RIESZ TRANSFORM CHARACTERIZATION OF HARDY SPACES ASSOCIATED WITH CERTAIN LAGUERRE EXPANSIONS

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Abstract

In this paper we prove Riesz transform characterizations for Hardy spaces associated with certain systems of Laguerre functions.

1 Introduction and statement of the results

Denote the Laguerre polynomials of order $\alpha > -1$ by

$$L_n^\alpha(x) = (n!)^{-1} e^x x^{-\alpha} \Big(\frac{d}{dx}\Big)^n (e^{-x} x^{n+\alpha}),\ n = 0, 1, 2, ...$$

In this paper we consider the following two systems of Laguerre functions on $(0, \infty)$

(1) $$\varphi_n^\alpha(x) \ = \ \sqrt{2}\, c_{n,\alpha}\, e^{-x^2/2} x^{\alpha+1/2} L_n^\alpha(x^2), \quad n = 0, 1, 2 \dots,$$

(2) $$\mathfrak{L}_n^\alpha(x) \ = \ c_{n,\alpha}\, e^{-x/2} x^{\alpha/2} L_n^\alpha(x), \quad n = 0, 1, 2, \dots ,$$

where $c_{n,\alpha} = (\Gamma(n+1)/\Gamma(n+1+\alpha))^{1/2}$. It is well known that, for every $\alpha > -1$, each of the systems $\{\varphi_n^\alpha\}_{n=0}^\infty$ and $\{\mathfrak{L}_n^\alpha\}_{n=0}^\infty$ is complete and orthonormal on $L^2((0, \infty), dx)$. Moreover, these functions are eigenvectors, respectively, of the differential operators

$$L_\alpha = \frac{1}{2}\Big(- \frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\Big(\alpha^2 - \frac{1}{4}\Big)\Big), \ \ \mathfrak{L}_\alpha = -\Big(x\frac{d^2}{dx^2} + \frac{d}{dx} - \Big(\frac{x}{4} + \frac{\alpha^2}{4x}\Big)\Big),$$

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satisfying

$$L_\alpha \varphi_n^\alpha = (2n+\alpha+1)\varphi_n^\alpha \quad \text{and} \quad \mathfrak{L}_\alpha(\mathfrak{L}_n^\alpha) = (n+(\alpha+1)/2)\,\mathfrak{L}_n^\alpha\,.$$

As in [6, 7], the operators L_α and $\mathfrak{L}_\alpha$ can be factored as

$$L_\alpha \;=\; \frac{1}{2}\,D_\alpha^* D_\alpha + \alpha + 1 \quad \text{and} \quad \mathfrak{L}_\alpha = \delta_\alpha^*\delta_\alpha + \frac{\alpha+1}{2},$$

where

$$D_\alpha = \frac{d}{dx} + x - \frac{\alpha+1/2}{x} \quad \text{and} \quad \delta_\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}\Big(\sqrt{x} - \frac{\alpha}{\sqrt{x}}\Big)\,,$$

and where D_α^* and δ_α^* denote, respectively, the formal adjoint operators to D_α and δ_α in $L^2((0,\infty),dx)$. Corresponding *Riesz transforms* are defined in $L^2((0,\infty),dx)$ by

$$R_\alpha = D_\alpha L_\alpha^{-1/2} \quad \text{and} \quad \mathfrak{R}_\alpha = \delta_\alpha \mathfrak{L}_\alpha^{-1/2}\,,$$

that is, they act on the basis elements by

$$(3) \qquad R_\alpha\varphi_n^\alpha = -\frac{2\sqrt{n}}{\sqrt{2n+\alpha+1}}\,\varphi_{n-1}^{\alpha+1}, \quad \mathfrak{R}_\alpha\mathfrak{L}_n^\alpha = -\frac{\sqrt{n}}{\sqrt{n+(\alpha+1)/2}}\,\mathfrak{L}_{n-1}^{\alpha+1}\,.$$

There exist kernels $R_\alpha(x,y)$ and $\mathfrak{R}_\alpha(x,y)$ such that

$$R_\alpha f(x) = \lim_{\varepsilon\to 0}\int_{0,\,|x-y|>\varepsilon}^\infty R_\alpha(x,y)f(y)\,dy, \quad \mathfrak{R}_\alpha f(x) = \lim_{\varepsilon\to 0}\int_{0,\,|x-y|>\varepsilon}^\infty \mathfrak{R}_\alpha(x,y)f(y)\,dy.$$

One can easily deduce from (1), (2) and (3) that these kernels are related by

$$(4) \qquad \mathfrak{R}_\alpha(x,y) = 2^{-3/2}(xy)^{-1/4}R_\alpha(\sqrt{x},\sqrt{y}), \quad x,y\in(0,\infty).$$

Riesz tranforms for Laguerre systems were defined and studied by Nowak and Stempak [7], and by Harboure, Torrea and Viviani [6], who proved that R_α for $\alpha\ge -1/2$ and $\mathfrak{R}_\alpha$ for $\alpha \ge 0$ extend as bounded linear operators on $L^p(0,\infty)$ when $1<p<\infty$ and are of weak type (1,1). Our goal in the present paper is to characterize the spaces

$$H^1_{\text{Riesz}}(L_\alpha) = \big\{f\in L^1(0,\infty)\;;\;\|R_\alpha f\|_{L^1}<\infty\big\} \quad \text{for} \quad \alpha > -1/2,$$

and

$$H^1_{\text{Riesz}}(\mathfrak{L}_\alpha) = \;\big\{f\in L^1(0,\infty)\;;\;\|\mathfrak{R}_\alpha f\|_{L^1}<\infty\big\} \quad \text{for} \quad \alpha > 0.$$

In [3], the second-named author considered Hardy spaces $H^1_{\max}(L_\alpha)$ and $H^1_{\max}(\mathfrak{L}_\alpha)$ defined by means of the maximal functions associated with the semigroups generated by $-L_\alpha$ and $-\mathfrak{L}_\alpha$, respectively. To be more precise, if

$$W_t^\alpha(x,y) = \sum_{n=0}^\infty e^{-(2n+\alpha+1)t}\varphi_n^\alpha(x)\varphi_n^\alpha(y), \quad \mathfrak{W}_t^\alpha(x,y) = \sum_{n=0}^\infty e^{-t(n+(\alpha+1)/2)}\mathfrak{L}_n^\alpha(x)\mathfrak{L}_n^\alpha(y)$$

denote the integral kernels of the semigroups $\{e^{-tL_\alpha}\}_{t>0}$ and $\{e^{-t\mathfrak{L}_\alpha}\}_{t>0}$, we say that a function f in $(0,\infty)$ belongs to $H^1_{\max}(L_\alpha)$ when the maximal function

$$W^\alpha_* f(x) = \sup_{t>0} \Big| \int_0^\infty W^\alpha_t(x,y) f(y)dy \Big|$$

belongs to $L^1(0,\infty)$. Then we set $\|f\|_{H^1_{\max}(L_\alpha)} = \|W^\alpha_* f\|_{L^1}$. Analogously, we define the maximal function $\mathfrak{W}^\alpha_*$, the space $H^1_{\max}(\mathfrak{L}_\alpha)$ and the norm $\|\cdot\|_{H^1_{\max}(\mathfrak{L}_\alpha)}$. It was proved in [3] that the spaces $H^1_{\max}(L_\alpha)$, $\alpha > -1/2$, and $H^1_{\max}(\mathfrak{L}_\alpha)$, $\alpha > 0$, admit atomic decompositions. The notion of atom for these spaces depends on the following auxiliary functions

$$\rho_{L_\alpha}(x) = \frac{1}{8}\min(x, 1/x) \quad \text{and} \quad \rho_{\mathfrak{L}_\alpha}(x) = \frac{1}{8}\min(x,1).$$

A measurable function $b : (0,\infty) \to \mathbb{C}$ is said to be an $H^1(L_\alpha)$-atom if there exists a ball $B = B(y_0, R) = \{|y_0 - y| < R\}$ with $R \le \rho_{L_\alpha}(y_0)$ such that

$$\operatorname{supp} b \subset B, \qquad \|b\|_\infty \le |B|^{-1} \quad \text{and}$$
$$\text{if } R \le \rho_{L_\alpha}(y_0)/2 \quad \text{then} \quad \int b(y)dy = 0.$$

The space $H^1_{\mathrm{at}}(L_\alpha)$ consists of all measurable functions f on $(0,\infty)$ of the form

$$f = \sum_{j=1}^\infty \lambda_j b_j,$$

where b_j are $H^1(L_\alpha)$-atoms, $\lambda_j \in \mathbb{C}$ and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm in $H^1_{\mathrm{at}}(L_\alpha)$ is defined by

$$\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} = \inf \sum_{j=1}^\infty |\lambda_j|,$$

where the infimum is taken over all decompositions $f = \sum_{j=1}^\infty \lambda_j b_j$, where b_j are $H^1(L_\alpha)$-atoms and $\lambda_j \in \mathbb{C}$. Similarly we define the space $H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)$ and the norm $\|\ \|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)}$, the only difference being that the function $\rho_{\mathfrak{L}_\alpha}$ replaces the function ρ_{L_α} in the definition of $H^1(\mathfrak{L}_\alpha)$-atoms. The main result in [3] was to show that

$$H^1_{\max}(L_\alpha) = H^1_{\mathrm{at}}(L_\alpha) \text{ for } \alpha > -1/2 \quad \text{and} \quad H^1_{\max}(\mathfrak{L}_\alpha) = H^1_{\mathrm{at}}(\mathfrak{L}_\alpha) \text{ for } \alpha > 0,$$

with equivalence of the corresponding norms. Our goal in this paper is to characterize these spaces by means of the Riesz transforms R_α and $\mathfrak{R}_\alpha$. More precisely, we shall prove the following theorems.

THEOREM 1.1. *If $\alpha > -1/2$, then $H^1_{\mathrm{Riesz}}(L_\alpha) = H^1_{\mathrm{at}}(L_\alpha)$. Moreover, there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} \le \|R_\alpha f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\mathrm{at}}(L_\alpha)}. \tag{5}$$

THEOREM 1.2. *If $\alpha > 0$, then $H^1_{\mathrm{Riesz}}(\mathfrak{L}_\alpha) = H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)$. Moreover, there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)} \le \|\mathfrak{R}_\alpha f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)}. \tag{6}$$

2 Hardy spaces $H^1(L_\alpha)$ associated with Laguerre operators L_α

In the present section, we shall prove Theorem 1.1. To do this, we recall the equivalence between Riesz and atomic definitions for the Hardy space associated with the Hermite operator,

$$H = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2),$$

which were established in [4]. First we let

$$\rho_H(y) = (1 + |y|)^{-1}. \tag{7}$$

It is easily seen that there exist constants $C, c > 0$ such that

$$c\rho_H(x)\big(1 + |x-y|/\rho_H(x)\big)^{-1} \le \rho_H(y) \le C\rho_H(x)\big(1 + |x-y|/\rho_H(x)\big)^{1/2}. \tag{8}$$

A function $a : \mathbb{R} \to \mathbb{C}$ is an $H^1(H)$-atom if there exists a ball $B = B(y_0, R) = \{y \in \mathbb{R}; |y - y_0| < R\}$ with $R \le \rho_H(y_0)$ such that

$$\operatorname{supp} a \subset B, \qquad \|a\|_{L^\infty} \le |B|^{-1} \quad \text{and}$$

$$\text{if } R \le \rho_H(y_0)/2 \quad \text{then} \quad \int a(y)\,dy = 0.$$

The atomic Hardy space $H^1_{\mathrm{at}}(H)$ and the norm $\|\ \|_{H^1_{\mathrm{at}}(H)}$ are defined in the standard way. On the other hand, a Riesz transform R^H can be defined in $L^2(\mathbb{R})$ by

$$R^H = \Big(\frac{d}{dx} + x\Big)H^{-1/2},$$

motivated by the factorization of the Hermite operator

$$H = -\frac{1}{4}\Big[\Big(\frac{d}{dx} + x\Big)\Big(\frac{d}{dx} - x\Big) + \Big(\frac{d}{dx} - x\Big)\Big(\frac{d}{dx} + x\Big)\Big].$$

To obtain a kernel expression for R^H, recall first the Mehler formula for Hermite functions (cf. [10, Lemma 1.1.1]), which asserts that the integral kernel $W^H_t(x, y)$ of the Hermite semigroup $\{e^{-tH}\}_{t>0}$ is given by

$$W^H_t(x, y) \;=\; \Big[\frac{e^{-t}}{\pi(1 - e^{-2t})}\Big]^{1/2} \exp\Big(-\frac{1}{2}\Big(\frac{1 + e^{-2t}}{1 - e^{-2t}}\Big)(x^2 + y^2) + 2xy\frac{e^{-t}}{1 - e^{-2t}}\Big) \tag{9}$$

when $t > 0$ and $x, y \in \mathbb{R}$. Using the formula $H^{-1/2} = \pi^{-1/2} \int_0^\infty e^{-tH} t^{-1/2} dt$, we can express the Riesz transform R^H as a principal value singular integral operator of the form

$$R^H(f)(x) = \lim_{\varepsilon \to 0} \int_{y \in \mathbb{R}\,:\, |x-y| > \varepsilon} R^H(x,y) f(y) dy\,,$$

with the kernel given by

$$\begin{aligned} R^H(x,y) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \Big(\frac{d}{dx} + x\Big) W_t^H(x,y) \frac{dt}{\sqrt{t}} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d}{dx} W_t^H(x,y) \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_0^\infty x W_t^H(x,y) \frac{dt}{\sqrt{t}} \\ &= R_1^H(x,y) + R_2^H(x,y). \end{aligned} \tag{10}$$

It is not difficult to prove using (9) and (10) that

$$\sup_{y \in \mathbb{R}} \int_{-\infty}^\infty |R_2^H(x,y)|\, dx < \infty, \quad \sup_{x \in \mathbb{R}} \int_{-\infty}^\infty |R_2^H(x,y)|\, dy < \infty \tag{11}$$

(see Section 4). Therefore, denoting $R_2^H = xH^{-1/2}$, we have

$$\|R_2^H f\|_{L^1(\mathbb{R})} \leq C \|f\|_{L^1(\mathbb{R})} \tag{12}$$

(see also [2, Theorem 4.5]). It was proved by Thangavelu [9] that the operator R^H is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover, Theorem 1.2 of Zhong [11] asserts that the operator $R_1^H = (d/dx)H^{-1/2}$ is a Calderón-Zygmund operator, hence it is of weak type (1,1) (see also [8] for a proof based on analysis of the Melher kernel). The above facts could also be deduced from the following lemma.

Lemma 2.1. *Let $\psi \in C_c^\infty(-2^{-4}, 2^{-4})$ be such that $\psi(x) = 1$ for $|x| < 2^{-5}$. Then there exists a constant $c_0 \neq 0$ and a kernel $h(x,y)$ such that*

$$R^H(x,y) = \frac{c_0}{x-y} \psi\Big(\frac{x-y}{\rho_H(x)}\Big) + h(x,y), \tag{13}$$

$$\sup_{y \in \mathbb{R}} \int_{-\infty}^\infty |h(x,y)| dx + \sup_{x \in \mathbb{R}} \int_{-\infty}^\infty |h(x,y)| dy < \infty. \tag{14}$$

This lemma is known, but a self-contained proof based on analysis of the Mehler kernel will be presented in Section 4. We set

$$H^1_{\mathrm{Riesz}}(H) = \big\{ f \in L^1(\mathbb{R})\,;\, \|R^H f\|_{L^1(\mathbb{R})} < \infty \big\}.$$

In view of (12), an L^1-function f belongs to $H^1_{\mathrm{Riesz}}(H)$ if and only if $(d/dx)H^{-1/2} f$ belongs to $L^1(\mathbb{R})$. From this remark and the results in [4], it follows that

$$H^1_{\mathrm{Riesz}}(H) = H^1_{\mathrm{at}}(H)$$

and there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{H^1_{\rm at}(H)} \le \|R^H f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\rm at}(H)}. \tag{15}$$

Having established the Riesz and atomic characterizations of the Hardy space associated with the Hermite operator, we continue our preparation for the proof of Theorem 1.1.

For a function f defined on $(0,\infty)$, we denote $R^H_{\rm loc} f = R^H_{1,{\rm loc}} f + R^H_{2,{\rm loc}} f$, where

$$R^H_{j,{\rm loc}} f(x) = \lim_{\varepsilon\to 0} \int_{x/2,\ |x-y|>\varepsilon}^{2x} R^H_j(x,y) f(y)\, dy, \quad x>0,\ j=1,2.$$

PROPOSITION 2.2. *For $f \in L^1(0,\infty)$, let f_o denote its odd extension. Then $R^H_1 f_o \in L^1(\mathbb{R})$ if and only if $R^H_{1,{\rm loc}} f$ is in $L^1(0,\infty)$. Moreover, there exists $C > 0$ such that*

$$\|R^H_1 f_o - R^H_{1,{\rm loc}} f\|_{L^1(0,\infty)} \le C\|f\|_{L^1(0,\infty)}.$$

PROOF. Set $r = r(t) = e^{-t} \in (0,1)$. According to (9) and (10), we have

$$R^H_1(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{r}\,\big(2ry - (1+r^2)x\big)}{(1-r^2)^{3/2}} \exp\Big(- \frac{1+r^2}{2(1-r^2)}(x^2+y^2) + \frac{2r}{1-r^2}xy \Big) \frac{dt}{\sqrt{t}}. \tag{16}$$

Note that $\|R^H_1 f_o\|_{L^1(\mathbb{R})} = 2\|R^H_1 f_o\|_{L^1(0,\infty)}$, because $R^H_1 f_o$ is an even function. Moreover,

$$R^H_1 f_o(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^\infty \big(R^H_1(x,y) - R^H_1(x,-y)\big) f(y) dy, \quad \text{a.e. } x \in (0,\infty).$$

Further,

$$\begin{aligned} R^H_1 f_o(x) - R^H_{1,{\rm loc}} f(x) &= \int_0^{x/2} \big(R^H_1(x,y) - R^H_1(x,-y)\big) f(y) dy \\ &\quad + \int_{2x}^\infty \big(R^H_1(x,y) - R^H_1(x,-y)\big) f(y) dy \\ &\quad - \int_{x/2}^{2x} R^H_1(x,-y) f(y) dy \\ &= \sum_{j=1}^3 T_j(f)(x), \ \ \textit{a.e. } x \in (0,\infty). \end{aligned} \tag{17}$$

It suffices to show that the operators T_j, $j = 1,2,3$, are bounded on $L^1((0,\infty), dx)$. To deal with T_1 and T_2, we estimate the difference $D^H(x,y) = |R^H_1(x,y) - R^H_1(x,-y)|$ for $x,y > 0$. By (16)

$$\begin{aligned} D^H(x,y) &\le C \int_0^\infty \frac{\sqrt{r}\,x}{(1-r^2)^{3/2}} \Big(\exp\big(\frac{2r}{1-r^2}xy\big) - \exp\big(-\frac{2r}{1-r^2}xy\big)\Big) \\ &\qquad \times \exp\Big(- \frac{1+r^2}{2(1-r^2)}(x^2+y^2)\Big) \frac{dt}{\sqrt{t}} \\ &\quad + C \int_0^\infty \frac{\sqrt{r}\,y}{(1-r^2)^{3/2}} \exp\Big(- \frac{1+r^2}{2(1-r^2)}(x^2+y^2)\Big) \exp\Big(\frac{2r}{1-r^2}xy\Big) \frac{dt}{\sqrt{t}}. \end{aligned} \tag{18}$$

Applying the mean value theorem in the first integral, we can assert that

(19)

$$\begin{aligned}D^H(x,y) &\\ &\le C\int_0^\infty \frac{\sqrt{r}}{(1-r^2)^{3/2}}\Big(\frac{rx^2y}{1-r^2}+y\Big)\exp\Big(-\frac{1+r^2}{2(1-r^2)}(x^2+y^2)\Big)\exp\Big(\frac{2r}{1-r^2}xy\Big)\frac{dt}{\sqrt{t}}\\ &= C\int_0^\infty \frac{\sqrt{r}}{(1-r^2)^{3/2}}\Big(\frac{rx^2y}{1-r^2}+y\Big)\exp\Big(-\frac{1+r^2}{2(1-r^2)}(x-y)^2\Big)\exp\Big(-\frac{1-r}{1+r}xy\Big)\frac{dt}{\sqrt{t}}.\end{aligned}$$

It is now not difficult to verify using (19) that

$$D^H(x,y)\le\begin{cases}Cyx^{-2} & \text{for } x>2y,\\ Cy^{-1} & \text{for } 2x<y.\end{cases} \tag{20}$$

The estimate (20) easily implies $\|T_1f\|_{L^1(0,\infty)}+\|T_2f\|_{L^1(0,\infty)}\le C\|f\|_{L^1(0,\infty)}$. Moreover, from (16), we conclude

$$|R_1^H(x,-y)|\le C\Big(xe^{-cx^2}\int_1^\infty e^{-t}dt+x\int_0^1\frac{1}{t^2}e^{-cx^2/t}dt\Big)\le\frac{C}{y}\quad\text{for } x/2<y<2x.$$

Hence T_3 is a bounded operator from $L^1(0,\infty)$ into itself. □

Proposition 2.3. *Let $\alpha>-1/2$, $f\in L^1(0,\infty)$ and f_o be the odd extension of f to $\mathbb{R}$. Then $R_\alpha f$ is in $L^1(0,\infty)$ if and only if R^Hf_o is in $L^1(\mathbb{R})$. Moreover, there exists $C>0$ such that*

$$\begin{aligned}C^{-1}(\|f_o\|_{L^1(\mathbb{R})}+\|R^Hf_o\|_{L^1(\mathbb{R})}) &\le \|f\|_{L^1(0,\infty)}+\|R_\alpha f\|_{L^1(0,\infty)}\qquad \textit{and}\\ \|f\|_{L^1(0,\infty)}+\|R_\alpha f\|_{L^1(0,\infty)} &\le C(\|f_o\|_{L^1(\mathbb{R})}+\|R^Hf_o\|_{L^1(\mathbb{R})}).\end{aligned}$$

Proof. According to [1, Lemma 2.13], we have

$$\begin{aligned}|R_\alpha(x,y)| &\le Cx^{\alpha+3/2}y^{-(\alpha+5/2)} && \text{for } 0<2x<y<\infty,\\ |R_\alpha(x,y)| &\le Cy^{\alpha+1/2}x^{-(\alpha+3/2)} && \text{for } 0<y<x/2, \text{ and}\\ |R_\alpha(x,y)-R^H(x,y)| &\le \frac{C}{y}\Big(1+\frac{(xy)^{1/4}}{|x-y|^{1/2}}\Big) && \text{for } 0<x/2<y<2x.\end{aligned} \tag{21}$$

Each of the Hardy operators

$$H_\alpha(g)(x)=x^{-\alpha-3/2}\int_0^x y^{\alpha+1/2}g(y)dy,\quad x>0$$

and

$$H^\alpha(g)(x)=x^{\alpha+1/2}\int_x^\infty y^{-\alpha-3/2}g(y)dy,\quad x>0$$

are bounded on $L^1(0,\infty)$ when $\alpha > -1/2$. Moreover, the operator N defined by

$$Nf(x) = \int_{x/2}^{2x} \frac{1}{y}\Big(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\Big) f(y)dy$$

is also bounded in $L^1(0,\infty)$. Hence, by (21), (11) and Proposition 2.2, we obtain

$$\begin{aligned}
\|R_\alpha f - R^H f_o\|_{L^1(0,\infty)} &\le \|R_\alpha f - R^H_{\rm loc} f\|_{L^1(0,\infty)} + \|R^H_{\rm loc} f - R^H f_o\|_{L^1(0,\infty)} \\
&\le C\big(\|N\,|f|\,\|_{L^1(0,\infty)} + \|H^{\alpha+1}|f|\,\|_{L^1(0,\infty)} + \|H_\alpha |f|\,\|_{L^1(0,\infty)}\big) \\
&\quad + \|R^H_{1,\rm loc} f - R^H_1 f_o\|_{L^1(0,\infty)} + \|R^H_{2,\rm loc} f\|_{L^1(0,\infty)} + \|R^H_2 f_o\|_{L^1(0,\infty)} \\
&\le C\|f\|_{L^1(0,\infty)}.
\end{aligned}$$

□

The next elementary lemma will be used below.

LEMMA 2.4. *Let $b : (0,\infty) \to \mathbb{C}$ be an $H^1(L_\alpha)$-atom. Then, its odd extension b_o satisfies*

$$\|b_o\|_{H^1_{\rm at}(H)} \le 36.$$

PROOF. Let $B = B(y,R) \subset (0,\infty)$ be a ball associated with b, that is, $R \le \rho_{L_\alpha}(y)$, $\operatorname{supp} b \subset B$ and $\|b\|_\infty \le |B|^{-1}$. Moreover, $\int b(y)dy = 0$ if $R \le \rho_{L_\alpha}(y)/2$. In this last case, since $\rho_{L_\alpha}(y) \le \rho_H(y)/2$, the function $b(x)$ (extended as 0 when $x \le 0$) is an $H^1(H)$-atom, and hence so is $-b(-x)$. Thus $\|b_o\|_{H^1_{\rm at}(H)} \le 2$.

Suppose now that $\rho_{L_\alpha}(y)/2 < R \le \rho_{L_\alpha}(y)$. We distinguish two cases. If $y \in (0, 8/9)$ then

$$\operatorname{supp} b_o \subset B(0, y+R) \subset B(0, 9y/8) \equiv B_o.$$

Since $\int_{\mathbb{R}} b_o = 0$ and $\|b_o\|_\infty \le \rho_{L_\alpha}(y)^{-1} = 18/|B_o|$, it follows that $b_o/18$ is an $H^1(H)$-atom associated with the ball B_o, and hence $\|b_o\|_{H^1_{\rm at}(H)} \le 18$. In the second case, i.e. $y > 8/9$, we may regard $b/18$ as an $H^1(H)$-atom associated with the ball $B(y, \rho_H(y))$, since

$$\operatorname{supp} b \subset B(y,\rho_H(y)) \quad \text{and} \quad \|b\|_\infty \le (2R)^{-1} \le 18|B(y,\rho_H(y))|^{-1}.$$

Similarly, $b(-x)/18$ is an $H^1(H)$-atom associated with the ball $B(-y, \rho_H(-y))$. We conclude that $\|b_o\|_{H^1_{\rm at}(H)} \le 36$, establishing the lemma. □

Proof of Theorem 1.1. Assume that f is in $H^1_{\rm at}(L_\alpha)$. Then f can be written as $\sum_j c_j b_j$, where b_j are $H^1(L_\alpha)$-atoms and $\sum_j |c_j| \sim \|f\|_{H^1_{\rm at}(L_\alpha)}$. By the previous lemma, the odd extension f_o of f belongs to $H^1_{\rm at}(H)$ and $\|f_o\|_{H^1_{\rm at}(H)} \le 36\|f\|_{H^1_{\rm at}(L_\alpha)}$. Applying Proposition 2.3 and using (15), we obtain

$$\|R_\alpha f\|_{L^1(0,\infty)} \le C(\|f_o\|_{L^1(\mathbb{R})} + \|R^H f_o\|_{L^1(\mathbb{R})}) \le C'\|f_o\|_{H^1_{\rm at}(H)} \le C''\|f\|_{H^1_{\rm at}(L_\alpha)}.$$

To prove the converse, assume that f is in $H^1_{\rm Riesz}(L_\alpha)$. Again, using Proposition 2.3 combined with (15), we obtain $f_o \in H^1_{\rm Riesz}(H) = H^1_{\rm at}(H)$ and

$$\|f_o\|_{H^1_{\rm at}(H)} \leq C(\|f_o\|_{L^1(\mathbb{R})} + \|R^H f_o\|_{L^1(\mathbb{R})}) \leq C(\|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)}).$$

Hence $f_o(x) = \sum_j c_j a_j(x)$, where a_j are $H^1(H)$-atoms and $\sum_j |c_j| \sim \|f_o\|_{H^1_{\rm at}(H)}$. Letting $b_j = a_j\big|_{(0,\infty)}$, one easily verifies the inequality $\|b_j\|_{H^1_{\rm at}(L_\alpha)} \leq C$. Thus f is in $H^1_{\rm at}(L_\alpha)$ and $\|f\|_{H^1_{\rm at}(L_\alpha)} \leq C'(\|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)})$. □

REMARK 2.5. Using a similar analysis based on a comparison of the kernels $W_t^\alpha(x,y)$ and $W_t^H(x,y)$ (see [1, Lemma 2.11]), one can prove that $W_*^H f_o$ belongs to $L^1(\mathbb{R})$ if and only if $W_*^\alpha f$ belongs to $L^1(0,\infty)$ and $\|f_o\|_{L^1(\mathbb{R})} + \|W_*^H f_o\|_{L^1(\mathbb{R})} \sim \|f\|_{L^1(0,\infty)} + \|W_*^\alpha f\|_{L^1(0,\infty)}$.

3 Hardy spaces $H^1(\mathfrak{L}_\alpha)$ associated with Laguerre operators $\mathfrak{L}_\alpha$.

In this section we prove Theorem 1.2. The proof is based on the following estimates for the kernel $\mathfrak{R}_\alpha(x,y)$.

PROPOSITION 3.1. *Let ψ be as in Lemma 2.1. Then, for every $\alpha > 0$, there exists a kernel $K(x,y)$ such that*

$$\mathfrak{R}_\alpha(x,y) = \frac{c_0}{\sqrt{2}(x-y)}\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big) + K(x,y), \quad x,y \in (0,\infty), \tag{22}$$

$$\sup_{y>0} \int_0^\infty |K(x,y)|dx < \infty, \tag{23}$$

where c_0 is the constant from (13).

PROOF. Set

$$K(x,y) = \mathfrak{R}_\alpha(x,y) - \frac{c_0}{\sqrt{2}(x-y)}\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big). \tag{24}$$

If $x < y/4$ or $y < x/4$, then $K(x,y) = \mathfrak{R}_\alpha(x,y)$. From (4) and (21), we conclude

$$|K(x,y)| \leq \begin{cases} Cx^{(\alpha+1)/2}y^{-(\alpha+3)/2} & \text{if } \ 4x < y < \infty \\ Cy^{\alpha/2}x^{-(\alpha+2)/2} & \text{if } \ 0 < y < x/4. \end{cases} \tag{25}$$

Hence

$$\sup_{y>0}\Big(\int_0^{y/4} |K(x,y)|\, dx + \int_{4y}^\infty |K(x,y)|\, dx\Big) < \infty. \tag{26}$$

In order to deal with the kernel $K(x,y)$ in the local part $y/4 \le x \le 4y$, we set

$$E(x,y) = \mathfrak{R}_\alpha(x,y) - 2^{-3/2}(xy)^{-1/4}R^H(\sqrt{x},\sqrt{y}),$$

$$G(x,y) = 2^{-3/2}\Big((xy)^{-1/4}\frac{c_0}{\sqrt{x}-\sqrt{y}}\psi\Big(\frac{\sqrt{x}-\sqrt{y}}{\rho_H(\sqrt{x})}\Big) - \frac{2c_0}{x-y}\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big)\Big).$$

Then, by (4) and Lemma 2.1, we have

$$K(x,y) = E(x,y) + 2^{-3/2}(xy)^{-1/4}h(\sqrt{x},\sqrt{y}) + G(x,y). \tag{27}$$

According to (21), we get

$$|E(x,y)| \le C\frac{(xy)^{-1/4}}{\sqrt{y}}\Big(1+\frac{(xy)^{1/8}}{|\sqrt{x}-\sqrt{y}|^{1/2}}\Big) \le C\frac{1}{y}\Big(1+\frac{\sqrt{x}}{|x-y|^{1/2}}\Big) \tag{28}$$

for $y/4 \le x \le 4y$. Trivially, using (28) and (14), we obtain

$$\int_{y/4}^{4y}\Big(|E(x,y)| + (xy)^{-1/4}|h(\sqrt{x},\sqrt{y})|\Big)\,dx \le C. \tag{29}$$

The proof will be complete if we show the inequality

$$\int_{y/4}^{4y}|G(x,y)|\,dx \le C. \tag{30}$$

Let us note that

$$G(x,y) = \frac{2^{-3/2}c_0}{x-y}\Big[\frac{\sqrt{x}+\sqrt{y}}{(xy)^{1/4}}\psi\Big(\frac{x-y}{(\sqrt{x}+\sqrt{y})\rho_H(\sqrt{x})}\Big) - 2\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_a}(x)}\Big)\Big]. \tag{31}$$

If $y > 10$, $y/4 \le x \le 4y$ and $|x-y| > 1$, then $G(x,y) = 0$. If $y > 10$, $y/4 < x < 4y$ and $|x-y| \le 1$, then, by the mean value theorem, $|G(x,y)| \le C$. Thus (30) is satisfied for $y > 10$. If $0 < y \le 10$ and $y/4 \le x \le 4y$, then applying the mean value theorem we deduce $|G(x,y)| \le Cy^{-1}$ and, consequently, (30) holds. □

Before we turn to the proof of Theorem 1.2, we state some results from the theory of local Hardy spaces [5]. Fix $l > 0$. We say that a function b is an atom for the local Hardy space $\mathbf{h}^1_l(\mathbb{R})$ if there exists a ball $B(y_0,R)$ with $R < l$ such that $\operatorname{supp} b \subset B(y_0,R)$, $\|b\|_\infty \le (2R)^{-1}$, and if $R \le l/2$, then $\int b(y)\,dy = 0$. A function f belongs to the space $\mathbf{h}^1_l$ if there exist a sequence b_j of $\mathbf{h}^1_l$-atoms and $\lambda_j \in \mathbb{C}$ with $\sum_j|\lambda_j| < \infty$ such that

$$f = \sum_j \lambda_j b_j. \tag{32}$$

The atomic norm in $\mathbf{h}^1_l$ is defined in a standard way, that is, $\|f\|_{\mathbf{h}^1_l} = \inf \sum_j |\lambda_j|$, where the infimum is taken over all decompositions (32). Moreover, if $f \in \mathbf{h}^1_l$ and $\operatorname{supp} f \subset B(y_0, l)$, then there exists decomposition (32) of f such that $\operatorname{supp} b_j \subset B(y_0, 10l/9)$ and $\sum_j |\lambda_j| \le C\|f\|_{\mathbf{h}^1_l}$. We define a local Hilbert transform

$$\mathcal{H}_l f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{c_0}{\sqrt{2}(x-y)} \psi\Big(\frac{x-y}{l}\Big) f(y)\, dy,$$

where c_0 and ψ are as in Lemma 2.1. The following result was actually proved in [5]. There exists a constant $C > 0$ independent of l such that

$$C^{-1}\|f\|_{\mathbf{h}^1_l} \le \|\mathcal{H}_l f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{\mathbf{h}^1_l}. \tag{33}$$

Proof of Theorem 1.2. Since $\mathfrak{R}_\alpha$ maps continuously $L^1(0,\infty)$ into the space of distributions, to prove the second inequality in (6), it suffices to verify that there exists a constant $C > 0$ such that, for every $H^1(\mathfrak{L}_\alpha)$-atom b, one has

$$\|\mathfrak{R}_\alpha b\|_{L^1} \le C. \tag{34}$$

Let b be an $H^1(\mathfrak{L}_\alpha)$-atom with associated ball $B(y_0, R)$. Clearly, letting $l = \rho_{\mathfrak{L}_\alpha}(y_0)$, we see that b is also an $\mathbf{h}^1_l$-atom. By Proposition 3.1,

$$\begin{aligned} \mathfrak{R}_\alpha b(x) = & \int K(x,y) b(y)\, dy + \mathcal{H}_l b(x) \\ & + \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{c_0}{\sqrt{2}(x-y)} \Big(\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big) - \psi\Big(\frac{x-y}{l}\Big)\Big) \chi_{B(y_0,l)}(y) b(y)\, dy. \end{aligned} \tag{35}$$

The kernel

$$U(x,y) = \frac{c_0}{\sqrt{2}(x-y)} \Big(\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big) - \psi\Big(\frac{x-y}{l}\Big)\Big) \chi_{B(y_0,l)}(y),$$

as a function of (x,y), is supported by $B(y_0, 3l) \times B(y_0, l)$. Moreover, $|U(x,y)| \le Cl^{-1}$, which implies $\sup_{y>0} \int |U(x,y)| dx < \infty$. Therefore, (34) holds by applying (23) and (33).

We now turn to prove the first inequality in (6). We define the intervals $\{I_j\}_{j\in\mathbb{Z}}$, $I_j = (\beta_j, \beta_{j+1})$, $\beta_j = (9/8)^j$ for $j \le 1$, and $\beta_j = 1 + j/8$ for $j \ge 1$. Set $l_j = \rho_{\mathfrak{L}_\alpha}(\beta_j)$. Let η_j be a family of smooth functions such that

$$0 \le \eta_j \le 1, \quad \operatorname{supp} \eta_j \subset I_j^*, \quad \Big|\frac{d}{dx}\eta_j(x)\Big| \le C l_j^{-1}, \quad \sum_j \eta_j(x) = 1 \quad \text{for } x > 0, \tag{36}$$

where $I_j^* = [\beta_{j-1}, \beta_{j+2}]$. Set $I_j^{**} = [\beta_{j-2}, \beta_{j+3}]$. Then $\sum_j \chi_{I_j^{**}} \le 5$. Fix $f \in L^1(0,\infty)$ such that $\|\mathfrak{R}_\alpha f\|_{L^1} < \infty$. We shall verify that

$$\sum_j \|\mathcal{H}_{l_j}(\eta_j f)\|_{L^1} \le C(\|\mathfrak{R}_\alpha f\|_{L^1} + \|f\|_{L^1}) \tag{37}$$

with a constant $C > 0$ independent of f. To this end, note that

(38)
$$\begin{aligned}\mathcal{H}_{l_j}(\eta_j f)(x) &= \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\Big(\eta_j(y)-\eta_j(x)\Big)\frac{c_0}{\sqrt{2}(x-y)}\psi\Big(\frac{x-y}{l_j}\Big)f(y)\,dy+\eta_j(x)\mathcal{H}_{l_j}f(x)\\ &= \Xi_j f(x)+\eta_j(x)\mathcal{H}_{l_j}f(x).\end{aligned}$$

Observe that the kernel

$$\Big|\Big(\eta_j(y)-\eta_j(x)\Big)\frac{c_0}{\sqrt{2}(x-y)}\psi\big(\frac{x-y}{l_j}\big)\Big|,$$

as a function of (x,y), is supported by $I_j^{**}\times I_j^{**}$ and bounded by Cl_j^{-1}. Since each $y>0$ belongs to at most 5 intervals I_j^{**}, and $|I_j^{**}|\sim l_j$, we can easily obtain

$$\sum_j\int|\Xi_j f(x)|\,dx\le C\|f\|_{L^1}. \tag{39}$$

Now we shall deal with $\eta_j(x)\mathcal{H}_{l_j}f(x)$, defined by

$$\begin{aligned}\eta_j(x)\mathcal{H}_{l_j}f(x) &= \int\eta_j(x)\Big[\psi\Big(\frac{x-y}{l_j}\Big)-\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha(x)}}\Big)\Big]\frac{c_0}{\sqrt{2}(x-y)}f(y)\,dy\\ &\quad+\eta_j(x)\mathfrak{R}_\alpha f(x)-\eta_j(x)\int K(x,y)f(y)\,dy.\end{aligned}\tag{40}$$

The integral kernel

$$\Big|\eta_j(x)\Big[\psi\Big(\frac{x-y}{l_j}\Big)-\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big)\Big]\frac{c_0}{\sqrt{2}(x-y)}\Big|,$$

as a function of (x,y), is supported by $I_j^{*}\times I_j^{**}$ and bounded by Cl_j^{-1}. Hence

$$\sup_{y>0}\int_0^\infty\sum_j\Big|\eta_j(x)\Big(\psi\Big(\frac{x-y}{l_j}\Big)-\psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big)\Big)\frac{c_0}{\sqrt{2}(x-y)}\Big|dx<\infty. \tag{41}$$

Using (40), (41), we obtain

$$\sum_j\|\eta_j\mathcal{H}_{l_j}f\|_{L^1}\le C(\|f\|_{L^1}+\|\mathfrak{R}_\alpha f\|_{L^1}), \tag{42}$$

which combined with (38), (39) and (36) gives (37). Having (37) already proved, we are in a position to complete the proof of the first inequality in (6). Applying (37) together with the results from the theory of local Hardy spaces stated in this section, we have

$$f=\sum_j(\eta_j f)=\sum_j\Big(\sum_i\lambda_{ij}a_{ij}\Big), \tag{43}$$

where a_{ij} are $\mathbf{h}^1_{l_j}$-atoms supported by I_j^{**}, and $\sum_{ij}|\lambda_{ij}| \le C(\|\mathfrak{R}_\alpha f\|_{L^1}+\|f\|_{L^1})$. The proof will be complete once we observe that each of these atoms is either an $H^1(\mathfrak{L}_\alpha)$-atom, or can be written as a sum of at most 20 such atoms. Indeed, fix an $h^1_{l_j}$-atom a supported in I_j^{**}. Then, for some $0 < R_0 < l_j$ and $y_0 \in I_j^{**}$ we have $\operatorname{supp} a \subset B(y_0, R_0) \subset I_j^{**}$, $\|a\|_\infty \le (2R_0)^{-1}$, and if $R_0 \le l_j/2$ then also $\int a(x)dx = 0$. Notice that, by construction,

$$\rho_{\mathfrak{L}_\alpha}(y) \le 2\rho_{\mathfrak{L}_\alpha}(y'), \quad \text{for all } \ y, y' \in I_j^{**} = [\beta_{j-2}, \beta_{j+3}].$$

If $R_0 \le l_j/2 = \rho_{\mathfrak{L}_\alpha}(\beta_j)/2$ then $\int a = 0$ and $R_0 \le \rho_{\mathfrak{L}_\alpha}(y_0)$, and therefore a is also an $H^1(\mathfrak{L}_\alpha)$-atom. If $R_0 > l_j/2$, then

$$I_j^{**} = \bigcup_{k=0}^{4} I_{j-2+k} \quad \text{with} \quad |I_{j-2+k}| = \rho_{\mathfrak{L}_\alpha}(\beta_{j-2+k}) \ ,$$

and using again $\rho_{\mathfrak{L}_\alpha}(\beta_{j+2}) \le 2\rho_{\mathfrak{L}_\alpha}(\beta_j)$ we see that

$$\|a\chi_{I_{j-2+k}}\|_\infty \le (2R_0)^{-1} \le \rho_{\mathfrak{L}_\alpha}(\beta_j)^{-1} \le 2|I_{j-2+k}|^{-1}.$$

Hence, each piece $a\chi_{I_{j-2+k}}/4$ is an $H^1(\mathfrak{L}_\alpha)$-atom for the ball $B(\beta_{j-2+k}, \rho_{\mathfrak{L}_\alpha}(\beta_{j-2+k}))$ and, consequently, $\|a\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)} \le 20$.

4 Proof of (11) and Lemma 2.1

During the proof we set $r = e^{-t} \in (0,1)$. We can rewrite (9) as

$$(44) \qquad W_t^H(x,y) = \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big) \exp\Big(-\frac{1-r}{1+r}xy\Big),$$

for all $x, y \in \mathbb{R}$. A simple computation using (44) or (9) gives

$$(45) \qquad W_t^H(x,y) \le \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{4}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big).$$

Let us note that, for every $N > 0$, there exists a constant C_N such that

$$(46) \qquad W_t^H(x,y) \le C_N \frac{e^{-t/3}}{\sqrt{(1-r^2)}} \Big(1 + \frac{t}{\rho_H(x)^2}\Big)^{-N}.$$

Indeed, if $|x-y| > |x|/2$, then

$$(47) \quad W_t^H(x,y) \le \frac{e^{-t/2}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{8}\Big(\frac{1+r^2}{1-r^2}\Big)x^2\Big) \le C_N \frac{e^{-t/3}}{\sqrt{(1-r^2)}} \Big(1 + \frac{t}{\rho_H(x)^2}\Big)^{-N}.$$

If $|x-y| \le |x|/2$, then $xy \sim x^2$ and, using (44), we get

$$(48) \qquad W_t^H(x,y) \le C \frac{e^{-t/2}}{\sqrt{1-r^2}} \exp\Big(-c(1-r)x^2\Big) \le C_N \frac{e^{-t/3}}{\sqrt{1-r^2}} \Big(1 + \frac{t}{\rho_H(x)^2}\Big)^{-N}.$$

Applying (45) and (46) combined with the fact that $W_t^H(x,y) = W_t^H(y,x)$, we obtain

$$(49) \quad W_t^H(x,y) \le C_N \frac{e^{-t/3}}{\sqrt{1-e^{-2t}}} \exp\Big(- \frac{|x-y|^2}{12(1-e^{-2t})}\Big)\Big(1 + \frac{t}{\rho(x)^2}\Big)^{-N}\Big(1+\frac{t}{\rho(y)^2}\Big)^{-N}.$$

We are now in a position to prove (11). If $|x-y| \le C\rho_H(y)$, then by (10) and (49) we have

(50)

$$\begin{aligned}|R_2^H(x,y)| &\le C_N\Big(\int_0^{|x-y|^2} |x| \Big(\frac{t}{|x-y|^2}\Big)^N \frac{dt}{t} + \int_{|x-y|^2}^{C^2\rho_H(y)^2} |x| \frac{dt}{t} + \int_{C^2\rho_H(y)^2}^{\infty} |x| \Big(\frac{\rho_H(y)^2}{t}\Big)^N \frac{dt}{t}\Big) \\ &\le C_N\Big(|x| + |x| \ln\Big(\frac{C\rho_H(y)}{|x-y|}\Big)\Big).\end{aligned}$$

If $|x-y| \ge C\rho_H(y)$, then we use again (49) and get

(51)

$$\begin{aligned}|R_2^H(x,y)| &\le C_N\Big(\int_0^{C^2\rho_H(y)^2} |x| \Big(\frac{t}{|x-y|^2}\Big)^N \frac{dt}{t} + \int_{C^2\rho_H(y)^2}^{|x-y|^2} |x| \Big(\frac{t}{|x-y|^2}\Big)^N \Big(\frac{t}{\rho_H(y)^2}\Big)^{-2N} \frac{dt}{t} \\ &\qquad + \int_{|x-y|^2}^{\infty} |x| \Big(\frac{\rho_H(y)^2}{t}\Big)^N \frac{dt}{t}\Big) \\ &\le C_N \frac{|x|\rho_H(y)^{2N}}{|x-y|^{2N}} \\ &\le C_N\Big(\frac{|x-y|\rho_H(y)^{2N}}{|x-y|^{2N}} + \frac{|y|\rho_H(y)^{2N}}{|x-y|^{2N}}\Big).\end{aligned}$$

Now the first inequality in (11) is a consequence of (50) and (51). Similarly to (50) and (51), we also conclude that

$$(52) \qquad |R_2^H(x,y)| \le \begin{cases} C\Big(|x| + |x| \ln\big(C\rho_H(x)/|x-y|\big)\Big) & \text{for } |x-y| \le C\rho_H(x) \\ C_N |x| \rho_H(x)^N / |x-y|^N & \text{for } |x-y| > C\rho_H(x), \end{cases}$$

from which we easily obtain the second inequality in (11).

Having (11) already established, we now turn to prove Lemma 2.1. By (44),

(53)

$$\begin{aligned}\frac{\partial}{\partial x} W_t^H(x,y) &= - \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \frac{1+r^2}{1-r^2}(x-y) \exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big) \exp\Big(-\frac{1-r}{1+r}xy\Big) \\ &\quad - y \frac{1-r}{1+r} \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big) \exp\Big(-\frac{1-r}{1+r}xy\Big).\end{aligned}$$

From (53) we deduce that, for $|x-y| > C\rho_H(y)$, we have

(54)

$$\Big|\frac{\partial}{\partial x} W_t^H(x,y)\Big| \le C_N\Big(\frac{1}{|x-y|} + |y|(1-r)\Big) \frac{e^{-t/3}}{\sqrt{1-r^2}} \exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N}.$$

Proceeding as in (51), we obtain

$$(55)\qquad \Big|\int_0^\infty \frac{\partial}{\partial x} W_t^H(x,y)\frac{dt}{\sqrt{t}}\Big| \le C_N\Big(\frac{1}{|x-y|}+|y|\Big)\frac{\rho_H(y)^{2N}}{|x-y|^{2N}} \quad \text{for } |x-y|>C\rho_H(y),$$

which leads to

$$(56)\qquad \sup_{y\in\mathbb{R}} \int_{|x-y|>C\rho_H(y)} |R_1^H(x,y)|\,dx \le C.$$

Our next step is to estimate $R_1^H(x,y)$ for $|x-y|\le C\rho_H(y)$. Note that (53) implies

(57)

$$\begin{aligned}\Big|\frac{\partial}{\partial x}W_t(x,y)\Big| &\le C_N\frac{e^{-t/3}}{\sqrt{1-r^2}}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|\exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N-1}\\ &\quad + C_N\frac{e^{-t/3}}{\sqrt{1-r^2}}|y|(1-r)\exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N-1}\\ &\le C_N\frac{e^{-t/4}}{1-r^2}\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N}.\end{aligned}$$

Consequently, using (57) we get

$$(58)\qquad \int_{\rho_H(y)^2}^\infty \Big|\frac{\partial}{\partial x}W_t(x,y)\Big|\frac{dt}{\sqrt{t}} \le C\rho_H(y)^{-1}.$$

In order to investigate the integral

$$\int_0^{\rho_H(y)^2} \frac{\partial}{\partial x}W_t(x,y)\frac{dt}{\sqrt{t}},$$

we study first the difference

$$Q(x,y)=\int_0^{\rho_H(y)^2}\frac{\partial}{\partial x}\Big(W_t^H(x,y)-P_t(x-y)\Big)\frac{dt}{\sqrt{t}},$$

where $P_t(x)=(2\pi t)^{-1/2}\exp(-x^2/2t)$ is the classical Gauss-Weierstrass kernel. The perturbation formula asserts that

$$Q(x,y)=-\frac{1}{2}\int_0^{\rho_H(y)^2}\int_0^t\int_{-\infty}^\infty \frac{\partial}{\partial x}P_{t-s}(x-z)z^2W_s^H(z,y)dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Therefore,

(59)

$$\begin{aligned}J &= \int_{|x-y|<C\rho_H(y)} |Q(x,y)|dx\\ &\le C\int_{|x-y|\le C\rho_H(y)}\int_0^{\rho_H(y)^2}\int_0^t\int_{-\infty}^\infty \frac{|x-z|}{t-s}P_{t-s}(x-z)(|z-x|^2+x^2)W_s^H(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx.\end{aligned}$$

Observe that $x^2 \le C\rho_H(y)^{-2}$ for $|x-y| \le C\rho_H(y)$. Substituting this inequality inside the above integral and then integrating with respect to dx and dz, we conclude

$$J \le C \int_0^{\rho_H(y)^2} \int_0^t \Big((t-s)^{1/2} + \frac{1}{(t-s)^{1/2}\rho_H(y)^2} \Big)\, ds \, \frac{dt}{\sqrt{t}} \le C\rho_H(y)^4 + C \le C. \tag{60}$$

Proceeding as in (55), we also get

$$|R_1^H(x,y)| \le C_N \rho_H(x)^{-1} \frac{\rho_H(x)^N}{|x-y|^N} \quad \text{for } |x-y| > C\rho_H(x),$$

and consequently,

$$\sup_{x\in\mathbb{R}} \int_{|x-y|>C\rho_H(x)} |R_1^H(x,y)|\, dy < \infty. \tag{61}$$

A similar procedure to that employed to estimate J gives

$$\sup_{x\in\mathbb{R}} \int_{|x-y|\le C\rho_H(x)} |Q(x,y)| dy \le C. \tag{62}$$

Finally, our analysis of the kernel $R_1^H(x,y)$ is reduced to the integral

$$\begin{aligned} \int_0^{\rho_H(y)^2} \frac{\partial}{\partial x} P_t(x-y) \frac{dt}{\sqrt{t}} &= -\int_0^{\rho_H(y)^2} \frac{x-y}{t} \frac{1}{\sqrt{2\pi t}} \exp(-|x-y|^2/2t) \frac{dt}{\sqrt{t}} \\ &= -\frac{2}{\sqrt{2\pi}(x-y)} \exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2} \Big). \end{aligned} \tag{63}$$

Taking into account (10), (55), (58), (60), (61), (62) and (63), we get

$$R_1^H(x,y) = -\frac{\sqrt{2}}{\pi(x-y)} \exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2} \Big) + h_1(x,y) \tag{64}$$

with

$$\sup_{y\in\mathbb{R}} \int_{-\infty}^{\infty} |h_1(x,y)| dx + \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |h_1(x,y)| dy < \infty. \tag{65}$$

To complete the proof, take any $\psi \in C_c^\infty(\mathbb{R})$ as in the statement of Lemma 2.1. Define a function $h_2(x,y)$ by

$$h_2(x,y) = \frac{\sqrt{2}}{\pi(x-y)} \psi\Big(\frac{x-y}{\rho_H(x)} \Big) - \frac{\sqrt{2}}{\pi(x-y)} \exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2} \Big), \quad x,y \in \mathbb{R}.$$

By (10), (64), (65) and (11), the lemma will be established once we show that, for some $C>0$ we have

$$\sup_{x\in\mathbb{R}} \int |h_2(x,y)|\, dy \le C \quad \text{and} \quad \sup_{y\in\mathbb{R}} \int |h_2(x,y)|\, dx \le C. \tag{66}$$

Set $A=\{(x,y)\in\mathbb{R}^2; |x-y|>\rho_H(x)\}$, $B=\{(x,y)\in\mathbb{R}^2; |x-y|\le\rho_H(x)\}$. Then

$$|h_2(x,y)|\le\frac{C}{|x-y|}\exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2}\Big)\chi_A(x,y)+C\Big(\frac{1}{\rho_H(x)}+\frac{|x-y|}{\rho_H(y)^2}\Big)\chi_B(x,y), \tag{67}$$

where the last summand is obtained by applying the mean value theorem. Using (8), we see that $\rho(y)^2\le c\rho(x)|x-y|$ when $(x,y)\in A$, and therefore

$$\begin{aligned}\int\frac{C}{|x-y|}\exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2}\Big)\chi_A(x,y)\,dy&\le\int\frac{C}{|x-y|}\exp\Big(-c\frac{|x-y|}{\rho_H(x)}\Big)\chi_A(x,y)\,dy\\&\le\int_{|u|>1}\exp(-c|u|)\frac{du}{|u|}\le C.\end{aligned}\tag{68}$$

On the other hand, $\rho_H(x)\sim\rho_H(y)$ when $(x,y)\in B$ (again by (8)), so we have

$$\int\Big(\frac{C}{\rho_H(x)}+\frac{C|x-y|}{\rho_H(y)^2}\Big)\chi_B(x,y)\,dy\le C,$$

which together with (68) implies the first inequality in (66). From (8) we also see that $A\subset\widetilde{A}=\{(x,y)\in\mathbb{R}^2; |x-y|>\varepsilon\rho_H(y)\}$ and $B\subset\widetilde{B}=\{(x,y)\in\mathbb{R}^2; |x-y|\le\rho_H(y)/\varepsilon\}$ for some $\varepsilon>0$. Using this fact, the second inequality in (66) follows by similar arguments. This completes the proof of Lemma 2.1.

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