) and Lemma 2.3 give

$$\begin{aligned} \left\| P_{A\setminus\Gamma}x \right\| &\le \tilde{\mu}_N \max_{A\setminus\Gamma} |\mathbf{e}_n^*(x)|\, \|\mathbf{1}_{\boldsymbol{\eta}\widetilde{\Gamma}}\| \\ &\le \tilde{\mu}_N \min_{\widetilde{\Gamma}} |\mathbf{e}_n^*(x-z)|\, \|\mathbf{1}_{\boldsymbol{\eta}\widetilde{\Gamma}}\| \\ &\le \tilde{\mu}_N\, \tilde{g}_N\, \|x - z\|. \end{aligned} \tag{3.5}$$

So, adding up (3.3) and (3.5) and taking the infimum over all $z \in \Sigma_N$ one obtains

$$\|x - Gx\| \le \left(k_{2N}^c + \tilde{\mu}_N\, \tilde{g}_N\right) \sigma_N(x),$$

as asserted in (1.6).

The estimate for $\widetilde{\mathbf{L}}_N$ is again similar: given a set A with $|A| = |\Gamma| = N$, we can replace (3.3) by

$$\|(I - P_{A\cup\Gamma})x\| = \|(I - P_{\Gamma\setminus A})(I - P_A)x\| \le g_N^c\, \|x - P_A x\|, \tag{3.6}$$

since $\Gamma\backslash A \in \mathscr{G}(x - P_A x, |\Gamma\backslash A|)$ and $|\Gamma\backslash A| \le N$. The second estimate in (3.5) is valid in this case setting $z = P_A x$ and $\widetilde{\Gamma} = \Gamma\backslash A$. Thus we conclude

$$\|x - G_N x\| \le \left(g_N^c + \tilde{\mu}_N \tilde{g}_N\right) \inf_{|A|=N} \|x - P_A x\|,$$

and as before, this last quantity coincides with $\widetilde{\sigma}_N(x)$ by [1, Lemma 3.4]. The optimality of the constants is a consequence of Example 5.2, that we discuss below. □

Remark 3.2 In (3.3) one could replace k_{2N}^c by $g_N^c\, k_N^c$, arguing as in (3.6). Thus, in the special case when $g_N^c = 1$ for some N, the bounds become

$$\mathbf{L}_N \le k_N^c + g_N \tilde{\mu}_N \quad \text{and} \quad \widetilde{\mathbf{L}}_N \le 1 + g_N \tilde{\mu}_N,$$

since $\tilde{g}_N = g_N$; see Lemma 2.1.

3.3 Proof of Theorem 1.8

The first estimate in (1.11) is implicit in the first papers in the topic (see e.g., [19,20] or [17, (1.8)]). We sketch below the elementary proof, as it also gives the second estimate. With the notation in (3.1), notice that

$$\begin{aligned} \left\| P_{A\backslash\Gamma} x \right\| &\le \sum_{m\in A\backslash\Gamma} |\mathbf{e}_m^*(x)| \|\mathbf{e}_m\| \le \sup_m \|\mathbf{e}_m\| \sum_{n\in\Gamma\backslash A} |\mathbf{e}_n^*(x)| \\ &\le \sup_{m,n} \|\mathbf{e}_m\| \|\mathbf{e}_n^*\|\; N\; \|x - z\|, \end{aligned} \tag{3.7}$$

since $\mathbf{e}_n^*(x) = \mathbf{e}_n^*(x - z)$ when $n \notin A$. Thus, using either (3.3) or (3.6) we see that

$$\mathbf{L}_N \le k_{2N}^c + \mathrm{K}\, N \quad \text{and} \quad \widetilde{\mathbf{L}}_N \le g_N^c + \mathrm{K}\, N. \tag{3.8}$$

Now (1.11) follows from (3.8) and the trivial upper bound

$$k_N \le k_1 N \implies g_N^c \le k_N^c \le 1 + k_1 N, \tag{3.9}$$

since $k_1 = \sup_{n\ge1} \|\mathbf{e}_n\| \|\mathbf{e}_n^*\| \le \mathrm{K}$. The optimality of the constants is a consequence of Example 5.1, that we discuss below. □

3.4 Proof of Corollary 1.4

The proof was already sketched in the introduction, except for the fact that $k_N^c \equiv 1$. This in turn follows from $k_{N_0}^c = 1$ and

$$1 \le k_1^c \le k_N^c \le (k_1^c)^N, \quad \forall\, N = 1, 2, \ldots \tag{3.10}$$

The last inequality in (3.10) is easily obtained from $I - P_A = \prod_{n \in A}(I - P_{\{n\}})$. For the middle inequality observe that, in an infinite dimensional space $\mathbb{X}$, for every compactly supported $x \in \mathbb{X}$ we can write $(I - P_{\{n\}})x = (I - P_A)x$ for a suitable $A = A_x$ of cardinality N. Thus, $\|(I - P_{\{n\}})x\| \leq k_N^c \|x\|$, and hence $k_1^c \leq k_N^c$.

3.5 Proof of Corollary 1.5

Since $\nu_1 \leq \nu_{N_0} = 1$ it follows that $\nu_1 = 1$. A simple induction argument, see [1, Lemma 2.1], shows that

$$\nu_N \leq (\nu_1)^N,$$

so we shall have $\nu_N \equiv 1$. From here, arguing as in [1, Theorem 2.3] one obtains $\widetilde{\mathbf{L}}_N \equiv 1$, and by (1.3) also $g_N^c \equiv 1$. Thus, we can invoke the last sentence in Remark 3.1 to obtain that $\mathbf{L}_N \leq k_N^c$. This together with the lower bound $\mathbf{L}_N \geq k_N^c$ in (1.3) establishes the Corollary. □

3.6 Proof of Corollary 1.6

We need an additional inequality to pass from $\tilde{\mu}_N$ to μ_N. Consider the new constant

$$\gamma_N = \sup \left\{ \frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \; : \; B \subset A, \; |A| \leq N, \; \boldsymbol{\varepsilon} \in \Upsilon \right\}, \tag{3.11}$$

and observe that $\gamma_N \leq \hat{g}_N$. We also have the following

Lemma 3.3 *Let $\kappa = 1$ or 2, if $\mathbb{X}$ is real or complex, respectively. Then,*

$$\left\|\mathbf{1}_{\varepsilon B}\right\| \leq 2\kappa\, \gamma_N \left\|\mathbf{1}_{\eta A}\right\|, \quad \forall\, B \subset A, \; |A| \leq N, \; \boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \Upsilon. \tag{3.12}$$

Proof Observe that changing the basis $\{\mathbf{e}_n\}$ to $\{\eta_n \mathbf{e}_n\}$ does not modify the value of γ_N. So we may assume in (3.12) that $\boldsymbol{\eta} \equiv 1$. We use the convexity argument in [6, Lemma 6.4]. First notice that (3.11) actually implies

$$\|x\| \leq \gamma_N \|\mathbf{1}_A\|, \quad \forall\, x \in S = \left\{ \sum_{A' \subset A} \theta_{A'} \mathbf{1}_{A'} \; : \; \sum_{A' \subset A} |\theta_{A'}| \leq 1 \right\}. \tag{3.13}$$

In the real case, splitting $B = B_+ \cup B_-$, with $B_\pm = \{n \in B \; : \; \varepsilon_n = \pm 1\}$, it is clear that $\mathbf{1}_{\varepsilon B} = \mathbf{1}_{B_+} - \mathbf{1}_{B_-} \in 2S$. In the complex case, a slightly longer argument as in [6, Lemma 6.4] gives that $\mathbf{1}_{\varepsilon B} \in 4S$. So, in both cases we obtain (3.12). □

Remark 3.4 Recalling [24, Def 3], $\mathscr{B}$ is *unconditional for constant coefficients* if $\|\mathbf{1}_{\varepsilon A}\| \approx \|\mathbf{1}_A\|$, for all finite A and $\boldsymbol{\varepsilon} \in \Upsilon$. Using Lemmas 2.7 and 3.3 one easily sees that this is the same as $\sup_N \gamma_N < \infty$. It is also a weaker notion than $\mathscr{B}$ being quasi-greedy; see Example 5.5 below.

Lemma 3.5 *Let κ be as in Lemma 3.3. Then,*

$$\tilde{\mu}_N \le 4\kappa^2 \gamma_N \mu_N, \quad \forall N = 1, 2, \dots \tag{3.14}$$

Proof Take $A, B \subset \mathbb{N}$ with $|A| = |B| \le N$ and $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \Upsilon$. We must show that

$$\|\mathbf{1}_{\boldsymbol{\varepsilon} A}\| \le 4\kappa^2 \gamma_N \mu_N \|\mathbf{1}_{\boldsymbol{\eta} B}\|. \tag{3.15}$$

In the real case, split $A = A_1 \mathbin{\dot\cup} A_2$ with $A_j = \{n \in A \ : \ \varepsilon_n = (-1)^j\}$, and pick any partition $B = B_1 \mathbin{\dot\cup} B_2$ such that $|B_j| = |A_j|$, $j = 1, 2$. Then

$$\|\mathbf{1}_{\boldsymbol{\varepsilon} A}\| \le \|\mathbf{1}_{A_1}\| + \|\mathbf{1}_{A_2}\| \le \mu_N \left[\|\mathbf{1}_{B_1}\| + \|\mathbf{1}_{B_2}\|\right] \le 4\gamma_N \mu_N \|\mathbf{1}_{\boldsymbol{\eta} B}\|,$$

using Lemma 3.3 in the last step. In the complex case, arguing as in (3.13) from the previous lemma, we have $\mathbf{1}_{\boldsymbol{\varepsilon} A} \in 4S$. Now given $x = \sum_{A' \subset A} \theta_{A'} \mathbf{1}_{A'} \in S$, we pick for each A' a subset $B' \subset B$ such that $|A'| = |B'|$. Again, we have

$$\|x\| \le \sum_{A' \subset A} |\theta_{A'}| \|\mathbf{1}_{A'}\| \le \mu_N \sum_{A' \subset A} |\theta_{A'}| \|\mathbf{1}_{B'}\| \le \mu_N \, 2\kappa \, \gamma_N \|\mathbf{1}_{\boldsymbol{\eta} B}\|,$$

using Lemma 3.3 at the last step. This easily gives (3.15). □

Proof of Corollary 1.6 By Theorem 1.3, (2.1) and Lemma 3.5, the last summand in (1.6) can now be controlled by

$$\tilde{g}_N \tilde{\mu}_N \le 2\hat{g}_N 4\kappa^2 \gamma_N \mu_N \le 8\kappa^2 \hat{g}_N^2 \mu_N.$$

This clearly implies (1.8) and (1.9). □

Remark 3.6 Observe that we actually have the more general bounds

$$\mathbf{L}_N \le k_{2N}^c + 8\kappa^2 \gamma_N \hat{g}_N \mu_N, \quad \text{and} \quad \widetilde{\mathbf{L}}_N \le g_N^c + 8\kappa^2 \gamma_N \hat{g}_N \mu_N. \tag{3.16}$$

In some cases, these estimates are strictly better than (1.8) and (1.9); see Example 5.5 below.

3.7 Proof of Corollary 1.7

First observe that

$$\frac{k_N}{2} \le \max\{1, k_N - 1\} \le k_N^c \le 1 + k_N \le 2k_N.$$

So, setting $\tilde{\mu} = \sup \tilde{\mu}_N$, from (1.3), (1.6) and (2.1) we see that

$$\frac{k_N}{2} \le k_N^c \le \mathbf{L}_N \le k_{2N}^c + \tilde{g}_N \tilde{\mu} \le 1 + 2k_N + k_N \tilde{\mu} \le (3 + \tilde{\mu}) k_N.$$

Similarly, one obtains

$$g_N^c \;\le\; \widetilde{\mathbf{L}}_N \;\le\; g_N^c + \tilde{g}_N\tilde{\mu} \;\le\; (1+2\tilde{\mu})g_N^c,$$

and as before $\frac{g_N}{2} \le g_N^c \le g_N + 1 \le 2g_N$. □

4 Lower bounds: Proof of Proposition 1.1

The lower bounds in (1.3) are quite elementary, and most of them have appeared before in the literature. We sketch the proof of those we did not find explicitly in this generality.

4.1 $\mathbf{L}_N \ge k_N^c$

This can be found in [9, Proposition 3.3].

4.2 $\widetilde{\mathbf{L}}_N \ge \nu_N$

Let $|A| = |B| \le N$, $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \Upsilon$, and $x \in \mathbb{X}$ such that $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ and $|x|_\infty \le 1$. We must show that

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A} + x\| \;\le\; \widetilde{\mathbf{L}}_N\,\|\mathbf{1}_{\boldsymbol{\eta}B} + x\|, \tag{4.1}$$

For every $j \ge 1$ we can find a set C_j with $|C_j| = N - |A|$, disjoint with $A \cup B$, and such that $\max_{n\in C_j} |\mathbf{e}_n^*(x)| \le 1/j$. We set

$$y_j = \mathbf{1}_{\boldsymbol{\varepsilon}A} + \mathbf{1}_{\boldsymbol{\eta}B} + (I - P_{C_j})x + \mathbf{1}_{C_j},$$

and select $G_N \in \mathscr{G}_N$ such that $G_N(y_j) = \mathbf{1}_{\boldsymbol{\eta}B} + \mathbf{1}_{C_j}$. Then

$$\begin{aligned}\|\mathbf{1}_{\boldsymbol{\varepsilon}A} + (I - P_{C_j})x\| = \|y_j - G_N(y_j)\| \;&\le\; \widetilde{\mathbf{L}}_N\,\widetilde{\sigma}_N(y_j)\\ &\le \widetilde{\mathbf{L}}_N\,\|(I - P_{A\cup C_j})y_j\| = \widetilde{\mathbf{L}}_N\|\mathbf{1}_{\boldsymbol{\eta}B} + (I - P_{C_j})x\|.\end{aligned}$$

Since $\lim_{j\to\infty} P_{C_j}x = 0$ we obtain (4.1).

4.3 $\widetilde{\mathbf{L}}_N \ge g_N^c$

We must show that for every $x \in \mathbb{X}$ and every $\Gamma \in \mathscr{G}(x,k)$ with $k \le N$, we have

$$\|x - P_\Gamma x\| \le \widetilde{\mathbf{L}}_N\|x\|. \tag{4.2}$$

Let $\alpha = \min_{n\in\Gamma} |\mathbf{e}_n^*(x)|$. Notice that for every $j \ge 1$ we can find a set $C_j \subset \Gamma^c$, with $|C_j| = N - k$, and $\max_{n\in C_j} |\mathbf{e}_n^*(x)| \le \alpha/j$. Let

$$y_j = x - P_{C_j}x + \alpha\mathbf{1}_{C_j},$$

so that $\Gamma \cup C_j \in \mathscr{G}(y_j, N)$. Thus

$$\|y_j - P_{\Gamma\cup C_j} y_j\| \le \widetilde{\mathbf{L}}_N\, \widetilde{\sigma}_N(y_j) \;\le\; \widetilde{\mathbf{L}}_N\, \|y_j - P_{C_j} y_j\|,$$

which is the same as

$$\|x - P_\Gamma x - P_{C_j} x\| \;\le\; \widetilde{\mathbf{L}}_N\, \|x - P_{C_j} x\|.$$

Since $\lim_{j\to\infty} P_{C_j} x = 0$ (in $\mathbb{X}$) we obtain (4.2).

4.4 $\nu_N \ge \max\{\mu_N, \tilde{\mu}_N^d\}$

This was shown in (2.2) above.

4.5 $\nu_N \ge \frac{1}{2\kappa}\tilde{\mu}_N$

Given $|A| = |B| \le N$ and $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \Upsilon$, we must show that

$$\|\mathbf{1}_{\eta B}\| \;\le\; 2\kappa\, \nu_N\, \|\mathbf{1}_{\varepsilon A}\|.$$

It is enough to prove it for $\boldsymbol{\varepsilon} \equiv 1$ (otherwise, apply the result to $\mathscr{B} = \{\varepsilon_n \mathbf{e}_n\}$). Recall from (3.13) (and [6, Lemma 6.4]) that $\mathbf{1}_{\eta B} \in 2\kappa S$, where

$$S = \Big\{ \sum_{B'\subset B} \theta_{B'} \mathbf{1}_{B'} \;:\; \sum_{B'\subset B} |\theta_{B'}| \le 1 \Big\},$$

so it suffices to show that

$$\|\mathbf{1}_{B'}\| \;\le\; \nu_N\, \|\mathbf{1}_A\|, \quad \forall\, B' \subset B.$$

Now if we write

$$\mathbf{1}_{B'} = \mathbf{1}_{B'\backslash A} + \mathbf{1}_{B'\cap A} =: z + x,$$

and observe that $|B'\backslash A| \le |B\backslash A| = |A\backslash B| \le |A\backslash B'| \le N$, we can apply the convexity Lemma 2.8 to obtain

$$\|\mathbf{1}_{B'}\| = \|\mathbf{1}_{B'\backslash A} + \mathbf{1}_{B'\cap A}\| \le \nu_N\, \|\mathbf{1}_{A\backslash B'} + \mathbf{1}_{B'\cap A}\| = \nu_N \|\mathbf{1}_A\|.$$

□

Remark 4.1 We do not know whether $\nu_N \ge \tilde{\mu}_N$ (or even $\mathbf{L}_N \ge \tilde{\mu}_N$) may hold in general.

5 Examples

5.1 The summing basis

Let $\mathbb{X}$ be the (real) Banach space of all sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ with

$$\|\mathbf{a}\| := \sup_{M\geq 1} \Big|\sum_{n=1}^{M} a_n\Big| < \infty. \tag{5.1}$$

The standard canonical basis $\{\mathbf{e}_n, \mathbf{e}_n^*\}$ satisfies $\|\mathbf{e}_m\| \equiv 1$, $\|\mathbf{e}_1^*\| = 1$ and $\|\mathbf{e}_n^*\| = 2$ if $n \geq 2$ (so $\mathtt{K} = 2$, with the notation in Theorem 1.8). The terminology comes from the fact that $\mathbb{X}$ is isometrically isomorphic[2] to the span of the "summing system" $\{\mathbf{s}_n := \sum_{k\geq n} \mathbf{e}_k\}_{n=1}^{\infty}$ in ℓ^∞; see [15, p. 20].

Proposition 5.1 *For this example we have*

- $\mu_N = 1$ *and* $\tilde{\mu}_N = N$
- $g_N = \tilde{g}_N = k_N = 2N$ *and* $g_N^c = k_N^c = 1 + 2N$
- $\nu_N = \widetilde{\mathbf{L}}_N = 1 + 4N$ *and* $\mathbf{L}_N = 1 + 6N$.

So, equalities hold everywhere in Theorem 1.8.

Proof It is clear that $\|\mathbf{1}_A\| = |A|$, so the basis is democratic and $\mu_N \equiv 1$. On the other hand, we trivially have

$$1 \leq \|\mathbf{1}_{\varepsilon A}\| \leq N, \quad \forall\, |A| = N,\ \varepsilon \in \Upsilon.$$

The upper bound is attained if $\varepsilon \equiv 1$, and the lower bound is attained in the explicit example $\|\sum_{n=1}^{N}(-1)^n \mathbf{e}_n\| = 1$. We conclude that $\tilde{\mu}_N = N$.

We know from (3.9) that $g_N \leq \tilde{g}_N \leq k_N \leq 2N$. To see the equality, pick the vector $\mathbf{a} = (-1, 2, -2, \ldots, 2, -2, 0, \ldots)$, which has $\|\mathbf{a}\| = 1$. Then $\Gamma = \{n : a_n = 2\} \in \mathscr{G}(\mathbf{a}, N)$ and

$$g_N \geq \|P_\Gamma \mathbf{a}\| = \|(0, 2, 0, \ldots, 2, 0, 0\ldots)\| = 2N.$$

Similarly, $g_N^c \leq k_N^c \leq 1 + 2N$ by (3.9), and setting $\Gamma' = \{n : a_n = -2\} \in \mathscr{G}(\mathbf{a}, N)$ we conclude

$$g_N^c \geq \|(I - P_{\Gamma'})\mathbf{a}\| = \|(1, 2, 0, \ldots, 2, 0, 0\ldots)\| = 1 + 2N.$$

Next we have $\nu_N \leq \widetilde{\mathbf{L}}_N \leq 1 + 4N$, by Proposition 1.1 and Theorem 1.8. For the lower bound we pick

$$x = \big(\overbrace{\tfrac{1}{2}, 0, \tfrac{1}{2}}\,;\ \ldots\ ;\ \overbrace{\tfrac{1}{2}, 0, \tfrac{1}{2}}\,;\ \tfrac{1}{2}, 0, 0, \ldots\big) \quad \text{and} \quad \mathbf{1}_B = \big(\overbrace{0, 1, 0}\,;\ \ldots\ ;\ \overbrace{0, 1, 0}\,;\ 0, \ldots\big)$$

[2] Via the map $\mathbf{a} \in \mathbb{X} \mapsto T\mathbf{a} = (\sum_{i=1}^{n} a_i)_{n\in\mathbb{N}} \in \ell^\infty$, since $T\mathbf{e}_n = \mathbf{s}_n$.

so that $\|x - \mathbf{1}_B\| = 1/2$, while $\|x + \mathbf{1}_A\| = \frac{1}{2} + 2N$ for any $|A| = N$. So,

$$\nu_N \geq \frac{\|x + \mathbf{1}_A\|}{\|x - \mathbf{1}_B\|} = 1 + 4N.$$

Finally, $\mathbf{L}_N \leq 1 + 6N$ by Theorem 1.8. To show equality, let

$$x = \big(\overbrace{\tfrac{1}{2}, 1, \tfrac{1}{2}}\,;\ \ldots\ ;\ \overbrace{\tfrac{1}{2}, 1, \tfrac{1}{2}}\,;\ \tfrac{1}{2};\ \overbrace{-1, 1},\ \ldots\ , \overbrace{-1, 1},\ 0, 0, \ldots\big),$$

and pick $\Gamma = \{n :\ x_n = -1\} \in \mathscr{G}(x, N)$. Then

$$\|x - P_\Gamma x\| = 3N + \tfrac{1}{2},$$

while

$$\sigma_N(x) \leq \big\|x - 2\big(\overbrace{0, 1, 0}\,;\ \ldots\ ;\ \overbrace{0, 1, 0}\,;\ 0, 0, \ldots\big)\big\| = \frac{1}{2}.$$

Thus, $\mathbf{L}_N \geq \|x - P_\Gamma x\| / \sigma_N(x) \geq 6N + 1.$ □

Remark 5.2 In this example one can also show that $\gamma_N = \lceil N/2 \rceil$. In particular, the factor 2κ in (3.12) cannot be removed (at least when $\mathbb{K} = \mathbb{R}$).

5.2 Canonical basis in $\ell^1 \oplus c_0$

Consider the space formed by pairs of sequences $(x, y) \in \ell^1 \times c_0$, endowed with the norm $\|(x, y)\| = \|x\|_1 + \|y\|_\infty$. Write the canonical basis as $\mathscr{B} = \{(\mathbf{e}_m, 0), (0, \mathbf{f}_n)\}_{m,n=1}^\infty$.

Proposition 5.3 *The canonical basis in $\ell^1 \oplus c_0$ satisfies*

- $\mu_N = \tilde{\mu}_N = N$
- $g_N = \tilde{g}_N = k_N = g_N^c = k_N^c = 1$
- $\nu_N = \widetilde{\mathbf{L}}_N = \mathbf{L}_N = 1 + \tilde{\mu}_N = 1 + N.$

So, equalities hold everywhere in Theorems 1.2 and 1.3.

Proof The second point is clear, since the canonical basis is 1-unconditional. For the first point just notice that

$$1 \leq \|\mathbf{1}_A\| = \|\mathbf{1}_{\varepsilon A}\| \leq |A|,$$

with the lower bound attained when $\mathbf{1}_A \in c_0$, and the upper bound when $\mathbf{1}_A \in \ell^1$. Finally, in view of Theorem 1.3 and the previous equalities, in the last point we only need to show that $\nu_N \geq N+1$. Let $\mathbf{1}_A = \sum_{n=1}^{N} \mathbf{e}_n$, $\mathbf{1}_B = \sum_{n=1}^{N} \mathbf{f}_n$, and $x = \mathbf{f}_{N+1}$, then

$$\nu_N \geq \frac{\|\mathbf{1}_A + x\|}{\|\mathbf{1}_B + x\|} = N+1.$$

□

5.3 Canonical basis in $\ell^1 \oplus \ell^q$, $1 \leq q < \infty$

This variant of the previous example also admits explicit Lebesgue constants, but equality fails in (1.6).

Proposition 5.4 *The canonical basis in $\ell^1 \oplus \ell^q$, $1 \leq q < \infty$ satisfies*

- $\mu_N = \tilde{\mu}_N = N^{1/q'}$
- $g_N = \tilde{g}_N = k_N = g_N^c = k_N^c = 1$
- $\nu_N = \widetilde{\mathbf{L}}_N = \mathbf{L}_N = (N+1)^{1/q'}$.

So equality holds in Theorem 1.2, but not in Theorem 1.3.

Proof We only prove the last part, the other two being easy. By Corollary 1.4, we only need to estimate ν_N. From below, we choose as before $\mathbf{1}_A = \sum_{n=1}^{N} \mathbf{e}_n$, $\mathbf{1}_B = \sum_{n=2}^{N+1} \mathbf{f}_n$, and $x = \mathbf{f}_1$, so that

$$\nu_N \geq \frac{\|\mathbf{1}_A + \mathbf{f}_1\|}{\|\mathbf{1}_B + \mathbf{f}_1\|} = \frac{N+1}{(N+1)^{\frac{1}{q}}} = (N+1)^{1/q'}.$$

From above, let $|A| = |B| = N$ and (x, y) have disjoint support with $A \cup B$. Then

$$\|(x, y) + \mathbf{1}_{\varepsilon A}\| \leq \|x\|_1 + \|y\|_q + N,$$

while if $k = |\operatorname{supp} P_{\ell^1}(\mathbf{1}_B)|$, then

$$\|(x, y) + \mathbf{1}_{\eta B}\| = \|x\|_1 + k + (\|y\|_q^q + N - k)^{\frac{1}{q}} \geq \|x\|_1 + (\|y\|_q^q + N)^{\frac{1}{q}}.$$

So,

$$\frac{\|(x, y) + \mathbf{1}_{\varepsilon A}\|}{\|(x, y) + \mathbf{1}_{\eta B}\|} \leq \frac{\|x\|_1 + \|y\|_q + N}{\|x\|_1 + (\|y\|_q^q + N)^{\frac{1}{q}}} \leq \frac{\|y\|_q + N}{(\|y\|_q^q + N)^{\frac{1}{q}}},$$

and the latter is easily seen to be maximized at $\|y\|_q = 1$. So $\nu_N \leq (1+N)^{\frac{1}{q'}}$, as asserted. □

Remark 5.5 With similar (but slightly more tedious) computations one can show that, for $\ell^p \oplus c_0$, $1 < p < \infty$, one has

$$\nu_N = \widetilde{\mathbf{L}}_N = \mathbf{L}_N = 1 + N^{\frac{1}{p}},$$

while $\tilde{\mu}_N = \mu_N = 1 + (N-1)^{\frac{1}{p}}$, so again equality fails in (1.6).

5.4 The trigonometric system

Consider $\mathscr{B} = \{e^{inx}\}_{n\in\mathbb{Z}}$ in $L^p(\mathbb{T})$, $1 \le p \le \infty$. In this case, neither (1.5) nor (1.6) give good estimates, even asymptotically. By a more direct approach, Temlyakov [19] showed the following

$$c_p N^{\left|\frac{1}{p}-\frac{1}{2}\right|} \le \mathbf{L}_N \le 1 + 3N^{\left|\frac{1}{p}-\frac{1}{2}\right|},$$

for some $c_p > 0$. More precisely, the following inequalities hold (if $p > 1$)

$$c_p N^{\left|\frac{1}{p}-\frac{1}{2}\right|} \le \gamma_N \le g_N^c \le k_N^c \le 1 + N^{\left|\frac{1}{p}-\frac{1}{2}\right|}, \tag{5.2}$$

and

$$c_p N^{\left|\frac{1}{p}-\frac{1}{2}\right|} \le \mu_N \le \tilde{\mu}_N = \tilde{\mu}_N^d \le \nu_N \le \widetilde{\mathbf{L}}_N \le \mathbf{L}_N \le 1 + 3N^{\left|\frac{1}{p}-\frac{1}{2}\right|}. \tag{5.3}$$

So all the involved constants have the same order of magnitude $N^{\left|\frac{1}{p}-\frac{1}{2}\right|}$. For the upper bounds in (5.2) and (5.3), see [19, Lemma 2.1 and Theorem 2.1]. The lower bounds are implicit in [19, Remark 2]; for instance if $1 < p \le 2$ and $N \in 2\mathbb{N}$ then

$$\mu_{N+1} \ge \frac{\|\mathbf{1}_{\{1,2,\dots,2^N\}}\|_p}{\|\mathbf{1}_{\{-N/2,\dots,N/2\}}\|_p} \ge c_p \frac{\sqrt{N}}{N^{1-\frac{1}{p}}} = c_p N^{\frac{1}{p}-\frac{1}{2}}, \tag{5.4}$$

since the Dirichlet kernel has norm $\|D_{N/2}\|_p \approx N^{1-\frac{1}{p}}$. Likewise, by (3.12)

$$\gamma_{N+1} \ge \frac{1}{4} \frac{\|\mathbf{1}_{\boldsymbol{\varepsilon}\{-N/2,\dots,N/2\}}\|_p}{\|\mathbf{1}_{\{-N/2,\dots,N/2\}}\|_p} \ge c'_p \frac{\sqrt{N}}{N^{1-\frac{1}{p}}} = c'_p N^{\frac{1}{p}-\frac{1}{2}}, \tag{5.5}$$

choosing in $\boldsymbol{\varepsilon}$ the signs of the corresponding Rudin–Shapiro polynomial. The case $p \ge 2$ is similar, replacing the roles of numerator and denominator.

We state separately the case $p = 1$, for which not all constants have the same order of magnitude.

Proposition 5.6 *The trigonometric system $\mathscr{B} = \{e^{inx}\}_{n\in\mathbb{Z}}$ in $L^1(\mathbb{T})$ satisfies*

- $\mathbf{L}_N \approx \widetilde{\mathbf{L}}_N \approx k_N \approx g_N \approx \sqrt{N}$.

- $\gamma_N \approx \mu_N \approx \tilde{\mu}_N \approx \frac{\sqrt{N}}{\log N}$.
- $\nu_N \approx \sqrt{N}$

Proof For the first point, the arguments in [19] are valid when $p = 1$, so we do not write them here. In the second point, the lower bound for each of the constants follows as in (5.4) and (5.5), using $\|D_{N/2}\|_1 \approx \log N$. The upper bound relies on $\|\mathbf{1}_{\eta B}\|_1 \leq \|\mathbf{1}_{\eta B}\|_2 = |B|^{\frac{1}{2}}$, and on the deeper result $\inf_{\boldsymbol{\varepsilon},|A|=N} \|\mathbf{1}_{\boldsymbol{\varepsilon} A}\|_1 \geq c \log N$, a famous problem posed by Littlewood and solved by Konyagin [13] and McGeehee–Pigno–Smith [16].

We now establish the third point. Since $\nu_N \leq \mathbf{L}_N \lesssim \sqrt{N}$, we only need to show the lower bound. For $N \in \mathbb{N}$ we pick $B = \{-N, \dots, N\}$ and an element $x \in L^1(\mathbb{T})$ so that

$$\mathbf{1}_{\{-N,\dots,N\}} + x = V_N,$$

where V_N denotes the de la Vallée-Poussin kernel (as in [18, p. 114]). Then $|x|_\infty \leq 1$, $\operatorname{supp} x \subset \{N < |k| < 2N\}$ and we have

$$\|\mathbf{1}_B + x\|_1 = \|V_N\|_1 \leq 3.$$

Next we pick $A = \{2^j : j_0 \leq j \leq j_0 + 2N\}$ where we choose $2^{j_0} \geq 4N$. We also notice that the operator $\mathscr{V}_{2N} : f \mapsto V_{2N} * f$, allows us to write $(I - \mathscr{V}_{2N})(\mathbf{1}_A + x) = \mathbf{1}_A$. Since the operator norm $\|I - \mathscr{V}_{2N}\| \leq 1 + \|V_{2N}\|_1 \leq 4$, we obtain

$$c_1\sqrt{N} \leq \|\mathbf{1}_A\|_1 \leq \|I - \mathscr{V}_{2N}\| \, \|\mathbf{1}_A + x\|_1 \leq 4 \, \|\mathbf{1}_A + x\|_1.$$

Overall we conclude that

$$\nu_{2N+1} \geq \frac{\|\mathbf{1}_A + x\|_1}{\|\mathbf{1}_B + x\|_1} \geq \tfrac{c_1}{12} \sqrt{N}.$$

□

5.5 A superdemocratic and not quasi-greedy basis

Theorem 1.3 becomes asymptotically optimal when $\tilde{\mu}_N \approx 1$, as in this case $\mathbf{L}_N \approx k_N$ and $\widetilde{\mathbf{L}}_N \approx g_N$. We give a non-trivial example of this situation, which is a small variation of [3, Example 4.8]. This example has the additional interesting property of being *unconditional with constant coefficients* but not quasi-greedy.

Proposition 5.7 *For every $1 \leq q \leq \infty$, there exists $(\mathbb{X}, \mathscr{B})$ such that*

- $\nu_N \approx \tilde{\mu}_N \approx \gamma_N \approx 1$
- $g_N \approx \tilde{g}_N \approx k_N \approx (\log N)^{1/q'}$
- $\mathbf{L}_N \approx \widetilde{\mathbf{L}}_N \approx (\log N)^{1/q'}$

So, in this case Theorems 1.2, 1.3 and Remark 3.6 are asymptotically optimal.

Proof Let $\mathcal{D}_k$ denote the set of all dyadic intervals $I \subset [0,1]$ with length $|I| = 2^{-k}$, and $\mathcal{D} = \cup_{k\ge 0}\mathcal{D}_k$. Consider the space $\mathfrak{f}_1^q$ of all (real) sequences $\mathbf{a} = (a_I)_{I\in\mathcal{D}}$ such that

$$\|\mathbf{a}\|_{\mathfrak{f}_1^q} = \Big\| \Big[\sum_I |a_I \chi_I^{(1)}|^q \Big]^{\frac{1}{q}} \Big\|_{L^1} < \infty,$$

where $\chi_I^{(1)} = |I|^{-1}\chi_I$. It is well known that $\{\mathbf{e}_I\}_{I\in\mathcal{D}}$, the canonical basis, is unconditional and democratic in $\mathfrak{f}_1^q$; see e.g. [8,12]. In particular, for some $c_q \ge 1$ we have

$$\tfrac{1}{c_q}|A| \le \|\mathbf{1}_{\boldsymbol{\varepsilon}A}\|_{\mathfrak{f}_1^q} \le |A|, \quad \forall\, A \subset \mathcal{D}, \quad \boldsymbol{\varepsilon} \in \Upsilon.$$

From the definition we also have

$$\Big\| \sum_k b_k 2^{-k} \mathbf{1}_{\mathcal{D}_k} \Big\|_{\mathfrak{f}_1^q} = \Big(\sum_k |b_k|^q \Big)^{\frac{1}{q}},$$

since $2^{-k}\sum_{I\in\mathcal{D}_k}\chi_I^{(1)} = \chi_{[0,1]}$. For every $N \ge 1$ we shall pick a subset $\{k_1, \ldots k_N\} \subset \mathbb{N}_0$, and look at the finite dimensional space F_N consisting of sequences supported in $\cup_{j=1}^N \mathcal{D}_{k_j}$. We order the canonical basis by $\cup_{j=1}^N \{\mathbf{e}_I\}_{I\in\mathcal{D}_{k_j}}$, so we may as well write their elements as $\mathbf{a} = (a_j)_{j=1}^{\dim F_N}$. We also consider in F_N the James norm

$$\|(a_j)\|_{J_q} = \sup_{m_0=0<m_1<\ldots} \Big[\sum_{k\ge 0} \Big| \sum_{m_k<j\le m_{k+1}} a_j \Big|^q \Big]^{\frac{1}{q}}.$$

Note that $\|\mathbf{a}\|_{J_q} \le \|\mathbf{a}\|_{\ell^1}$, with equality iff all the a_j's have the same sign.[3] In particular,

$$\|\mathbf{1}_A\|_{J_q} = |A|.$$

Now set in F_N a new norm

$$|||\mathbf{a}||| = \max\Big\{ \|\mathbf{a}\|_{\mathfrak{f}_1^q},\ \|\mathbf{a}\|_{J_q} \Big\},$$

and observe that $1/c_q|A| \le |||\mathbf{1}_{\boldsymbol{\varepsilon}A}||| \le |A|$, with c_q independent of N and k_j. Also, the vector $x = \sum_{j=1}^N (-1)^{j+1} 2^{-k_j} \mathbf{1}_{\mathcal{D}_{k_j}}$ has

$$\|x\|_{\mathfrak{f}_1^q} = \|x\|_{J_q} = |||x||| = N^{\frac{1}{q}}.$$

[3] Note that $|a-b| < (a^q + b^q)^{\frac{1}{q}}$ if $a, b > 0$, so consecutive elements with different signs should be in different blocks of the James norm.

At this point we write $N = 2n$ and choose our k_j's as

$$k_{2j+1} = j \quad \text{and} \quad k_{2j+2} = n + j, \quad j = 0, \ldots, n-1.$$

Then if $P = \sum_{j \text{ odd}} 2^{k_j} = 2^n - 1$ we have $G_P x = \sum_{j \text{ odd}} 2^{-k_j} \mathbf{1}_{\mathcal{D}_{k_j}}$, which implies

$$\|G_P x\|_{\mathfrak{f}_1^q} = n^{\frac{1}{q}}, \quad \|G_P x\|_{J_q} = n, \quad \text{and} \quad \|\!|G_P x|\!\| = n.$$

Therefore

$$g_{2^n} \geq \|\!|G_P x|\!\| / \|\!|x|\!\| \geq n^{1-\frac{1}{q}}.$$

We turn to estimate the unconditionality constant k_m of the space F_N. Given $|A| = m$, we first claim that

$$\|P_A x\|_{\ell^1} \leq c'_q \, (\log |A|)^{1/q'} \, \|x\|_{\mathfrak{f}_1^q}. \tag{5.6}$$

This is clear when $q = 1$ (since $\mathfrak{f}_1^1 = \ell^1$). When $q = \infty$, it is a consequence e.g. of [8, Remark 5.6] (since $\mathfrak{f}_1^\infty$ is a 1-space, in the terminology of [8, (2.8)]). Thus one derives (5.6) by complex interpolation. From here

$$\|\!|P_A x|\!\| \leq \|P_A x\|_{\ell^1} \leq c'_q \, (\log |A|)^{1/q'} \, \|\!|x|\!\|,$$

which implies the bound $k_m \leq c'_q (\log m)^{1/q'}$.

Finally, we consider the space $\mathbb{X} = \oplus_{\ell^1} F_N$ with $\mathscr{B}$ the consecutive union of the natural bases in F_N. Then

$$\frac{1}{c_q} |A| \leq \|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A}|\!\| = \sum_N \|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A_N}|\!\| \leq |A|,$$

so $\mathscr{B}$ is superdemocratic. We claim further that $\nu_N = O(1)$. Let $|A| = |B| = N$ and $x \in \mathbb{X}$ have disjoint support with $A \cup B$. Assuming first that $\|\!|x|\!\| \geq 2N$, we have

$$\frac{\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A} + x|\!\|}{\|\!|\mathbf{1}_{\boldsymbol{\eta} B} + x|\!\|} \leq \frac{\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A}|\!\| + \|\!|x|\!\|}{\|\!|x|\!\| - \|\!|\mathbf{1}_{\boldsymbol{\eta} B}|\!\|} \leq \frac{3/2 \|\!|x|\!\|}{1/2 \|\!|x|\!\|} = 3,$$

since $\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A}|\!\|, \|\!|\mathbf{1}_{\boldsymbol{\eta} B}|\!\| \leq N \leq \|\!|x|\!\|/2$. Otherwise we have $\|\!|x|\!\| < 2N$, which implies

$$\frac{\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A} + x|\!\|}{\|\!|\mathbf{1}_{\boldsymbol{\eta} B} + x|\!\|} \leq \frac{\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A}|\!\| + \|\!|x|\!\|}{\sum_N \|\mathbf{1}_{\boldsymbol{\eta} B_N} + x_N\|_{\mathfrak{f}_1^q}} \leq \frac{3N}{\sum_N \|\mathbf{1}_{\boldsymbol{\eta} B_N}\|_{\mathfrak{f}_1^q}} \leq 3c_q,$$

since $\sum_N \|\mathbf{1}_{\boldsymbol{\eta} B_N}\|_{\mathfrak{f}_1^q} \geq c_q \sum_N |B_N| = N$. Thus $\nu_N \lesssim 1$ as asserted. A similar argument shows that

$$\gamma_N \leq \frac{\|\!|\mathbf{1}_{\boldsymbol{\varepsilon} A}|\!\|}{\|\!|\mathbf{1}_{\boldsymbol{\eta} B}|\!\|} \leq \frac{N}{\sum_N \|\mathbf{1}_{\boldsymbol{\eta} B_N}\|_{\mathfrak{f}_1^q}} \leq c_q.$$

Finally, observe that $k_m^{\mathbb{X}} \le \max_N k_m^{F_N} \le c'_q (\log m)^{1/q'}$, while if $N = 2n$ we have

$$g_{2^n}^{\mathbb{X}} \ge g_{2^n}^{F_N} \ge n^{1/q'}.$$

This completes the proof of Proposition 5.7. □

Remark 5.8 The above construction enjoys the following remarkable property: $\exists\, c_1, c_2 > 0$ such that

$$c_1 \min_{n\in A} |a_n| \, |||\mathbf{1}_A||| \le |||\sum_A a_n \mathbf{e}_n||| \le c_2 \max_{n\in A} |a_n| \, |||\mathbf{1}_A|||, \tag{5.7}$$

for all finite sets A and all scalars a_n. Indeed, the right hand side is a consequence of $\gamma_N \approx 1$, (3.12) and convexity (as in Sect. 2.3). The left hand inequality is true for the norm $\|\cdot\|_{\mathfrak{f}_1^q}$, and since $\|\mathbf{1}_A\|_{\mathfrak{f}_1^q} \approx |A| \approx |||\mathbf{1}_A|||$, it will also hold for the norm $|||\cdot|||$. The fact that a non quasi-greedy basis may satisfy (5.7) seems to have been unnoticed before.

6 Further questions

As shown in Example 5.4, the multiplicative bounds in Theorems 1.2 and 1.3 are not so good when both g_N and $\tilde{\mu}_N$ go to infinity.

Q1: *Find bounds for $\mathbf{L}_N$ and $\widetilde{\mathbf{L}}_N$ which depend **additively** on k_N, $\tilde{\mu}_N$ or ν_N. More precisely, determine in what cases it can be true that*

$$\mathbf{L}_N \lesssim k_N + \nu_N \quad \text{or} \quad \mathbf{L}_N \lesssim k_N + \tilde{\mu}_N.$$

This is for instance the case for the trigonometric system, and the other examples in Sect. 5. In this respect, we can mention the results of Oswald [17], who obtains additive estimates of the form $\mathbf{L}_N \approx k_N + B_N$, but with constants B_N of a more complicated nature.

Related to the previous one can ask

Q2: *Find examples such that k_N and ν_N grow independently to infinity.*
Example 5.5 shows that one can have $\nu_N \approx 1$ and $\mathbf{L}_N \approx k_N \to \infty$. We do not know whether it is possible to have $\nu_N \approx N^\alpha$ and $k_N \approx N^\beta$ for arbitrary $0 < \alpha, \beta \le 1$. A similar question, posed as Problem 4.4 in [1], asks whether it could be possible to have $\nu_N \equiv 1$ and $k_N \to \infty$.

The new constant γ_N in (3.11) is a natural replacement for g_N in some situations. Example 5.5 (and also Example 5.4 with $p = 1$) show that this improvement may be strict and the ratio g_N/γ_N as large as $\log N$.

Q3: *Find examples with $\gamma_N \approx 1$ and g_N as large as possible.*

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