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Discrete radar ambiguity problems

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Abstract

In this paper, we pursue the study of the radar ambiguity problem started in [Ph. Jaming, Phase retrieval techniques for radar ambiguity functions, J. Fourier Anal. Appl. 5 (1999) 313–333; G. Garrigós, Ph. Jaming, J.-B. Poly, Zéros de fonctions holomorphes et contre-exemples en théorie des radars, in: Actes des rencontres d'analyse complexe, Atlantique, Poitiers, 2000, pp. 81–104, available on http://hal.ccsd.cnrs.fr/ccsd-00007482]. More precisely, for a given function u we ask for all functions v (called *ambiguity partners*) such that the ambiguity functions of u and v have same modulus. In some cases, v may be given by some elementary transformation of u and is then called a *trivial partner* of u, otherwise we call it a *strange partner*. Our focus here is on two discrete versions of the problem. For the first one, we restrict the problem to functions u of the Hermite class, $u = P(x)e^{-x^2/2}$, thus reducing it to an algebraic problem on polynomials. Up to some mild restriction satisfied by quasi-all and almost-all polynomials, we show that such a function has only trivial partners. The second discretization, restricting the problem to pulse type signals, reduces to a combinatorial problem on matrices of a special form. We then exploit this to obtain new examples of functions that have only trivial partners. In particular, we show that most pulse type signals have only trivial partners. Finally, we clarify the notion of *trivial partner*, showing that most previous counterexamples are still trivial in some restricted sense.

Keywords: Radar ambiguity problem; Phase retrieval problems; Hermite functions; Pulse type signals

1. Introduction

Phase retrieval problems arise naturally in the applied study of signals [8,11,17,28]. They are based on the ambiguity for the phase choice in a signal with fixed frequency amplitude. To be more precise, let us denote the Fourier transform of $u \in L^1(\mathbb{R})$ (with the usual extension to $L^2(\mathbb{R})$) by $\mathcal{F}$

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} u(x)\, e^{ix\xi}\, \mathrm{d}x, \quad \xi \in \mathbb{R}.$$

The *phase retrieval problem* then amounts to solving the following:

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Problem 1 *(Phase retrieval). Given $u \in L^2(\mathbb{R})$, find all $v \in L^2(\mathbb{R})$ such that for all $x \in \mathbb{R}$,*

$$\left|\mathcal{F}u(x)\right| = \left|\mathcal{F}v(x)\right|. \tag{1}$$

This problem admits always the trivial solutions $v(x) = c\,u(x-\alpha)$ and $v(x) = c\overline{u(-x-\alpha)}$, where $|c| = 1$ and $\alpha \in \mathbb{R}$.

In applied problems, one may usually further restrict the class of functions to which u and v should belong. A typical example would be to ask for u and v to be compactly supported. In this case, there are usually many non-trivial solutions and a complete description of them is available in terms of the zeros of the holomorphic functions $\mathcal{F}u$ and $\mathcal{F}v$ (see [12,23,28] for a complete description of these solutions). For further information on phase retrieval problems, we refer to these articles as well as [13], the surveys [17,20], the book [11] and references therein.

In this paper we shall deal with a different, although closely related type of phase retrieval problem, having its origin in the analysis of radar signals. Following Woodward [30], a radar antenna emits a signal $u \in L^2(\mathbb{T})$ that is reflected by a target and modified by Doppler effect. It then returns to the antenna where it is correlated by the emitted signal, so that, under certain physical conditions (the so-called narrow-band approximation), the radar measures the quantity:

$$A(u)(x,y) = \int_{\mathbb{R}} u(t)\overline{u(t-x)}e^{iyt}\,\mathrm{d}t, \quad x, y \in \mathbb{R}, \tag{2}$$

and $A(u)$ is called the *radar ambiguity function of u*. As usually happens, receivers are not able to read the phase, but only the amplitude $|A(u)(x,y)|$, giving rise to the following *radar ambiguity problem*:

Problem 2 *(Radar ambiguity). Let $u \in L^2(\mathbb{R})$, then find all $v \in L^2(\mathbb{R})$ such that*

$$\left|A(u)(x,y)\right| = \left|A(v)(x,y)\right|, \quad x, y \in \mathbb{R}. \tag{3}$$

Note that, for each $x \in \mathbb{R}$, $A(u)(x,\cdot) = \mathcal{F}[u(\cdot)\bar{u}(\cdot - x)]$, so that Eq. (3) is actually a family of phase retrieval problems as described in (1). Two functions u and v satisfying (3) are said to be (*radar*) *ambiguity partners*. The reader may find a comprehensive historical introduction and further references to this problem in [12,21]. Properties of $A(u)$ that we may use in this paper can all be found there and in [2,26,29] (note that we slightly change the normalization for $A(u)$ from [2]).

It is not difficult to verify that trivial solutions to the equation in (3) are given by

$$v(t) = ce^{i\beta t}u(t-\alpha) \quad \text{and} \quad v(t) = ce^{-i\beta t}u(-t-\alpha), \quad |c| = 1,\ \alpha, \beta \in \mathbb{R}. \tag{4}$$

The first set of solutions corresponds to a unitary representation of the Heisenberg group, while the second is just a composition with the isometry $Zf(t) = f(-t)$. So, following [12], we say that u and v are *trivial partners* when they satisfy (4). If u and v are ambiguity partners that are not trivial partners, we will say that they are *strange partners* and in [5,9,12], examples of signals having strange partners are given. In the opposite direction, there exist signals for which every ambiguity partner is trivial.

The aim of this paper is to get some insight on which functions may or may not have strange partners. To tackle this problem we appeal to two different discrete (finite dimensional) versions of the problem, both being also of practical interest.

The first discretization is the restriction of the problem to Hermite functions, that is to functions of the form $P(t)e^{-t^2/2}$ where P is a polynomial. There are several reasons for this: first it was proposed by Wilcox in his pioneering paper [29], since it is a dense class of functions which are best localized in the time–frequency plane and are thus well adapted for numerical analysis. Second, in some sense this class is "extremal" for the *uncertainty principle*, so one can show that all solutions to Problem 2 are necessarily Hermite functions $v(t) = Q(t)e^{-t^2/2}$ for some polynomial Q (except perhaps for trivial transformations; see [5] or Lemma 2.1 below). Finally, Hermite functions are of theoretical importance for the problem considered. Indeed, Bueckner [4] associated to each function $u \in L^2(\mathbb{R})$ an Hilbert–Schmidt operator K_u in a way that finding all solutions for the ambiguity problem for u amounts to finding all functions v such that $K_u^* K_u = K_v^* K_v$. He then proved that K_u is of finite rank if and only if u is a Hermite function. Moreover, the following conjecture was proposed:

Conjecture. *(See [4].) If u is a Hermite function, then u has only trivial partners.*

Indeed, Bueckner was considering the *bilinear* version of (3)

$$\left|A(u_1, u_2)\right| = \left|A(v_1, v_2)\right|, \tag{5}$$

where $A(u_1, u_2)$ is the bilinear functional associated with $A(u)$. He proved that for almost every couple of functions of the form

$$(u_1, u_2) = \left(P_1(x)e^{-x^2/2}, P_2(x)e^{-x^2/2}\right)$$

(P_1, P_2 polynomials), the solutions to (5) are trivial partners of (u_1, u_2). However, his techniques depend on a certain criterion that excludes the quadratic case, and hence do not say anything about Problem 2.

In this paper we will prove, using a simple algebraic approach, the following result about ambiguity partners of Hermite functions:

Theorem A. *For almost all and quasi-all polynomials P, the function $u(x) = P(x)e^{-x^2/2}$ has only trivial partners.*

Here almost all (respectively quasi-all) refers to Lebesgue measure (respectively Baire category) when one identifies the set of polynomials of fixed degree n with $\mathbb{C}^{n+1}$.

The problem has also been considered by de Buda [5], who obtained some partial results in an unpublished report which unfortunately are not always complete. Although our approach shares some common features with his, it is essentially distinct as we introduce a new argument by using the fact that $A(u)$ has some factorization if u has non-trivial partners. Some technical difficulties remain as our use of Bezout's theorem forces us to assume that some polynomial associated to u has only simple non-symmetric zeros in order to prove that u has only trivial partners.

The second class of functions we consider is the restriction to signals of *pulse type*

$$u(t) = \sum_{j=-\infty}^{\infty} a_j H(t-j), \quad x \in \mathbb{R}, \tag{6}$$

where $H \in L^2(\mathbb{T})$ has $\operatorname{supp} H \subset [0, \frac{1}{2}]$, and $\{a_j\}_{j\in\mathbb{Z}}$ is a sequence of complex numbers (of finite support). This class of functions is very common in radar signal design (see, e.g., [27, p. 285]). It also leads naturally to a discretization of Problem 2. Indeed, a simple computation shows that, for all $k \in \mathbb{Z}$, $y \in \mathbb{R}$ and $k - \frac{1}{2} \leqslant x \leqslant k + \frac{1}{2}$, one has:

$$A(u)(x, y) = \left(\sum_{j\in\mathbb{Z}} a_j \overline{a_{j-k}} e^{ijy}\right) A(H)(x-k, y). \tag{7}$$

This following *discrete ambiguity problem* was proposed in [9]:

Problem 3 *(Discrete Radar Ambiguity Problem). Given $a = \{a_j\} \in \ell^2(\mathbb{Z})$, find all sequences $b \in \ell^2(\mathbb{Z})$ such that, for every $k \in \mathbb{Z}$ and $y \in \mathbb{R}$,*

$$\left|\mathcal{A}(a)(k, y)\right| = \left|\mathcal{A}(b)(k, y)\right|, \tag{8}$$

where

$$\mathcal{A}(a)(k, y) = \sum_{j\in\mathbb{Z}} a_j \overline{a_{j-k}} e^{ijy}.$$

Again, a sequence b, solution to (8), is called an *ambiguity partner of a*. It is easy to see that trivial solutions to (8) are given by

$$b_j = ce^{i\beta j} a_{j-k} \quad \text{and} \quad b_j = ce^{i\beta j} a_{-j-k}, \quad |c| = 1,\ \beta \in \mathbb{R},\ k \in \mathbb{Z}.$$

Such solutions are again called *trivial partners of a* and solutions that are not of this type are called *strange partners*. The main result of [9] shows that a finite sequence $a = \{a_j\} \in \mathbb{C}^{d+1}$ has only trivial partners, except perhaps for a's in a semialgebraic set of real codimension 1 in $\mathbb{C}^{d+1}$ (see Theorem 4.3 below). This was done by adapting Bueckner's method to the Discrete Radar Ambiguity Problem, and then adapting a careful analysis to the obtained combinatorial equation of matrices. The form of these matrices was also exploited to produce new constructions of non-trivial solutions in the exceptional set. A few other points about such constructions, which were only announced in [9], are proven here in full detail (see Section 4.3).

It was not investigated, however, how to translate these discrete results into uniqueness statements for the general ambiguity problem, i.e. to Problem 2. This step is now different from the corresponding one for Hermite functions, since the class of pulse type signals is not extremal for the uncertainty principle. In this paper, we introduce new techniques for this class based on complex analysis and distribution theory, which allows us to prove the following theorem:

Theorem B. *Let $0 < \eta \leqslant \frac{1}{3}$, and let $a = (a_0, a_1, \dots, a_N) \in \mathbb{C}^{N+1}$ that has only trivial partners. Then the pulse type signal*

$$u(t) = \sum_{j=0}^{N} a_j \chi_{[j,j+\eta]}(t)$$

has only trivial partners.

We do not know whether the condition $\eta \leqslant 1/3$ is optimal. It was essential in the proof to ensure that v is also of pulse type.

Next, we clarify the notion of *trivial solutions*. There are numerous phase retrieval problems in the literature and we think that a natural definition of a trivial solution is to be a linear or anti-linear operator that associates to each function a solution of the given phase retrieval problem. Using Theorem A, we will show that those trivial solutions described in Eq. (4) are indeed the only trivial solutions in the previous sense:

Theorem C. *The only linear (or anti-linear) bounded transformations $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ so that*

$$\bigl|A(Tu)(x,y)\bigr| = \bigl|A(u)(x,y)\bigr| \quad \text{for all } u \in L^2(\mathbb{R})$$

are those described in (4).

We do not know of an earlier proof of that simple fact. This theorem is also reminiscent of Wigner's Unitary-Antiunitary Theorem (see also, e.g., [19,22,24]) which can be stated as follows. Let T be an operator on a Hilbert space H and assume that T preserves the modulus of the scalar product:

$$\bigl|\langle Tx, Ty\rangle\bigr| = \bigl|\langle x, y\rangle\bigr| \quad \text{for all } x, y \in H.$$

Then T is of the form $Tx = \omega(x)Ux$ where ω is a scalar valued function on H such that $|\omega(x)| = 1$ and U is either unitary or anti-unitary operator on H. Here we are in a slightly different situation and Wigner's theorem cannot be applied. It does nevertheless ask whether the (anti)linearity assumption in Theorem C may be removed.

Finally, we also consider a further restriction of the Discrete Ambiguity Problem by considering sequences in $\ell^2(\Lambda)$ for some $\lambda \subset \mathbb{Z}$. This is natural since most of the known examples of signals with strange partners are of the form

$$u(t) = \sum_{j\in\Lambda} c_j \chi_{[0,\eta]+j},$$

at least when Λ has "enough gaps" (see, e.g., [12]). Indeed, partners of $u(t)$ can be easily obtained by multiplying each c_j by a unimodular constant $\exp(i\omega_j)$. Here we clarify the nature of these "gaps" in terms of arithmetic conditions which appear in the classical theory of trigonometric series with gaps. More precisely, we assume that Λ is a B_2 or a B_3-set (see Remark 3.7 for precise definitions). In particular, and as a consequence of our results we obtain the following

Theorem D. *Let $u(t) = \sum_{j\in\Lambda} c_j \chi_{[0,\eta]+j}$. Then, if Λ is a B_2-set, then for all real ω_j,*

$$v(t) = \sum_{j\in\Lambda} e^{i\omega_j} c_j \chi_{[0,\eta]+j}$$

is a partner of $u(t)$. Moreover, if $0 \leqslant \eta \leqslant \frac{1}{3}$ and if Λ is a finite B_3-set these are all partners of u.

Nevertheless, recall from [9] or formula (50) below, that already when $\Lambda = \{0, 1, 2, 3\}$ there exist exceptional cases when strange solutions cannot be classified in terms of gaps.

The article is organized as follows. In the next section, we concentrate on the continuous problem for Hermite functions, and we prove Theorem A. The following section is devoted to the characterization of trivial solutions, both in the discrete case and in the continuous case. The last section is devoted to the case of pulse type signals. We start by proving Theorem B and conclude by recalling and completing the main results of [9].

2. The ambiguity problem for Hermite functions

We now prove Theorem A. We will need a certain number of steps in the proof. The two first ones are mainly due to de Buda [5] and [6]. In particular, de Buda has established the stability of the class of Hermite signals for the ambiguity problem using an elementary proof (which is not complete in [5]). It can also be obtained as a consequence of the *uncertainty principle for ambiguity functions*, as it is mentioned in [3].

2.1. Stability of Hermite functions for the ambiguity problem

Lemma 2.1. *Let $u(t) = P(t)e^{-t^2/2}$, where $P(t)$ is a polynomial. Then, except perhaps for a trivial transformation, every ambiguity partner v of u is of the form $v(t) = Q(t)e^{-t^2/2}$, where $Q(t)$ is a polynomial with* $\deg P = \deg Q$.

Proof. Using the fact $\mathcal{F}(e^{-t^2/2})(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$, an elementary computation shows

$$A(u)(x, y) = e^{-i\frac{xy}{2}}\widetilde{P}(x, y)e^{-\frac{x^2+y^2}{4}},$$

where $\widetilde{P}(x, y)$ is a polynomial of 2 variables of total degree $2\deg P$ (see, e.g., [29, Theorem 7.2] or (11) below). Then,

$$\left|A(v)(x, y)\right|^2 = \left|A(u)(x, y)\right|^2 = \left|\widetilde{P}(x, y)\right|^2 e^{-\frac{x^2+y^2}{2}},$$

so we can use the uncertainty principle in [3, Proposition 6.2] (see also [10]) to conclude $v(t) = Q(t)e^{i\omega t}e^{-\frac{(t-a)^2}{2}}$, for a polynomial Q and two real constants ω, a. We only need to show that $\deg Q = \deg P$, but this follows easily from

$$\left|A(v)(x, y)\right| = \left|\widetilde{Q}(x, y)\right|e^{-\frac{x^2+y^2}{4}} = \left|\widetilde{P}(x, y)\right|e^{-\frac{x^2+y^2}{4}},$$

and the fact $2\deg Q = \deg \widetilde{Q} = \deg \widetilde{P} = 2\deg P$. $\square$

2.2. Reformulation of the ambiguity problem as an algebraic problem

Let us first give some notation that we will use in this section.

Notation. We say that a polynomial is *monic* when the coefficient of its term of higher degree is equal to 1.

For a polynomial $\Pi \in \mathbb{C}[Z]$, we will write Π^* the polynomial given by $\Pi^*(z) = \overline{\Pi(\bar{z})}$.

For a polynomial Π of degree n, that is, $\Pi \in \mathbb{C}_n[Z]$, we write $\check{\Pi}(z) = (-1)^n\Pi(-z)$. Note that $(\Pi')^\vee = (\check{\Pi})'$. We will thus write unambiguously $\check{\Pi}'$. Remark also that $\check{\Pi}$ is monic when Π is.

For Π, Ψ two polynomials, we write

$$\{\Pi, \Psi\}_- = \Pi\check{\Psi} - \check{\Pi}\Psi, \qquad \{\Pi, \Psi\}_+ = \Pi\check{\Psi} + \check{\Pi}\Psi. \tag{9}$$

We shall prove that the ambiguity problem for Hermite functions is equivalent to an algebraic problem, which we state now. For $\mathcal{P} \in \mathbb{C}_n[Z]$, we define its ambiguity polynomial as the polynomial in two variables given by

$$A_{\mathcal{P}}(z, w) := \sum_{m=0}^{n} \frac{1}{m!}\mathcal{P}^{(m)}(z)\mathcal{P}^{*(m)}(w).$$

Note that $A_{\mathcal{P}} = A_{\mathcal{Q}}$ if and only if there exists some unimodular constant c such that $\mathcal{P} = c\mathcal{Q}$.

The ambiguity problem for Hermite functions will then be reduced to the following one:

The algebraic ambiguity problem. *For a given polynomial $\mathcal{P}$ of degree n, find all polynomials $\mathcal{Q}$ for which one has the following identity*:

$$A_{\mathcal{P}}(z,w)A_{\mathcal{P}}(-z,-w)=A_{\mathcal{Q}}(z,w)A_{\mathcal{Q}}(-z,-w). \tag{10}$$

Again, our question is the following: Does there exist other partners than the trivial ones, given by $c\mathcal{P}$ and $c\check{\mathcal{P}}$, with c a unimodular constant?

We first prove the equivalence between the two problems.

Let us denote by

$$H_k(x)=(-1)^k e^{x^2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\big(e^{-x^2}\big),\quad k=0,1,2,\dots,$$

the Hermite polynomials. We recall that, with the normalizing constant $\gamma_k=(\sqrt{\pi}2^k k!)^{\frac12}$, the system

$$\psi_k(x)=\frac{1}{\gamma_k}H_k(x)e^{-x^2/2},\quad k=0,1,2,\dots,$$

is an orthonormal basis of $L^2(\mathbb{R})$, called the *Hermite basis* of L^2.

Let $\mathcal{B}$ be the linear map on $\mathbb{C}[Z]$ defined by $\mathcal{B}(H_k)=2^{k/2}Z^k$ (i.e. $\mathcal{B}$ is the Bargmann transform).

The equivalence between the two problems is given by the following lemma, which is essentially contained in [6].

Lemma 2.2. *Let P and Q be two polynomials. Then $Pe^{-t^2/2}$ and $Qe^{-t^2/2}$ are ambiguity partners if and only if $\mathcal{B}(P)$ and $\mathcal{B}(Q)$ are partners for the algebraic ambiguity problem.*

Proof. First of all, an explicit computation gives the well-known formula (see, e.g., [29, Theorem 7.2])

$$A\big(H_je^{-t^2/2},H_ke^{-t^2/2}\big)(x,y)=\mathcal{L}_{jk}(x/\sqrt{2},y/\sqrt{2})e^{-(x^2+y^2)/4}e^{i\frac{xy}{2}}, \tag{11}$$

where $\mathcal{L}_{j,k}$ is the Laguerre polynomial defined by

$$\mathcal{L}_{j,k}(x,y)=\gamma_j\gamma_k\sqrt{\frac{k!}{j!}}(x+iy)^{j-k}\sum_{\ell=0}^{k}\binom{j}{k-\ell}\frac{(-1)^\ell}{\ell!}\big(x^2+y^2\big)^\ell,\quad \text{if } j\geqslant k$$

(and $\mathcal{L}_{j,k}(x,y)=\mathcal{L}_{k,j}(-x,y)$ if $j<k$). We can write this formula in a unified way as

$$\mathcal{L}_{j,k}(x,y)=\sqrt{\pi 2^{j+k}}\,j!k!\sum_{m=0}^{j\wedge k}\frac{(x+iy)^{j-m}(-x+iy)^{k-m}}{(j-m)!(k-m)!m!}.$$

Thus, defining the new variable $z=x+iy$ we have

$$\begin{aligned}A\big(H_je^{-t^2/2},H_ke^{-t^2/2}\big)(x,y)&=\sqrt{\pi}\,j!k!\left(\sum_{m=0}^{j\wedge k}\frac{2^m}{m!}\frac{z^{j-m}(-\bar z)^{k-m}}{(j-m)!(k-m)!}\right)e^{-|z|^2/4}e^{i\frac{xy}{2}}\\&=\sqrt{\pi}\,j!k!\left(\sum_{m=0}^{j\wedge k}\frac{2^m}{m!}\frac{\partial^m}{\partial t^m}\left(\frac{t^j}{j!}\right)\bigg|_{t=z}\frac{\partial^m}{\partial t^m}\left(\frac{t^k}{k!}\right)\bigg|_{t=-\bar z}\right)e^{-|z|^2/4}e^{i\frac{xy}{2}}.\end{aligned}$$

Now, consider the expansion of P and Q in terms of the basis of Hermite polynomials

$$P=\sum_{j=0}^{n}\alpha_jH_j\quad\text{and}\quad Q=\sum_{j=0}^{n}\beta_jH_j \tag{12}$$

with $\alpha_n\neq 0$, $\beta_n\neq 0$. Then, calling $\mathcal{P}:=\mathcal{B}(P)=\sum_{j=0}^n\alpha_j2^{j/2}Z^j$, and using the bilinearity of the operator A we have

$$\big|A(u)(x,y)\big|=\sqrt{\pi}\left|\sum_{m=0}^{n}\frac{1}{m!}\mathcal{P}^{(m)}(\sqrt{2}z)\overline{\mathcal{P}^{(m)}(-\sqrt{2}z)}\right|e^{-|z|^2/4}.$$

Calling $\mathcal{Q} := \mathcal{B}(Q) = \sum_{j=0}^{n} \beta_j 2^{j/2} Z^j$, the fact that u and v are partners is equivalent to the identity

$$\left| \sum_{m=0}^{n} \frac{1}{m!} \mathcal{P}^{(m)}(z) \overline{\mathcal{P}^{(m)}(-z)} \right|^2 = \left| \sum_{m=0}^{n} \frac{1}{m!} \mathcal{Q}^{(m)}(z) \overline{\mathcal{Q}^{(m)}(-z)} \right|^2 \tag{13}$$

for all complex numbers z. Since two holomorphic polynomials in two complex variables z, w coincide when they coincide for $z = -\overline{w}$, this is equivalent to the identity

$$\begin{aligned} &\left(\sum_{m=0}^{n} \frac{1}{m!} \mathcal{P}^{(m)}(z)\, \mathcal{P}^{*\,(m)}(w) \right) \left(\sum_{m=0}^{n} \frac{1}{m!} \mathcal{P}^{(m)}(-z)\, \mathcal{P}^{*\,(m)}(-w) \right) \\ &\quad = \left(\sum_{m=0}^{n} \frac{1}{m!} \mathcal{Q}^{(m)}(z)\, \mathcal{Q}^{*\,(m)}(w) \right) \left(\sum_{m=0}^{n} \frac{1}{m!} \mathcal{Q}^{(m)}(-z)\, \mathcal{Q}^{*\,(m)}(-w) \right). \end{aligned} \tag{14}$$

We recognize the algebraic ambiguity problem, which finishes the proof of the lemma. □

Remark 2.3. Note that the highest order coefficient in (14) is $|\alpha_n|^4 = |\beta_n|^4$, so that $|\beta_n| = |\alpha_n|$. Replacing $\mathcal{Q}$ by its trivial partner $\widetilde{\mathcal{Q}} = \frac{\alpha_n}{\beta_n} \mathcal{Q}$, we may thus assume that $\beta_n = \alpha_n$. Then, using the homogeneity of Eq. (14), there is no loss of generality to assume that $\beta_n = \alpha_n = 1$.

2.3. Solution of the algebraic ambiguity problem in the generic case

Definition. By a generic polynomial $\mathcal{P}$ we mean a polynomial that has only simple roots and has no common root with $\check{\mathcal{P}}$, that is, $\mathcal{P}$ has only simple non-symmetric roots.

Of course, almost all and quasi-all polynomials are generic.

We will now prove the following theorem which implies Theorem A.

Theorem 2.4. *Assume that the polynomial $\mathcal{P}$ is generic and let $\mathcal{Q}$ be a partner of $\mathcal{P}$. Then $\mathcal{Q}$ is a trivial partner, that is, there exists a unimodular constant c such that either $\mathcal{Q} = c\mathcal{P}$ or $\mathcal{Q} = c\check{\mathcal{P}}$.*

The proof is divided into two steps. In the first one, we will directly use Eq. (14) to get substantial information on $\mathcal{Q}$. The second step will consist in exploiting the factorization that $\mathcal{A}(\mathcal{P})$ would have if $\mathcal{Q}$ were not a trivial partner.

First step. As explained in Remark 2.3, we can assume that $\mathcal{P}$ and $\mathcal{Q}$ are monic polynomials and write

$$\mathcal{P} := Z^n + p_1 Z^{n-1} + \cdots + p_{n-1} Z + p_n, \qquad \mathcal{Q} := Z^n + q_1 Z^{n-1} + \cdots + q_{n-1} Z + q_n.$$

Equation (14) can as well be written

$$A_{\mathcal{P}} A_{\check{\mathcal{P}}} = A_{\mathcal{Q}} A_{\check{\mathcal{Q}}}.$$

Looking at $A_{\mathcal{P}} \in \mathbb{C}[Z, W]$ as a polynomial in W with coefficients in $\mathbb{C}[Z]$, we can write

$$A_{\mathcal{P}} \equiv \mathcal{P} W^n + (\bar{p}_1 \mathcal{P} + n\mathcal{P}') W^{n-1} + \left(\bar{p}_2 \mathcal{P} + (n-1)\bar{p}_1 \mathcal{P}' + \frac{n(n-1)}{2} \mathcal{P}'' \right) W^{n-2}$$

modulo terms of smaller degree. Looking at the coefficient of W^{2n} in (14), we get

$$\mathcal{P}\check{\mathcal{P}} = \mathcal{Q}\check{\mathcal{Q}}, \tag{15}$$

which in particular implies that

$$\mathcal{P}'\check{\mathcal{P}} + \mathcal{P}\check{\mathcal{P}}' = \mathcal{Q}'\check{\mathcal{Q}} + \mathcal{Q}\check{\mathcal{Q}}' \tag{16}$$

and

$$\mathcal{P}''\check{\mathcal{P}} + \mathcal{P}\check{\mathcal{P}}'' + 2\mathcal{P}'\check{\mathcal{P}}' = \mathcal{Q}''\check{\mathcal{Q}} + \mathcal{Q}\check{\mathcal{Q}}'' + 2\mathcal{Q}'\check{\mathcal{Q}}'.$$

Then, looking at the coefficient of W^{2n-2} in (14),[1] an elementary computation which uses the previous identities leads to

$$n\mathcal{P}'\check{\mathcal{P}}' + \bar{p}_1(\mathcal{P}\check{\mathcal{P}}' - \check{\mathcal{P}}\mathcal{P}') = n\mathcal{Q}'\check{\mathcal{Q}}' + \bar{q}_1(\mathcal{Q}\check{\mathcal{Q}}' - \check{\mathcal{Q}}\mathcal{Q}'). \tag{17}$$

The highest order term in this equation gives $|q_1| = |p_1|$.

From (15) we deduce that there exist two monic polynomials A and B such that

$$\mathcal{P} = AB \quad \text{and} \quad \mathcal{Q} = A\check{B}. \tag{18}$$

Let us further write

$$A := Z^k + a_1Z^{k-1} + \cdots + a_k \quad \text{and} \quad B := Z^l + b_1Z^{l-1} + \cdots + a_l.$$

Then $p_1 = a_1 + b_1$, $q_1 = a_1 - b_1$ and $|q_1| = |p_1|$ is equivalent to

$$a_1\bar{b}_1 + \bar{a}_1b_1 = 0. \tag{19}$$

These relations, written for all possible decompositions of $\mathcal{P}$ as a product AB, is sufficient to prove that the set of coefficients $p_1, \dots, p_n$ is contained in a real analytic variety of codimension 1 in $\mathbb{C}^n$, and imply Theorem A. We will not give details for this reduction since we have more information, as stated in Theorem 2.4.

Note that, using the notations defined by (9), (17) may as well be written as

$$2\bar{a}_1A\check{A}\{B', B\}_- + 2\bar{b}_1B\check{B}\{A', A\}_- + n\{A', A\}_-\{B', B\}_- = 0. \tag{20}$$

Remark that the condition $\{A', A\}_- = 0$, which may be written as well as $\frac{A'}{A} = \frac{\check{A}'}{\check{A}}$, is equivalent to the fact that $\check{A} = A$. If a_1 is 0, then either $\{A', A\}_- = 0$, which means that $Q = \check{P}$, or $2\bar{b}_1B\check{B} + n\{B', B\}_- = 0$. This last identity is only possible when $b_1 = 0$, and thus $\{B', B\}_- = 0$. So $Q = P$. In particular, we have proved the following. At this point, $\mathcal{P}$ is not necessarily generic.

Proposition 2.5. *Assume that the polynomial $\mathcal{P}$ is such that $p_1 = 0$. Let $\mathcal{Q}$ be a partner of $\mathcal{P}$. Then $\mathcal{Q}$ is a trivial partner, that is, there exists a unimodular constant c such that either $\mathcal{Q} = c\mathcal{P}$ or $\mathcal{Q} = c\check{\mathcal{P}}$.*

We will now concentrate on the case when a_1 and b_1 are different from zero, and $\mathcal{P}$ (thus $\mathcal{Q}$) is generic. As A (respectively B) has no multiple or symmetric zeros, then $A\check{A}$ and $\{A', A\}_-$ (respectively $B\check{B}$ and $\{B', B\}_-$) are mutually prime. Moreover, zeros of $A\check{A}$ and $B\check{B}$ are different. It follows from (20) that $2\bar{b}_1B\check{B} + n\{B', B\}_-$ can be divided by $A\check{A}$, while $2\bar{a}_1A\check{A} + n\{A', A\}_-$ can be divided by $B\check{B}$. So A and B have the same degree. We conclude directly that there is a contradiction when n is odd. From now on, we assume that $n = 2k$. Then A and B have degree k. Moreover, looking at terms of higher degree, we conclude that

$$n\{B', B\}_- = 2\bar{b}_1(A\check{A} - B\check{B}). \tag{21}$$

Differentiating (21), we obtain

$$2\bar{b}_1\big(\{A', A\}_+ - \{B', B\}_+\big) = n\{B'', B\}_-. \tag{22}$$

We can exchange the roles of A and B in the previous identities. In particular, we get that

$$\bar{b}_1\{A', A\}_- + \bar{a}_1\{B', B\}_- = 0. \qquad \square \tag{23}$$

Second step. We will now work with polynomials in two variables. For $\Pi \in \mathbb{C}[Z, W]$, we define $\check{\Pi}$ as before, the degree of a polynomial being taken as the total degree. Using the fact that $A_{\mathcal{P}}\check{A}_{\mathcal{P}} = A_{\mathcal{Q}}\check{A}_{\mathcal{Q}}$, we know that there exists a factorization with polynomials C, D in two variables, such that

$$A_{\mathcal{P}} = CD, \qquad A_{\mathcal{Q}} = C\check{D}. \tag{24}$$

[1] The coefficient of W^{2n-1} only leads to Eq. (16).

Let us consider C and D as polynomials in the variable W with coefficients that are polynomials in Z, and write

$$C \equiv C_0 W^\alpha \quad \text{(modulo polynomials in } W \text{ of lower degree)},$$
$$D \equiv D_0 W^\beta \quad \text{(modulo polynomials in } W \text{ of lower degree)}.$$

Then $\mathcal{P} = C_0 D_0$, while $\mathcal{Q} = \varepsilon C_0 \check{D}_0$, with $\varepsilon = (-1)^{\deg D + \deg D_0 + \beta}$. The assumption that $\mathcal{P}$ is generic implies that there is uniqueness in the factorization (18). So C_0 is equal to A (up to a constant) and D_0 is equal to B (up to a constant). Exchanging the role of the two variables, we see that $\alpha = \beta = k$. So $\varepsilon = 1$, and we can assume that C_0 and D_0 are monic, so that $C_0 = A$ and $D_0 = B$.

These considerations allow us to write

$$C(z,w) \equiv A(z)A^*(w) + C_1(z)w^{k-1}, \qquad D \equiv B(z)B^*(w) + D_1(z)w^{k-1} \tag{25}$$

(modulo polynomials in W of lower degree). Moreover, C_1 and D_1 have degree at most $k-1$.

We shall now identify A_1 and B_1.

Writing $A_{\mathcal{P}}$ as a product, we have that

$$A_{\mathcal{P}}(z,w) \equiv \mathcal{P}(z)\mathcal{P}^*(w) + \big[A(z)D_1(z) + C_1(z)B(z)\big]w^{n-1}$$

(modulo polynomials in W of lower degree), whereas a direct computation, using the fact that $\mathcal{P} = AB$ shows that

$$A_{\mathcal{P}}(z,w) \equiv \mathcal{P}(z)\mathcal{P}^*(w) + n\big[A(z)B'(z) + A'(z)B(z)\big]w^{n-1}$$

(modulo polynomials in W of lower degree). Comparing both expressions leads to

$$(nA' - C_1)B + (nB' - D_1)A = 0. \tag{26}$$

Our assumption on the zeros of $\mathcal{P}$ implies that A and B are mutually prime so that, using the information on the degrees of C_1, D_1, we get that

$$C_1 = nA' \quad \text{and} \quad D_1 = nB'.$$

Symmetry considerations now imply that

$$C(z,w) \equiv A(z)A^*(w) + 2A'(z)A'^*(w) + C_2(z)w^{k-2},$$
$$D(z,w) \equiv B(z)B^*(w) + 2B'(z)B'^*(w) + D_2(z)w^{k-2}$$

(modulo polynomials in W of lower degree). Moreover, C_2 and D_2 have degree at most $k-2$.

It then follows that

$$\begin{aligned} A_{\mathcal{P}}(z,z) &\equiv \mathcal{P}(z)\mathcal{P}^*(w) + \mathcal{P}'(w)\mathcal{P}'^*(w) + \big(A(z)\big[(\bar{a}_1 - \bar{b}_1)B'(z) + D_2(z)\big] \\ &\qquad + B(z)\big[(\bar{b}_1 - \bar{a}_1)A'(z) + C_2(z)\big] + n^2 A'(z)B'(z)\big)w^{n-2} \\ &\equiv \mathcal{P}(z)\mathcal{P}^*(w) + \mathcal{P}'(z)\mathcal{P}'^*(w) + \frac{n(n-1)}{2}\big(A''(z)B(z) + 2A'(z)B'(z) + A(z)B''(z)\big)w^{n-2} \end{aligned}$$

(modulo polynomials in W of lower degree). It follows that

$$(\bar{a}_1 - \bar{b}_1)(AB' - A'B) + AF + BE + nA'B' = 0, \tag{27}$$

where $E := C_2 - \frac{n(n-1)}{2}A''$ and $F := D_2 - \frac{n(n-1)}{2}B''$. Exploiting the expressions of $A_{\mathcal{Q}}$, that is changing B into $\check{B}$ (thus also b_1 into $-b_1$ and F into $\check{F}$), we get

$$(\bar{a}_1 + \bar{b}_1)(A\check{B}' - A'\check{B}) + A\check{F} + \check{B}E + nA'\check{B}' = 0. \tag{28}$$

Let us multiply the left-hand side of (27) by $\check{B}$ and the left-hand side of (28) by B, and take the difference. We obtain that

$$A\big(\bar{a}_1\{B', B\}_- - \bar{b}_1\{B', B\}_+ + \{F, B\}_-\big) + A'\big(2\bar{b}_1 B\check{B} + n\{B', B\}_+\big) = 0.$$

Using (21) and (23), we can write that

$$A'\big(2\bar{b}_1 B\check{B} + n\{B', B\}_+\big) = 2\bar{b}_1 AA'\check{A} = A\big(\bar{b}_1\{A', A\}_+ - \bar{a}_1\{B', B\}_-\big).$$

Finally, using (22), we obtain the identity

$$\left\{F + \frac{n}{2}B'', B\right\}_- = 0.$$

Since B and $\check{B}$ are mutually prime by assumption, this means that $F = -\frac{n}{2}B''$.

We could as well prove that $E = -\frac{n}{2}A''$. If we compute the coefficient of the term of higher degree in the left-hand side of (27), we obtain $|a_1|^2 + |b_1|^2 + n$, which cannot vanish. This concludes for the proof. □

Remark 2.6. We will need the following: for all integers $n \neq m$ and $a \in \mathbb{C}$, then $\psi_n + a\psi_m$ has only trivial partners for the ambiguity problem (equivalently, $Z^n + aZ^m$ has only trivial partners for the algebraic ambiguity problem). Indeed, for $|n - m| \geqslant 2$ this is a consequence of Proposition 2.5. For $|n - m| = 1$, this follows directly from (15).

3. Trivial solutions and constructions of special strange partners

3.1. The discrete case

We refer to Problem 3 as Problem (P). In this setting, two sequences a and b are said to be *discrete ambiguity partners* (*or* (P)*-partners*) whenever (8) holds.

We start by defining the dual problem of (P), when 2π-periodic functions, rather than sequences in $\mathbb{Z}$, are considered. Here $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \equiv [0, 2\pi)$, and for $f \in L^2(\mathbb{T})$ we let

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-int}\, \mathrm{d}t, \quad n \in \mathbb{Z}.$$

In this way, one can write $f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$ in the usual $L^2(\mathbb{T})$ sense (and a.e.). We shall also identify $L^2(\mathbb{T})$ with $\ell^2(\mathbb{Z})$ via the correspondence: $f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$. This gives the following equivalent formulation of (P).

($\hat{\mathrm{P}}$) The Periodic Ambiguity Problem. *For $f \in L^2(\mathbb{T})$ define the periodic ambiguity function by*

$$\hat{\mathcal{A}}(f)(k,t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\overline{f(s-t)}e^{-iks}\, \mathrm{d}s, \quad (k,t) \in \mathbb{Z}\times\mathbb{T}.$$

We want to find all $g \in L^2(\mathbb{T})$ such that

$$\left|\hat{\mathcal{A}}(f)(k,t)\right| = \left|\hat{\mathcal{A}}(g)(k,t)\right| \quad \textit{for all } (k,t) \in \mathbb{Z}\times\mathbb{T}.$$

Two functions f and g as above are called ($\hat{\mathrm{P}}$)*-partners.*

Note that f and g are $\hat{P}$-partners if and only if the sequences of their Fourier coefficients $\{\hat{f}(n)\}$ and $\{\hat{g}(n)\}$ are (P)-partners in the sense of (8) since Parseval's formula gives

$$\hat{\mathcal{A}}(f)(k,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{f}(n-k)}e^{int} = \mathcal{A}\big(\{\hat{f}(n)\}\big)(k,t). \tag{29}$$

In the sequel, we will therefore write $\mathcal{A}(f)$ instead of $\hat{\mathcal{A}}(f)$ to simplify the notation.

Let us give a precise definition of trivial solutions, as announced in the introduction. Intuitively these should be simple transformations of the data function that always give solutions to the functional equation proposed. The definition below, given for ($\hat{\mathrm{P}}$) easily adapts to other problems.

Definition. A *trivial solution for* ($\hat{\mathrm{P}}$) is a bounded linear operator $R : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ preserving ($\hat{\mathrm{P}}$)-partners, i.e., such that for every $f \in L^2(\mathbb{T})$, f and Rf are ($\hat{\mathrm{P}}$)-partners. We denote by $\mathcal{T}$ the semi-group of all such operators.

Example. Let $\mathbb{H} \equiv \mathbb{T} \times \mathbb{Z} \times \mathbb{T}$, and define for $h = (\alpha, k, \beta) \in \mathbb{H}$

$$R_h f(t) = e^{i\beta} e^{ikt} f(t+\alpha) \quad \text{and} \quad \widetilde{R}_h f(t) = e^{i\beta} e^{-ikt} f(-t+\alpha).$$

Then R_h and $\widetilde{R}_h$ are trivial solutions for $(\hat{\mathrm{P}})$. Note that R_h is a unitary representation of the periodized Heisenberg group $\mathbb{H}$, with the product defined by

$$h \cdot h' = (\alpha, k, \beta) \cdot (\alpha', k', \beta') = (\alpha + \alpha', k + k', \beta + \beta' + k'\alpha)$$

while as before $\widetilde{R}_h = Z R_h$.

Let us prove that there are no other trivial solutions.

Proposition 3.1. *Let R be a trivial solution for $(\hat{\mathrm{P}})$, then there exists $h \in \mathbb{H}$ such that, either $R = R_h$ or $R = \widetilde{R}_h$. In particular, $\mathcal{T}$ can be identified with the group $\{-1, 1\} \times \mathbb{H}$.*

Proof. For $n \in \mathbb{Z}$, let $f_n(t) = e^{int}$. Then, $|\mathcal{A}(f_n)(k,t)| = \delta_{0,k}$, where $\delta_{0,k}$ is the usual Krönecker symbol. Moreover,

$$\left|\mathcal{A}(Rf_n)(k,t)\right| = \left|\sum_{\ell \in \mathbb{Z}} \widehat{Rf_n}(\ell) \overline{\widehat{Rf_n}(\ell - k)} e^{i\ell t}\right| = \delta_{0,k}$$

implies that there exists a unique $m(n)$ such that $\widehat{Rf_n}(m(n)) \neq 0$, that is $Rf_n(t) = c_n e^{im(n)t}$ with $|c_n| = 1$. Note that, if $n_1 \neq n_2$, then $m(n_1)$ and $m(n_2)$ are different. Indeed, if they were equal, the non-zero function $g := c_{n_2} f_{n_1} - c_{n_1} f_{n_2}$ would have a zero radar ambiguity function, a clear contradiction.

We wish to show that either $m(n) - n$ or $m(n) + n$ is a constant. Let us consider the test functions $g(t) = e^{in_1t} + e^{in_2t}$, for distinct $n_1, n_2 \in \mathbb{Z}$. Then $Rg(t) = c_{n_1} e^{im(n_1)t} + c_{n_2} e^{im(n_2)t}$, and therefore,

$$\left|\mathcal{A}(g)(0,t)\right| = \left|e^{in_1t} + e^{in_2t}\right| = \left||c_{n_1}|^2 e^{im(n_1)t} + |c_{n_2}|^2 e^{im(n_2)t}\right| = \left|\mathcal{A}(Rg)(0,t)\right|.$$

This implies, $|m(n_1) - m(n_2)| = |n_1 - n_2|$, which is an isometry of the integers, and therefore of the form $m(n) = m(0) + \varepsilon n$, with a constant $\varepsilon = \pm 1$. In particular, when $\varepsilon = 1$ we have

$$(Rf_n)(t) = c_n e^{im(0)t} f_n(t). \tag{30}$$

We shall show that actually $R = R_h$ for some $h \in \mathbb{H}$. The case $\varepsilon = -1$, then follows by replacing R by RZ.

So, assuming (30), let us establish the dependence of c_n on n. Testing with $h_n(t) = e^{int} + e^{i(n+1)t} + e^{i(n+2)t}$, we obtain

$$\left|\mathcal{A}(h_n)(1,t)\right| = \left|1 + e^{it}\right| = \left|1 + c_n \overline{c_{n+1}}^2 c_{n+2} e^{it}\right| = \left|\mathcal{A}(Rh_n)(1,t)\right|.$$

Therefore $\overline{c_{n+1}} c_{n+2} = \overline{c_n} c_{n+1}$. Writing $c_n = e^{i\gamma(n)}$, this relation can be expressed as

$$\gamma(n+2) - \gamma(n+1) = \gamma(n+1) - \gamma(n) = \cdots = \gamma(1) - \gamma(0) \pmod{2\pi}.$$

Hence, for some $\alpha \in \mathbb{T}$, we must have

$$\gamma(n) = \alpha + \gamma(n-1) = \cdots = n\alpha + \gamma(0) \pmod{2\pi},$$

concluding that

$$(Rf_n)(t) = c_0 e^{in\alpha} e^{im(0)t} f_n(t) = c_0 e^{im(0)t} f_n(t + \alpha).$$

Then, the linearity and boundedness of R give $R = R_h$, where $h = (\alpha, m(0), \gamma(0))$. □

Remark 3.2. It is worthwhile to notice that, from the above proof, an anti-linear bounded operator R cannot preserve $(\hat{\mathrm{P}})$-partners. Indeed, in the last step of the proof one may test with a function $f(t) = 1 + e^{it} + ce^{2it}$, for $|c| = 1$. Then $|\mathcal{A}(f)(1,t)| = |1 + ce^{it}|$, whereas if R were antilinear, $|\hat{\mathcal{A}}(f)(1,t)| = |\mathcal{A}(Rf)(1,t)| = |1 + \bar{c}e^{it}|$. This excludes anti-linear operators to give trivial solutions for $(\hat{\mathrm{P}})$.

A normalization remark. Let $f \in L^2(\mathbb{T})$ be a trigonometric polynomial. Then, up to a change $f \mapsto e^{ikt} f$, we may assume that $\operatorname{supp} \hat{f} \subset \{0, \ldots, N\}$ for some integer N and that $\hat{f}(0) \neq 0$, $\hat{f}(N) \neq 0$. We then say that $f \in \mathcal{P}_N$.

The next lemma shows in particular that there is no loss of generality if we restrict the study of the discrete radar ambiguity problem to functions in $\mathcal{P}_N$ when dealing with trigonometric polynomials.

Lemma 3.3. *Let $f \in L^2(\mathbb{T})$ and let $\Lambda = \operatorname{supp} f$. Then*

$$\operatorname{supp} \mathcal{A}(f) := \{k\colon\ \mathcal{A}(f)(k,t) \text{ is not identically } 0\} = \Lambda - \Lambda.$$

In particular, if $f \in \mathcal{P}_N$ for some $N \in \mathbb{N}$, and if g is a (P̂)*-partner of f then, up to replacing g by a trivial partner, we may also assume that $g \in \mathcal{P}_N$.*

Proof. The nth Fourier coefficients of $t \mapsto \mathcal{A}(f)(k,t)$, namely $\hat{f}(n)\overline{\hat{f}(n-k)}$, will vanish unless $n, n-k \in \Lambda$, so that $\operatorname{supp}\mathcal{A}(f) = \Lambda - \Lambda$.

If $f \in \mathcal{P}_N$ then $\underline{\Lambda \subset}$ $\{0,\dots,N\}$, thus $\operatorname{supp}\mathcal{A}\underline{(f)} \subset \{0,\dots,N\} - \{0,\dots,N\} = \{-N,\dots,N\}$. Obviously $\mathcal{A}(f)(-N,t) = \hat{f}(0)\overline{\hat{f}(N)} \neq 0$, $\mathcal{A}(f)(N,t) = \hat{f}(N)\overline{\hat{f}(0)}e^{iNt} \neq 0$, thus $\operatorname{supp}\mathcal{A}(f)$ cannot be included in a smaller interval.

Now, if $g \in L^2(\mathbb{T})$ is a $(\hat{P})$-partner of f, then $\Lambda' := \operatorname{supp}\hat{g}$ is such that $\Lambda' - \Lambda'$ is finite, thus Λ' itself is finite. Thus g is a trigonometric polynomial, thus we may assume that $g \in \mathcal{P}_M$ for some M. The first part of the proof then shows that $M = N$. $\square$

Finally, it is obvious from the definition that if $f \in \mathcal{P}_N$ with $N = 0$ or $N = 1$, then f has only trivial partners.

3.2. Restricted discrete problems

In this section we consider the discrete radar ambiguity problem $(\hat{P})$ restricted to the subspaces $L^2_\Lambda(\mathbb{T})$. Recall that, for Λ a subset of $\mathbb{Z}$, this space consists of all functions $f \in L^2(\mathbb{T})$ with $\operatorname{supp}\hat{f} \subset \Lambda$. The discrete radar ambiguity problem may then be restricted in two ways:

The Ambiguity Problems in $L^2_\Lambda(\mathbb{T})$. *Given $f \in L^2_\Lambda(\mathbb{T})$,*

$\hat{P}_\Lambda$ *find all $g \in L^2(\mathbb{T})$ such that for all $(k,t) \in \mathbb{Z}\times\mathbb{T}$*

$$|\mathcal{A}(f)(k,t)| = |\mathcal{A}(g)(k,t)| \tag{$\hat{P}_\Lambda$}$$

and such a g will be called a $\hat{P}_\Lambda$-partner of f;

$\hat{P}_{\Lambda,\Lambda}$ *find all $g \in L^2_\Lambda(\mathbb{T})$ such that for all $(k,t) \in \mathbb{Z}\times\mathbb{T}$*

$$|\mathcal{A}(f)(k,t)| = |\mathcal{A}(g)(k,t)|. \tag{$\hat{P}_{\Lambda,\Lambda}$}$$

Such a g will be called a $\hat{P}_{\Lambda,\Lambda}$-partner of f.

In other words, the $\hat{P}_\Lambda$-ambiguity problem is just the $\hat{P}$-ambiguity problem for functions in $L^2_\Lambda(\mathbb{T})$ whereas in the $\hat{P}_{\Lambda,\Lambda}$-ambiguity problem one further seeks for the solutions of the $\hat{P}$-ambiguity partners to be in $L^2_\Lambda(\mathbb{T})$.

Restricted trivial solutions may now be defined in two natural ways:

- an operator $R : L^2_\Lambda(\mathbb{T}) \to L^2(\mathbb{T})$ such that, for every $f \in L^2_\Lambda(\mathbb{T})$, f and $g = Rf$ satisfy $(\hat{P}_\Lambda)$ will be called a *$\hat{P}_\Lambda$-trivial solution*;
- an operator $R : L^2_\Lambda(\mathbb{T}) \to L^2_\Lambda(\mathbb{T})$ such that, for every $f \in L^2_\Lambda(\mathbb{T})$, f and $g = Rf$ satisfy $(\hat{P}_{\Lambda,\Lambda})$ will be called a *$\hat{P}_{\Lambda,\Lambda}$-trivial solution.*

Of course, every $\hat{P}_{\Lambda,\Lambda}$-trivial solution is also a $\hat{P}_\Lambda$-trivial solution. The converse may not be true as the trivial solutions $R_{0,k,0}$ and $\widetilde{R}_{0,k,0}$ do not preserve $L^2_\Lambda(\mathbb{T})$ in general. Note also that every trivial solution is a $\hat{P}_\Lambda$-trivial solution. Again the converse may be false as the example below will show.

It is a remarkable fact that the more *lacunary* a sequence Λ is, the more trivial solutions the problem admits.

Notation. For $\Lambda \subset \mathbb{Z}$ and $\mathbf{c} = \{c(n)\}_{n\in\Lambda}$ a sequence of unimodular numbers, we define the (multiplier) operator $R_{\mathbf{c}} : L^2_\Lambda(\mathbb{T}) \to L^2_\Lambda(\mathbb{T})$ by

$$R_{\mathbf{c}} f(t) = \sum_{n\in\Lambda} c(n)\hat{f}(n)e^{int}, \quad t \in \mathbb{T}.$$

This operator is extended to $L^2(\mathbb{T})$ in the obvious way: $R_{\mathbf{c}}e^{int} = 0$ if $n \notin \Lambda$.

Example. Let $\Lambda = \{2^j\}_{j=0}^\infty$. Then, any multiplier $R_{\mathbf{c}}$ is a trivial solution for $(\hat{P}_\Lambda)$, but in general not for $(\hat{P})$. This is due to the fact that $\mathcal{A}f(0,t) = \mathcal{A}R_{\mathbf{c}}f(0,t)$, while

$$\left|\mathcal{A}f\left(2^{k_1} - 2^{k_2}, t\right)\right| = \left|\hat{f}\left(2^{k_1}\right)\overline{\hat{f}\left(2^{k_2}\right)}\right| = \left|\mathcal{A}R_{\mathbf{c}}f\left(2^{k_1} - 2^{k_2}, t\right)\right|,$$

for non-negative integers $k_1 \neq k_2$.

In general, we have the following result:

Proposition 3.4. *Let $\Lambda \subset \mathbb{Z}$. An operator $R : L^2_\Lambda(\mathbb{T}) \to L^2(\mathbb{T})$ is a $(\hat{P}_\Lambda)$-trivial solution if and only if it is of the form $R = SR_{\mathbf{c}}$, where $S \in \mathcal{T}$ and $\mathbf{c} = \{c(n)\}_{n\in\Lambda}$ is a sequence of unimodular constants satisfying*

$$c(n_1)\overline{c(n_2)} = c(n_3)\overline{c(n_4)}, \quad \text{whenever } n_1 - n_2 = n_3 - n_4,\ n_i \in \Lambda,\ i = 1,2,3,4. \tag{31}$$

Proof. The sufficiency is easy to check. Indeed, just notice that for $R = R_{\mathbf{c}}$, using condition (31) one obtains

$$\left|\mathcal{A}(Rf)(k,t)\right| = \left|\sum_{n,n-k\in\Lambda} c(n)\hat{f}(n)\overline{c(n-k)\hat{f}(n-k)}e^{int}\right| = \left|\mathcal{A}(f)(k,t)\right|$$

since $c(n)\overline{c(n-k)}$ depends only on k and is of modulus 1.

For the necessity, it is easy to see that the operator R will act on the exponentials $f_n(t) = e^{int}$ by

$$\text{either} \quad Rf_n(t) = c(n)e^{imt}f_n(t) \quad \text{or} \quad Rf_n(t) = c(n)e^{-imt}f_{-n}(t),$$

for some $m \in \mathbb{Z}$ and $|c(n)| = 1$, $n \in \Lambda$. Indeed, the part of the proof of Proposition 3.1 to give (30) can be used here. Factoring out the corresponding $(\hat{P}_\Lambda)$-trivial operator $S = R_h$ or $S = \widetilde{R}_h$ with $h = (0,m,0) \in \mathbb{H}$, we may assume that $R = R_{\mathbf{c}}$. It remains to determine the relations in (31) among the $c(n)$'s.

Excluding the trivial cases, we have only to check (31) when $n_1 > n_2$ and $n_1 > n_3 > n_4$. This leaves only two possibilities:

Case 1. $n_2 = n_3$. Then testing with $g(t) = e^{in_1t} + e^{in_2t} + e^{in_4t}$, we obtain

$$\left|\mathcal{A}(Rg)(n_1 - n_2, t)\right| = \left|c(n_1)\overline{c(n_2)}e^{in_1t} + c(n_2)\overline{c(n_4)}e^{in_2t}\right| = \left|e^{in_1t} + e^{in_2t}\right|,$$

and consequently, $c(n_1)\overline{c(n_2)} = c(n_2)\overline{c(n_4)}$.

Case 2. $n_2 \neq n_3$. Then, the n_is are all different and we may test with $h(t) = e^{in_1t} + e^{in_2t} + e^{in_3t} + e^{in_4t}$, obtaining:

$$\left|\mathcal{A}(Rh)(n_1 - n_3, t)\right| = \left|e^{in_1t} + e^{in_2t} + \hat{h}(n_3)\overline{\hat{h}(2n_3 - n_1)}e^{in_3t}\right|.$$

Note that $\hat{h}(2n_3 - n_1) \neq 0$ only if $2n_3 - n_1 = n_2$ or n_4. But the last choice implies $n_2 = n_3$, which is not possible. If instead $n_3 - n_1 = n_2 - n_3$, then the previous case gives us the equality

$$c(n_3)\overline{c(n_1)} = c(n_2)\overline{c(n_3)}.$$

Therefore

$$\left|\mathcal{A}(Rh)(n_1 - n_3, t)\right| = \left|c(n_1)\overline{c(n_3)}\left(e^{in_1t} + e^{in_3t}\right) + c(n_2)\overline{c(n_4)}e^{in_2t}\right| = \left|e^{in_1t} + e^{in_2t} + e^{in_3t}\right|,$$

from which we obtain $c(n_1)\overline{c(n_3)} = c(n_2)\overline{c(n_4)}$. When, on the contrary, $\hat{h}(2n_3 - n_1) = 0$, then the situation is simpler since

$$\left|\mathcal{A}(Rh)(n_1 - n_3, t)\right| = \left|c(n_1)\overline{c(n_3)}e^{in_1t} + c(n_2)\overline{c(n_4)}e^{in_2t}\right| = \left|e^{in_1t} + e^{in_2t}\right|,$$

leading to the same result. $\square$

Remark. We shall denote by $\mathcal{T}_\Lambda$ the set of all operators which are trivial solutions for $(\hat{P}_\Lambda)$. Note that now $\mathcal{T}_\Lambda$ is not a semi-group with the usual composition law, unless $\Lambda = \mathbb{Z}$.

The previous proposition, translated into the language of the periodic radar ambiguity problem $(\hat{\mathrm{P}})$, guarantees the existence of *many strange solutions* for every function in $L^2_\Lambda(\mathbb{T})$, provided Λ has enough gaps. Further, we obtain the following:

Corollary 3.5. *The set of functions $f \in L^2(\mathbb{T})$ admitting strange solutions to $(\hat{\mathrm{P}})$ is dense in $L^2(\mathbb{T})$.*

Proof. Consider, for every $N \geqslant 1$, functions with Fourier transform supported in $\Lambda_N = \{-N, \dots, N\} \cup \{3N+1\}$. It is clear that $\bigcup_{N=1}^\infty L^2_{\Lambda_N}(\mathbb{T})$ is dense in $L^2(\mathbb{T})$. Further, any function $f \in L^2_{\Lambda_N}(\mathbb{T})$ will have infinitely many $(\hat{\mathrm{P}})$-strange partners. Indeed, these are given by the $(\hat{P}_\Lambda)$-trivial solutions:

$$R_{\mathbf{c}_N} f(t) = \sum_{n=-N}^{N} \varepsilon \hat{f}(n) e^{int} + \varepsilon' \hat{f}(3N+1) e^{i(3N+1)t},$$

for $|\varepsilon| = |\varepsilon'| = 1$. Since the multiplier $\mathbf{c}_N = \{c(-N) = \cdots = c(N) = \varepsilon, c(3N+1) = \varepsilon'\}$ satisfies condition (iii) of Proposition 3.4, we must have $R_{\mathbf{c}_N} \in \mathcal{T}_{\Lambda_N}$, establishing our claim. $\square$

Here is one more consequence of our proposition, generalizing the example given above. We exclude the case $\mathrm{Card}(\Lambda) = 2$ for which one easily knows all solutions to $(\hat{\mathrm{P}})$ or $(\hat{P}_\Lambda)$.

Corollary 3.6. *Let $\Lambda \subset \mathbb{Z}$ be such that Card$(\Lambda) \geqslant 3$. Suppose that every $n \in \Lambda + \Lambda$ can be written uniquely* (*up to permutation*) *as $n = n_1 + n_2$, with $n_1, n_2 \in \Lambda$. Then $R : L^2_\Lambda(\mathbb{T}) \to L^2_\Lambda(\mathbb{T})$ is a $\hat{P}_{\Lambda,\Lambda}$-trivial solution if and only if it is of the form $R = R_{\mathbf{c}}$ with $\mathbf{c} \equiv \{c(n)\}_{n \in \Lambda} \in \mathbb{T}^\Lambda$.*

Further, if $f \in L^2_\Lambda(\mathbb{T})$ and Card$(\operatorname{supp} \hat{f}) \geqslant 3$*, then every solution to $\hat{P}_{\Lambda,\Lambda}$ is given by $R_{\mathbf{c}} f$, for some $\mathbf{c} \in \mathbb{T}^\Lambda$.*

In other words, this corollary states that the trivial solutions may be identified with $\mathbb{T}^\Lambda$ and that, if $f \in L^2_\Lambda(\mathbb{T})$ and $\mathrm{Card}(\operatorname{supp} \hat{f}) \geqslant 3$, every solution to $\hat{P}_{\Lambda,\Lambda}$ is a $\hat{P}_{\Lambda,\Lambda}$-trivial solution.

Proof. Under the assumption on Λ, condition (iii) of Proposition 3.4 always holds, since $n_1 - n_2 = n_3 - n_4$, for $n_i \in \Lambda$ implies $n_1 = n_3$ or $n_1 = n_2$. It follows that the $(\hat{P}_\Lambda)$-trivial solutions are all given by $R_{\mathbf{c}} R_{\alpha,k,\beta}$ or by $R_{\mathbf{c}} \widetilde{R}_{\alpha,k,\beta}$ for some $(\alpha, k, \beta) \in \mathbb{H}$ and some $\mathbf{c} = \{c(n)\}_{n \in \Lambda} \in (S^1)^\Lambda \equiv \mathbb{T}^\Lambda$. Among these operators, the only ones that preserve $L^2_\Lambda(\mathbb{T})$ are $R_{\mathbf{c}} \widetilde{R}_{\alpha,0,\beta} = R_{\widetilde{\mathbf{c}}}$ with $\widetilde{\mathbf{c}}_n = e^{i\beta + in\alpha} \mathbf{c}_n$.

We shall show that if $g \in L^2_\Lambda(\mathbb{T})$ is a $\hat{P}_{\Lambda,\Lambda}$-partner of f, and $\mathrm{Card}(\operatorname{supp} \hat{f}) \geqslant 3$, then $|\hat{f}(n)| = |\hat{g}(n)|$, for all $n \in \Lambda$. This will imply that $g = R_{\mathbf{c}} f$ for some multiplier $\mathbf{c} \in \mathbb{T}^\Lambda$ and establish the corollary.

From the assumptions on Λ, it follows that $|\mathcal{A}(f)(k,t)| = |\mathcal{A}(g)(k,t)|$ is a constant for each $k \in \mathbb{Z}$. For instance, if we fix $n_0 \in \operatorname{supp} \hat{f}$, then for every $n \in \Lambda \setminus \{n_0\}$, we get, for $k = n_0 - n \neq 0$

$$\bigl|\hat{f}(n_0)\overline{\hat{f}(n)}\bigr| = \bigl|\hat{a}(n_0)\overline{\hat{g}(n)}\bigr|. \tag{32}$$

Since $\mathrm{Card}(\operatorname{supp} \hat{f}) \geqslant 2$, we must have $\hat{g}(n_0) \neq 0$. Denoting $z = \frac{\hat{f}(n_0)}{\hat{g}(n_0)}$, and using $|\mathcal{A}(f)(0,t)|^2 = |\mathcal{A}(g)(0,t)|^2$ we obtain:

$$\left| \bigl|\hat{f}(n_0)\bigr|^2 e^{in_0 t} + \sum_{n \neq n_0} \bigl|\hat{f}(n)\bigr|^2 e^{int} \right|^2 = \left| \frac{1}{|z|^2} \bigl|\hat{f}(n_0)\bigr|^2 e^{in_0 t} + |z|^2 \sum_{n \neq n_0} \bigl|\hat{f}(n)\bigr|^2 e^{int} \right|^2.$$

Thus both trigonometric polynomials have same coefficients so that either $|z| = 1$ (which is what we wish), or

$$\bigl|\hat{f}(n_0)\bigr|^4 = |z|^4 \left| \sum_{\lambda \neq \lambda_0} \bigl|\hat{f}(n)\bigr|^2 e^{int} \right|.$$

Since $\mathrm{Card}(\operatorname{supp} \hat{f}) \geqslant 3$, we see that the latter cannot happen, so that $|z| = 1$. The corollary then follows from (32). $\square$

Remark 3.7. Sets Λ satisfying the condition of the corollary are usually called B_2-sets (or $B_2[1]$-sets) and have been extensively studied by Erdös and various collaborators (see, e.g., [7]), as well as their generalization, the B_k-sets, where sums of two integers are replaced by sums of k integers. One may show that a subset Λ of $\{1, \dots, N\}$ that is a B_k set has size at most $\operatorname{Card}\Lambda \leqslant CN^{1/k}$ and this bound is sharp [7]. A survey on the subject may be found on M. Koluntzakis' web page (see also [15,16]). B_k-sets are particular examples of $\Lambda(2k)$-*sets* for trigonometric series [25]. The two-dimensional version of these sets, contained in the lattice $\mathbb{Z}^2$, also appears in the study of certain phase retrieval problems arising from crystallography [8].

When the gaps of Λ are even larger, we will now prove that the problem $\hat{P}_\Lambda$ has *only* trivial solutions. To do so, we will need the following lemma which may be well known.

Lemma 3.8. *Let $\Lambda, \Lambda' \subset \mathbb{Z}$ and assume that every $n \in \Lambda + \Lambda + \Lambda$ can be written uniquely up to permutation as $n = n_1 + n_2 + n_3$ with $n_1, n_2, n_3 \in \Lambda$. Assume further that $\Lambda' - \Lambda' = \Lambda - \Lambda$, then $\Lambda' = \Lambda - m$ or $\Lambda' = m - \Lambda$ for some $m \in \mathbb{Z}$.*

Proof. Without loss of generality, we may assume that $0 \in \Lambda, \Lambda'$. Now, if $m \in \Lambda' \setminus \{0\}$, we may write $m = m - 0 = n_1 - n_2$ for some $n_1, n_2 \in \Lambda$. Assume that we may write $m = n_1' - n_2'$ with $n_1', n_2' \in \Lambda$, then $n_1 + n_2' + 0 = n_1' + n_2 + 0$. The property of Λ together with $m \neq 0$ then implies that $n_1' = n_1$ and $n_2' = n_2$. It follows that every $m \in \Lambda' \setminus \{0\}$ may be written in a unique way as $m = n_m - \widetilde{n}_m$ with $n_m \neq \widetilde{n}_m \in \Lambda$.

Further, fix $m_0 \in \Lambda' \setminus \{0\}$ and write $m_0 = n_0 - \widetilde{n}_0$ with $n_0 \neq \widetilde{n}_0 \in \Lambda$. Then, for $m \in \Lambda' \setminus \{0, m_0\}$, as $m - m_0 \in \Lambda' - \Lambda' = \Lambda - \Lambda$, there exist $n \neq \widetilde{n} \in \Lambda$ such that $m - m_0 = n - \widetilde{n}$. It follows that $n_m + \widetilde{n}_0 + \widetilde{n} = \widetilde{n}_m + n_0 + n$. As $m \neq 0$, we get $\widetilde{n}_m \neq n_m$ and as $m \neq m_0$, we get $\widetilde{n} \neq n$. The condition on Λ then implies that either $(n_m, \widetilde{n}_0, \widetilde{n}) = (n_0, n, \widetilde{n}_m)$ or $(n_m, \widetilde{n}_0, \widetilde{n}) = (n, \widetilde{n}_m, n_0)$. In the first case, $m = n_m - \widetilde{n}_m = n_0 - \widetilde{n}_m \in n_0 - \Lambda$ while in the second case $m = n_m - \widetilde{n}_m = n_m - \widetilde{n}_0 \in \Lambda - \widetilde{n}_0$.

It is now enough to prove that, for a given Λ', only one of these cases may occur.

If $\operatorname{Card}\Lambda' \leqslant 2$ this is trivial. If $\operatorname{Card}\Lambda' = 3$, the uniqueness of the decomposition $0 \neq m = n_m - \widetilde{n}_m$ implies that, if $m \in (\Lambda - \widetilde{n}_0) \cap (n_0 - \Lambda)$ then $m = m_0$. We may thus assume that $\operatorname{Card}\Lambda' \geqslant 4$.

Let $m \neq \widetilde{m} \in \Lambda' \setminus \{0, m_0\}$ and assume that we may write $m = n_0 - n$ and $\widetilde{m} = \widetilde{n} - \widetilde{n}_0$ with $n, \widetilde{n} \in \Lambda$. Again, as $\Lambda' - \Lambda' = \Lambda - \Lambda$, there exists $n_1 \neq n_2$ such that $m - \widetilde{m} = n_1 - n_2$. It follows that $n_0 + \widetilde{n}_0 + n_2 = n + \widetilde{n} + n_1$. The property of Λ with $n_1 \neq n_2$ then implies that only four cases may occur:

$$(n_0, \widetilde{n}_0, n_2) = (n, n_1, \widetilde{n}), \qquad (n_0, \widetilde{n}_0, n_2) = (n_1, \widetilde{n}, n),$$
$$(n_0, \widetilde{n}_0, n_2) = (\widetilde{n}, n_1, n) \quad \text{or} \quad (n_0, \widetilde{n}_0, n_2) = (n_1, n, \widetilde{n}).$$

The two first cases are respectively excluded with $m \neq 0$, i.e. $n_0 \neq n$, and $\widetilde{m} \neq 0$, i.e. $\widetilde{n}_0 \neq \widetilde{n}$. The two last cases are respectively excluded with $\widetilde{m} \neq m_0$, i.e. $n_0 \neq \widetilde{n}$, and $m \neq m_0$, i.e. $\widetilde{n}_0 \neq n$. This concludes the proof of the lemma. □

Sets Λ satisfying the condition of the lemma are usually called B_3-sets. See Remark 3.7 above.

Corollary 3.9. *Let $\Lambda \subset \mathbb{Z}$ be a B_3-set. Then every solution to $\hat{P}_\Lambda$ is a trivial solution, that is if $f \in L^2_\Lambda(\mathbb{T})$, then the solutions to* ($\hat{\mathrm{P}}$) *are all given by $SR_{\mathbf{c}}f$, for $\mathbf{c} \in \mathbb{T}^\Lambda$, $S \in \mathcal{T}$.*

Proof. Without loss of generality, we assume $0 \in \operatorname{supp}\hat{f}$ and $\operatorname{Card}(\operatorname{supp}\hat{f}) \geqslant 3$. Note that, since Λ satisfies the assumptions in Corollary 3.6, all the solutions to $(\hat{P}_{\Lambda,\Lambda})$ are given by $R_{\mathbf{c}}f$.

We shall show that if g is a ($\hat{\mathrm{P}}$)-partner of f, then $\operatorname{supp}\widehat{Sg} \subset \Lambda$, for some $S \in \mathcal{T}$. This will imply that f and Sg are $(\hat{P}_{\Lambda,\Lambda})$-partners, and hence $g = S^{-1}R_{\mathbf{c}}f$.

We denote $\Lambda_f = \operatorname{supp}\hat{f}$ and $\Lambda_g = \operatorname{supp}\hat{g}$. As f and g are ambiguity partners, $\mathcal{A}(f)$ and $\mathcal{A}(g)$ have same support and, with Lemma 3.3 this implies that $\Lambda_f - \Lambda_f = \Lambda_g - \Lambda_g$. From Lemma 3.8, we get that either $\Lambda_g = \Lambda_f - m$ or $\Lambda_g = m - \Lambda_f$ for some $m \in \mathbb{Z}$.

In the first case, it suffices to define $S \in \mathcal{T}$ by $Sg(t) = e^{-imt}g(t)$ while in the second case we consider $Sg(t) = e^{imt}g(-t)$. We then have $\operatorname{supp}\widehat{Sg} \subset \operatorname{supp}\hat{f}$ and, hence, f and Sg are $(\hat{P}_{\Lambda,\Lambda})$-partners. The proof of the corollary is then complete. □

To conclude this section, let us point out the existing relation between ($\hat{\mathrm{P}}$)-trivial solutions and "restricted" solutions to the ambiguity problem, as they were defined for the continuous case (3) in [12]. In the periodic situation, the question can be asked as follows:

($\hat{\mathbf{P}}_r$) The Restricted Ambiguity Problem. *For $f \in L^2(\mathbb{T})$, find all $g \in L^2(\mathbb{T})$ for which there is some family of unimodular constants η_k such that, for all $(k,t) \in \mathbb{Z} \times \mathbb{T}$*

$$\hat{\mathcal{A}}(f)(k,t) = \eta_k \hat{\mathcal{A}}(g)(k,t). \tag{33}$$

Two functions f and g as above are called restricted partners.

We have the following result:

Corollary 3.10. *Let $f \in L^2(\mathbb{T})$ and $\Lambda = \operatorname{supp} \hat{f}$. Then, all the restricted partners of f are of the form $R_{\mathbf{c}} f$, with $R_{\mathbf{c}} \in \mathcal{T}_\Lambda$, that is $\mathbf{c}$ is a sequence of unimodular constants supported in Λ that satisfies* (31).

Proof. It is clear that for each $R_{\mathbf{c}} \in \mathcal{T}_\Lambda$, with $\Lambda = \operatorname{supp} \hat{f}$, then $R_{\mathbf{c}} f$ is a restricted partner of f. Indeed, (33) holds with $\eta_k = c(n)\overline{c(n-k)}$, which by (31) does not depend on $n \in \Lambda$. Conversely, Equality (33) for $k = 0$ implies $|\hat{f}(n)| = |\hat{g}(n)|$ for all $n \in \mathbb{Z}$. Thus, $g \in L^2_\Lambda(\mathbb{T})$ and $g = R_{\mathbf{c}} f$ for a sequence of unimodular constant $\mathbf{c} = \{c(n)\}_{n\in\Lambda}$. It remains to show that condition (iii) in Proposition 3.4 holds. But this once more follows from (33), since for general values of $k \in \Lambda - \Lambda$, have $\eta_k = \overline{c(n)}c(n-k)$, for all $n, n-k \in \Lambda$. □

3.3. The continuous case

The definition of trivial solutions immediately adapts to the continuous radar ambiguity problem: a *trivial solution* to the continuous radar ambiguity problem is a linear or anti-linear continuous operator T on $L^2(\mathbb{R})$ such that for every $u \in L^2(\mathbb{R})$, u and Tu are ambiguity partners. We have the following description of these operators:

Proposition 3.11. *The trivial solutions of the continuous radar ambiguity are the operators of the form $Tu(t) = ce^{i\omega t}u(\varepsilon(t-a))$ with $c \in \mathbb{T}$, $\varepsilon = \pm 1$, $\omega, a \in \mathbb{R}$.*

Proof. Let T be a trivial solution and let ψ_n be the Hermite basis. According to Remark 2.6, ψ_n, $\psi_n + \psi_k$ have only trivial partners. Thus, for every n, there exists $c_n \in \mathbb{T}$, $\varepsilon_n = \pm 1$, $\omega_n, a_n \in \mathbb{R}$ such that

$$T\psi_n(t) = c_n e^{i\omega_n t}\psi_n\big(\varepsilon_n(t-a_n)\big) = c_n \varepsilon_n^n e^{i\omega_n t}\psi_n(t-a_n).$$

We want to prove that these constants do not depend on n: $a_n = a_0$, $\omega_n = \omega_0$ and either $c_n\varepsilon_n^n = c_0$ or $c_n\varepsilon_n^n = (-1)^n c_0$. If this is the case, then respectively $T\psi_n(t) = c_0 e^{i\omega_0 t}\psi_n(t-a_0)$ or $T\psi_n(t) = c_0 e^{i\omega_0 t}\psi_n(-t+a_0)$. By density of the span of the ψ_ns, linearity and continuity of T, it follows that $Tu(t) = c_0 e^{i\omega_0 t}u(\varepsilon_1(t-a_0))$ for all $u \in L^2$, as desired.

To do so, take $n \neq k$ and note that by additivity of T,

$$T(\psi_n + \psi_k) = T\psi_n + T\psi_k = c_n e^{-a_n^2/2}\varepsilon_n^n H_n(t-a_n)e^{(a_n+i\omega_n)t-t^2/2} + c_k e^{-a_k^2/2}\varepsilon_k^k H_k(t-a_k)e^{(a_k+i\omega_k)t-t^2/2}.$$

On the other hand, $\psi_n + \psi_k$ has only trivial partners, thus there exists constants $c_{k,n} \in \mathbb{T}$, $\varepsilon_{k,n} = \pm 1$, $\omega_{k,n}, a_{k,n} \in \mathbb{R}$ such that

$$T(\psi_n + \psi_k) = c_{k,n} e^{-a_{k,n}^2/2}\big[\varepsilon_{k,n}^n H_n(t-a) + \varepsilon_{k,n}^k H_k(t-a)\big]e^{(a_{k,n}+i\omega_{k,n})t-t^2/2}.$$

Comparing the growth at $\pm\infty$ and $\pm i\infty$ in these two expressions, we get that the exponential parts have to be the same, that is

$$a_n + i\omega_n = a_k + i\omega_k = a_{k,n} + i\omega_{k,n}$$

so that $a_{k,n} = a_n = a_k$ and $\omega_{k,n} = \omega_n = \omega_k$, i.e. for every n, $a_n = a_0$ and $\omega_n = \omega_0$ as desired. We are then left with

$$c_n\varepsilon_n^n H_n(t-a_0) + c_k\varepsilon_k^k H_k(t-a_0) = c_{k,n}\varepsilon_{k,n}^n H_n(t-a_0) + c_{k,n}\varepsilon_{k,n}^k H_k(t-a_0).$$

But, looking at the highest order term, this implies first that $c_n\varepsilon_n^n = c_{k,n}\varepsilon_{k,n}^n$ and then $c_k\varepsilon_k^k = c_{k,n}\varepsilon_{k,n}^k$. If n and k are both even then this reduces further to $c_n = c_{k,n} = c_k$, i.e. for every n even, $c_n = c_0$. If n and k are both odd, we get $c_n\varepsilon_n = c_{k,n}\varepsilon_{k,n} = c_k\varepsilon_k$, i.e. for every n odd, $c_n\varepsilon_n = c_1\varepsilon_1$. Finally, if $n = 0, k = 1$ we get $c_0 = c_{1,0}$ and $c_1\varepsilon_1 = c_{1,0}\varepsilon_{1,0}$. There are thus two alternatives, either $\varepsilon_{1,0} = 1$ or $\varepsilon_{1,0} = -1$. In the first case, $c_1\varepsilon_1 = c_0$ and then $T\psi_n(t) = c_0 e^{i\omega_0 t}\psi_n(t - a_0)$. In the second case $c_1\varepsilon_1 = -c_0$ so that $c_n\varepsilon_n^n = (-1)^n c_0$ and $T\psi_n(t) = c_0 e^{i\omega_0 t}\psi_n(-t + a_0)$ as desired. □

4. Pulse type signals

4.1. The stability of pulse type signals for the ambiguity problem

The main result in this section can be stated as follows:

Theorem 4.1. *Let $0 < \eta \leqslant \frac{1}{3}$ and $u(t) = \sum_{j=0}^N a_j \chi_{[j,j+\eta]}(t)$ for some $(a_0, a_1, \ldots, a_N) \in \mathbb{C}^{N+1}$. Then (modulo a trivial transformation) every solution $v(t) \in L^2(\mathbb{R})$ of the ambiguity problem (3) is necessarily of the form $v = \sum_{j=0}^N b_j \chi_{[j,j+\eta]}$, for some $(b_0, b_1, \ldots, b_N) \in \mathbb{C}^{N+1}$.*

This theorem may be seen as an "uncertainty principle" for pulse type signals, in analogy to Lemma 2.1 for Hermite signals. The techniques we use here, however, are different, containing ideas from phase retrieval and various limiting arguments. The role of $\eta \leqslant \frac{1}{3}$ is crucial in the proof, and one may conjecture that $\frac{1}{3}$ is critical to obtain such an uncertainty principle.

The following elementary lemma will be used in the sequel.

Lemma 4.2. *Let u, v be Lebesgue measurable functions and $[a, b] \subset \mathbb{R}$. Assume that for almost all $x \in [a, b]$, and almost every $t \in \mathbb{R}$, $u(t)v(t + x) = 0$. Then, if $t_0 \in \operatorname{supp} u$ we have $v(t) = 0$ for almost every $t \in t_0 + [a, b]$.*

Proof. Consider the set

$$A = \{(t, x) \in \mathbb{R} \times [a, b] \mid u(t)\, v(t + x) \neq 0\}.$$

By Tonelli's theorem and the assumption in the lemma

$$|A| = \int_{[a,b]} \left|\{t \in \mathbb{R} \mid u(t)\, v(t + x) \neq 0\}\right| \mathrm{d}x = 0.$$

Without loss of generality, we shall assume $t_0 = 0$. For $0 < \varepsilon < \frac{b-a}{2}$, let $U_\varepsilon = \{t \in (-\varepsilon, \varepsilon) \mid u(t) \neq 0\}$ and $V_\varepsilon = \{x \in [a + \varepsilon, b - \varepsilon] \mid v(x) \neq 0\}$. As $0 \in \operatorname{supp} u$, for every $\varepsilon > 0$, $|U_\varepsilon| > 0$.

Consider the set $A_\varepsilon = \bigcup_{t\in U_\varepsilon} \{t\} \times (V_\varepsilon - t)$ and note that

$$A_\varepsilon \subset \{(t, x) \in U_\varepsilon \times [a, b] \mid u(t)\, v(t + x) \neq 0\} \subset A.$$

Since $|A| = 0$, it follows that A_ε is measurable in $\mathbb{R}^2$ and $|A_\varepsilon| = 0$. Thus, using again Tonelli's theorem

$$|A_\varepsilon| = \int_{U_\varepsilon} |V_\varepsilon - t|\, \mathrm{d}t = |U_\varepsilon||V_\varepsilon| = 0.$$

As $|U_\varepsilon| > 0$ this implies that $|V_\varepsilon| = 0$ for every $\varepsilon > 0$, thus $|x \in [a, b] \mid v(x) \neq 0| = 0$. □

Proof of Theorem 4.1. We shall assume $a_0 a_N \neq 0$. Let $v \in L^2(\mathbb{R})$ be an ambiguity partner of u, that is

$$\left|\mathcal{F}^{-1}\big(v\overline{v(\cdot - x)}\big)(y)\right| = \left|\mathcal{F}^{-1}\big(u\overline{u(\cdot - x)}\big)(y)\right| = \sum_{k=-N}^{N} \left|\mathcal{A}a(k, y)\right| \left|\frac{\sin(\eta - |x - k|)y/2}{y/2}\right| \chi_{[-\eta,\eta]}(x - k), \tag{34}$$

for all $x, y \in \mathbb{R}$. We need to show that v is a pulse function of the same type as u. This will be obtained directly from (34) in various steps. To begin with we recall that, modulo a trivial transformation, we must have

$$\operatorname{conv}(\operatorname{supp} v) = \operatorname{conv}(\operatorname{supp} u) = [0, N + \eta] \tag{35}$$

(see, e.g., Lemma 1 in [12, 3.2.2]). In particular, v is compactly supported and (34) is an equality of continuous functions in x and y.

Step 1. *A bound for the support of v.*

From (34) it is clear that, for every $x \in [\eta, 1-\eta] + \mathbb{Z}$,

$$v(\cdot)\overline{v(\cdot + x)} = u(\cdot)\overline{u(\cdot + x)} = 0 \quad \text{a.e.} \tag{36}$$

Since $0 \in \operatorname{supp} v$, we conclude from Lemma 4.2 that $v(x) = 0$, for almost every $x \in [\eta, 1-\eta] + \mathbb{Z}$. Thus, there are some smallest intervals $I_j = [l_j, r_j] \subset j + [-\eta, \eta]$, $j = 0, \dots, N$, so that

$$\operatorname{supp} v \subset \bigcup_{j=0}^{N} I_j = \bigcup_{j=0}^{N} [l_j, r_j]. \tag{37}$$

Observe that $l_0 = 0$ and $r_N = N + \eta$ by (35). Further, we claim that our assumption $\eta \leqslant \frac{1}{3}$ actually implies $r_j - l_j \leqslant \eta$. Indeed, we already know this for $I_0 = [0, r_0] \subset [0, \eta]$. Let us now show it for $I_1 = [l_1, r_1] \subset [1-\eta, 1+\eta]$. Since $l_1 \in \operatorname{supp} v$, we can use again Lemma 4.2 and (36) to conclude

$$v(l_1 + x) = 0, \quad \text{for almost every } x \in [\eta, 1-\eta],$$

or equivalently, v vanishes in $l_1 + [\eta, 1-\eta]$. Now, this interval cannot be strictly contained in $[l_1, r_1]$ because the latter has length not exceeding 2η and the former (with left extreme $l_1 + \eta$) has length $1 - 2\eta \geqslant \eta$. Therefore, by the minimality of I_1 we must necessarily have $l_1 + \eta \geqslant r_1$, which gives our claim. One proceeds similarly with the other intervals I_j.

In particular, we have shown that

$$\operatorname{supp} v \subset \bigcup_{j=0}^{N} I_j \subset \bigcup_{j=0}^{N} [l_j, l_j + \eta].$$

Observe that we cannot exclude the possibility that some I_j may be empty. In this case, there is no loss in considering $l_j = r_j = j$.

Step 2. *The phase retrieval problem.*

Let us now fix $k \in \{0, \dots, N\}$ and $x \in k + (-\eta, \eta)$. We then study (35) as the phase retrieval problem

$$\left|\mathcal{F}\big[v(\cdot)\,\overline{v(\cdot - x)}\big](y)\right| = \left|\mathcal{F}\big[u(\cdot)\,\overline{u(\cdot - x)}\big](y)\right| = \left|\mathcal{A}a(k, y)\right| \left|\frac{\sin(\eta - |x-k|)y/2}{y/2}\right|. \tag{38}$$

By Walther's theorem ([28] or [12, Theorem 2]), the solution to this problem is necessarily of the form

$$\mathcal{F}\big[v(\cdot)\,\overline{v(\cdot - x)}\big](y) = e^{i\alpha(x)} e^{i\beta(x)y} \mathcal{A}a(k, y) \frac{\sin(\eta - |x-k|)y/2}{y/2} G_x(y), \tag{39}$$

where $\alpha(x), \beta(x)$ are real functions, and G_x is a unimodular function of the form

$$G_x(y) = \prod_{z \in J_x} \frac{(1 - \frac{y}{\bar{z}})e^{\frac{y}{\bar{z}}}}{(1 - \frac{y}{z})e^{\frac{y}{z}}},$$

for some set of (non-real) complex numbers J_x. The set J_x is a subset of the *complex* zeros of $z \mapsto \mathcal{F}[u(\cdot)\overline{u(\cdot - x)}](z)$. The effect of G_x is to take these zeros into their complex conjugates (the so called *zero-flipping*).

Since $z \mapsto \frac{\sin(\eta - |x-k|)z/2}{z/2}$ has only real zeros, flipping may only occur in the set $\mathcal{Z}_k$ of *non-real zeros* of $z \mapsto \mathcal{A}a(k, z)$ (where as usual, zeros are repeated according to multiplicity). We can partition $\mathcal{Z}_k = I_x \cup J_x$, with J_x the subset of zeros that "flip" in (38).

Our first claim is that, for each $k = 0, \dots, N$, J_x (and thus G_x) are actually independent of $x \in k + (-\eta, \eta)$. Indeed, given one such x_0 one notices that the holomorphic function $F_{x_0}(z) = \mathcal{F}[v(\cdot)\,\overline{v(\cdot - x_0)}](z)$ is not identically zero, and

$F_x \to F_{x_0}$ in $\mathcal{H}(\mathbb{C})$ when $x \to x_0$. Moreover, given any zero $z \in \mathcal{Z}(F_{x_0})$, by Rouché's theorem we obtain the equality of multiplicities $m(z, F_x) = m(z, F_{x_0})$, for all $|x - x_0| < \varepsilon$ provided $\varepsilon = \varepsilon(x_0, z) > 0$ is small enough.

Proceeding as before for every x_0, an easy compactness-connectedness argument gives $m(z, F_x) = m(z, F_k)$ for all $x \in k + (-\eta, \eta)$. Finally, repeating this argument with all zeros $z \in \mathcal{Z}(F_k)$ one concludes $J_x = J_k$ for all $x \in k + (-\eta, \eta)$. We will then write $G_k = G_x$ for such x.

Step 3. *Determination of the support of* $v(\cdot)\overline{v(\cdot - x)}$.

Let us now go back to (39) and define the bounded function

$$\hat{U}_x(y) = e^{i\alpha(x)}\, e^{i\beta(x)y}\, \mathcal{A}a(k, y)G_k(y), \tag{40}$$

so that U_x is a tempered distribution satisfying, for all $x \in k + (-\eta, \eta)$,

$$v(\cdot)\,\overline{v(\cdot - x)} = U_x * \chi_{[-\frac{\eta-|x-k|}{2}, \frac{\eta-|x-k|}{2}]}. \tag{41}$$

Next, we define another distribution $\widetilde{U}_k$ by

$$\widehat{\widetilde{U}}_k(y) = \mathcal{A}a(k, y)G_k(y),$$

so that, for $x \in k + (-\eta, \eta)$,

$$\widetilde{U}_k = e^{-i\alpha(x)} U_x\big(\cdot + \beta(x)\big). \tag{42}$$

Let us emphasize that, in this identity, $\widetilde{U}_k$ does not depend on x. Now, if we consider $k = 0$ and fix $x \in [0, \eta)$ we must have, using step 1,

$$\begin{aligned}\bigcup_{j=0}^{N} [l_j + x, l_j + \eta] \supset \operatorname{supp} v(\cdot)\overline{v(\cdot - x)} &= \operatorname{supp} U_x * \chi_{[-\frac{\eta-x}{2}, \frac{\eta-x}{2}]} \\ &= \operatorname{supp}(\widetilde{U}_0 * \chi_{[-\frac{\eta-x}{2}, \frac{\eta-x}{2}]}) + \beta(x).\end{aligned} \tag{43}$$

Now, as $v(\cdot)\,\overline{v(\cdot - x)}$ is supported in $[0, N + \eta]$, $z \mapsto \mathcal{F}[v(\cdot)\,\overline{v(\cdot - x)}](z)$ is entire of exponential type at most $N + \eta$ (for any x). It follows that β is a bounded function and thus we may find a sequence $x_m \nearrow \eta$ so that $\beta(x_m)$ has some limit, say β_+.

Next recall the following elementary fact: for every distribution $U \in \mathcal{S}'$ we have

$$\frac{1}{2\delta} U * \chi_{-(\delta,\delta)} \to U$$

when $\delta \to 0$ with convergence in $\mathcal{S}'$.

Then, letting $x_m \to \eta$ in (43) we easily obtain

$$\operatorname{supp} \widetilde{U}_0 \subset \bigcup_{j=0}^{N} \{l_j + \eta\} - \beta_+.$$

Further, observe that $\widehat{\widetilde{U}}_0$ is bounded (and hence cannot be a polynomial), which necessarily implies

$$\widetilde{U}_0 = \sum_{j=0}^{N} \gamma_j \delta_{l_j+\eta-\beta_+}, \tag{44}$$

for some complex numbers γ_j, $j = 0, \dots, N$. Thus, we conclude that, if $x \in (-\eta, \eta)$,

$$U_x = e^{i\alpha(x)} \sum_{j=0}^{N} \gamma_j \delta_{l_j+\eta+\beta(x)-\beta_+}, \tag{45}$$

and therefore

$$v(\cdot)\,\overline{v(\cdot - x)} = e^{i\alpha(x)} \sum_{j=0}^{N} \gamma_j \chi_{[-\frac{\eta-|x|}{2}, \frac{\eta-|x|}{2}]+l_j+\eta+\beta(x)-\beta_+}. \tag{46}$$

Step 4. *Determination of* $|v|$.

We begin by showing that $\gamma_0\gamma_N \neq 0$. Indeed, we test (46) with $x=0$, and using the property that $0 \in \operatorname{supp} v$, we find a smallest integer $j_0 \in \{0,\dots,N\}$ such that

$$0 \in \left[-\frac{\eta}{2}, \frac{\eta}{2}\right] + l_{j_0} + \eta + \beta(0) - \beta.$$

We claim that $j_0 = 0$. If not we must have $\frac{3\eta}{2} + \beta(0) - \beta < 0$ (since $l_0 = 0$), and thus $\beta(0) - \beta < -\frac{3\eta}{2}$. But now, since $N + \eta \in \operatorname{supp} v$ we also have

$$N + \eta \leqslant \frac{\eta}{2} + l_N + \eta + \beta(0) - \beta < \frac{3\eta}{2} + l_N - \frac{3\eta}{2} = l_N,$$

which is a contradiction (since $r_N = N + \eta \in \operatorname{supp} v$). Thus, $j_0 = 0$, which forces $\gamma_0 \neq 0$. A completely symmetrical argument gives $\gamma_N \neq 0$.

Next, we shall determine explicitly the function $\beta(x)$ in (39). Recall from (43) that

$$\operatorname{supp} v(\cdot)\overline{v(\cdot - x)} \subset \begin{cases} \bigcup_{j=0}^{N} [x + l_j, l_j + \eta], & \text{if } x \in [0, \eta), \\ \bigcup_{j=0}^{N} [l_j, l_j + \eta + x], & \text{if } x \in (-\eta, 0]. \end{cases}$$

Since $\gamma_0\gamma_N \neq 0$, we see from the (46) that the extreme points $-\frac{\eta - |x|}{2} + l_0 + \eta + \beta(x) - \beta_+$, $\frac{\eta - |x|}{2} + l_N + \eta + \beta(x) - \beta_+$ must belong to $\operatorname{supp} v(\cdot)\,\overline{v(\cdot - x)}$. Therefore, if $x \in [0, \eta)$, $x + l_0 \leqslant -\frac{\eta - x}{2} + l_0 + \eta + \beta(x) - \beta_+$ so that

$$-\frac{\eta - x}{2} \leqslant \beta(x) - \beta_+,$$

and $\frac{\eta - |x|}{2} + l_N + \eta + \beta(x) - \beta_+ \leqslant l_N + \eta$ so that

$$\beta(x) - \beta_+ \leqslant -\frac{\eta - x}{2}.$$

Thus, we conclude $\beta(x) = \beta_+ - \frac{\eta - x}{2}$, $x \in [0, \eta)$. Proceeding symmetrically with $x \in (-\eta, 0]$ one extends this identity to all $x \in (-\eta, \eta)$. In conclusion, going back to Eq. (46) with $x = 0$ we have shown that

$$|v|^2 = e^{i\alpha(0)} \sum_{j=0}^{N} \gamma_j\, \chi_{[l_j, l_j + \eta]}.$$

Next we shall determine explicitly the values of l_j. As we said in step 1, there is no loss in assuming $l_j = j$ when $\gamma_j = 0$. We will prove that we must also have $l_j = j$ when $\gamma_j \neq 0$. Indeed, we already know that $l_0 = 0$. Moreover, when $\gamma_j \neq 0$ we know from step 1 that

$$[l_j, l_j + \eta] \subset j + [-\eta, \eta],$$

from which it follows $j - \eta \leqslant l_j \leqslant j$. Assume by contradiction that for one such j we have $j - \eta \leqslant l_j \leqslant j - \varepsilon$, for some $0 < \varepsilon < \eta$. Then we can select

$$x = l_j - (\eta - \varepsilon) \in (j - 1) + [\eta, 1 - \eta],$$

so that by (36) it holds $v(\cdot)\overline{v(\cdot - x)} = 0$, a.e. Now, when $t \in [l_j, l_j + \varepsilon]$ we also have $t - x \in [\eta - \varepsilon, \eta] \subset [0, \eta]$, and therefore, for $t \in [l_j, l_j + \varepsilon]$,

$$v(t)\,\overline{v(t - x)} = \gamma_j \overline{\gamma_0} \neq 0,$$

which is a contradiction. Thus, we have proven

$$|v|^2 = e^{i\alpha(0)} \sum_{j=0}^{N} \gamma_j\, \chi_{[j, j + \eta]}, \tag{47}$$

or more generally, looking at (46), for $x \in (-\eta, \eta)$

$$v(\cdot)\overline{v(\cdot - x)} = e^{i\alpha(x)} \sum_{j=0}^{N} \gamma_j \chi_{[-\frac{\eta-|x|}{2}, \frac{\eta-|x|}{2}]+j+\frac{x+\eta}{2}}. \tag{48}$$

Step 5. *Determination of the phase of v.*

From (47) we conclude that there are numbers $b_0, \dots, b_N \geqslant 0$, and a function $t \mapsto \phi(t)$ real such that

$$v(t) = e^{i\phi(t)} \sum_{j=0}^{N} b_j \chi_{[j,j+\eta]}.$$

Observe that we can modify v in a null set so that this equality holds in all points $t \in \mathbb{R}$. We want to show that the phase $\phi(t)$ is constant in each interval $[j, j+\eta]$ for which $b_j \neq 0$. When $x \in [0, \eta)$, using the expression in (48) we see that

$$v(\cdot)\overline{v(\cdot - x)} = e^{i\alpha(x)} \sum_{j=0}^{N} \gamma_j \, \chi_{[x,\eta]+j} = e^{i(\phi(t)-\phi(t-x))} \sum_{j=0}^{N} |b_j|^2 \, \chi_{[x,\eta]+j}.$$

Since by (47) $e^{i\alpha(0)}\gamma_j \geqslant 0$, we must have

$$\widetilde{\alpha}(x) \equiv \alpha(x) - \alpha(0) = \phi(t) - \phi(t-x) \pmod{2\pi}$$

whenever $x \in [0, \eta)$, $t \in [x, \eta] + j$ and $b_j \neq 0$. Choosing $t = x + j$ we see that $\phi(x+j) = \widetilde{\alpha}(x) + \phi(j) \pmod{2\pi}$, and therefore

$$\widetilde{\alpha}(x) = \widetilde{\alpha}(t) - \widetilde{\alpha}(t-x) \pmod{2\pi}, \quad x, t, t-x \in [0, \eta).$$

This is equivalent to

$$\widetilde{\alpha}(t+x) = \widetilde{\alpha}(t) + \widetilde{\alpha}(x) \pmod{2\pi}, \quad \text{when } x, t, t+x \in [0, \eta),$$

which by continuity of $\widetilde{\alpha}$ (by (39)) implies $\widetilde{\alpha}(t) = \omega t$, $t \in [0, \eta)$, for some real number ω. Thus, modulo 2π, $\phi(t+j) = \phi(j) + \omega t$, $t \in [0, \eta)$, so calling $\widetilde{b}_j = e^{i(\phi(j)-\omega j)} b_j$ we conclude

$$v(t) = e^{i\omega t} \sum_{j=0}^{N} \widetilde{b}_j \chi_{[j,j+\eta]}.$$

Therefore we have shown that, modulo a trivial transformation, v is a signal of pulse type of the same form as u, concluding the proof of the theorem. □

4.2. Rareness of pulse signals with non-trivial partners

Contrary to Section 3, from now on it will be more convenient to study the Discrete Radar Ambiguity Problem (P) for sequences rather than the Periodic Radar Ambiguity Problem ($\hat{\text{P}}$). Let us first note that there is no difficulty to transpose Proposition 3.1 to this context. Note that the trivial solutions are generated by the two representations of the periodized Heisenberg group $\mathbb{H} = \mathbb{T} \times \mathbb{T} \times \mathbb{Z}$ on $\ell^2(\mathbb{Z})$ given as follows. For $h = (\beta, \omega, l) \in \mathbb{H}$ and $a = (a_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$, define $b = S_h a$ by

$$b_j = e^{i\beta + ij\omega} a_{j-l},$$

and $\widetilde{b} = \widetilde{S}_h a$ by

$$\widetilde{b}_j = e^{i\beta + ij\omega} a_{-j-l}.$$

Further, when looking for partners of a finite sequence a, we may replace a by a trivial partner and assume that $a = (a_0, \dots, a_N)$ for some integer N and that $a_0 a_N \neq 0$. We will then write $a \in \mathcal{S}(N)$. Transposing Lemma 3.3 from trigonometric polynomials to finite sequences, every partner of a may then also be assumed to be in $\mathcal{S}(N)$.

In view of Theorem 4.1, the study of Problem 3 for pulse type signals of finite length is then reduced to the following finite dimensional ambiguity problem, where N is a fixed positive integer.

Ambiguity problem in $\mathcal{S}(N)$. *Given $a=(a_0,a_1,\dots,a_N)\in\mathcal{S}(N)$, find all $b=(b_0,b_1,\dots,b_N)\in\mathcal{S}(N)$ such that*

$$\big|\mathcal{A}(b)(j,y)\big| = \big|\mathcal{A}(a)(j,y)\big| \quad \textit{for all } j\in\mathbb{Z},\ y\in\mathbb{T}. \tag{49}$$

We will now use the following notation.

Notation. If b is a trivial ambiguity partner of a, we write $b\equiv a$. If $b\simeq a$ (b a partner of a) but $b\not\equiv a$, we call b a *strange partner* of a and write $b\sim a$.

The goal is to describe the class of all signals a which only admit trivial partners b. Several results in this direction have already appeared in [9], which we describe now. We shall denote the complementary of the searched class by

$$\mathcal{E}(N)=\big\{a\in\mathcal{S}(N)\colon\ a \text{ admits strange partners}\big\}.$$

It is easy to see that $\mathcal{E}(N)=\emptyset$ for $N=0,1,2$. The main result in [9] establishes that for larger values of N this set cannot be too large.

Theorem 4.3. *For every $N\geqslant 3$, $\mathcal{E}(N)$ is a non-empty semi-algebraic variety of real dimension at most $2N+1$.*

We recall that a semi-algebraic variety is a set defined by polynomial equalities and/or inequalities. The theorem says that $\mathcal{E}(N)$ has this structure, and moreover is contained in a real algebraic variety (i.e., finite unions of polynomial zero sets) of real dimension $2N+1$. This implies that $\mathcal{E}(N)$ has Lebesgue measure 0 in $\mathbb{C}^{N+1}$ and is also thin in the Baire sense.

Corollary 4.4. *For every $N\geqslant 0$, quasi-all and almost all elements of $\mathcal{S}(N)$ have only trivial partners.*

A full description of $\mathcal{E}(N)$ for $N=3,4$ can be found in [9]. In particular, $\mathcal{E}(3)$ contains sequences with all $a_j\neq 0$, $j=0,1,2,3$. This shows that sequences with strange partners do not necessarily have to contain “gaps,” a remarkable fact in view of the results in Section 3. In [9], a general argument showing the non-emptiness of $\mathcal{E}(N)$ for $N\geqslant 3$ was only sketched. The object of the next section is to prove it in full detail.

4.3. Construction of strange partners

A simple way to construct strange ambiguity partners when $N=2K+1$ is odd is as follows: take $\alpha=(\alpha_0,\dots,\alpha_K)$ be any sequence of length K. A direct computation of their ambiguity functions shows that for $\lambda\in\mathbb{C}$, the sequences

$$a_k=\begin{cases}\alpha_p & \text{when } k=2p,\\ \lambda\alpha_p & \text{when } k=2p+1\end{cases} \quad\text{and}\quad b_k=\begin{cases}\bar\lambda\alpha_p & \text{when } k=2p,\\ \alpha_p & \text{when } k=2p+1\end{cases} \tag{50}$$

are ambiguity partners. In general, these are non-trivial partners (see [9, p. 102]). Since this method is restricted to N odd, we will now describe another method that gives elements of $\mathcal{E}(N)$ as soon as $N\geqslant 4$.

First recall from [9] that when $a\in\mathcal{S}(N)$ one can reformulate (49) as an equivalent combinatorial problem on matrices. Namely, if we let K_a be the matrix with entries

$$d_{j,k}=\begin{cases}a_{\frac{j+k}{2}}a_{\frac{j-k}{2}} & \text{if } j,k \text{ have same parity},\\ 0 & \text{else},\end{cases}$$

then we have the following

Proposition 4.5. *Two sequences $a,b\in\mathcal{S}(N)$ are ambiguity partners if and only if*

$$K_a^*K_a=K_b^*K_b.$$

Example. If $a \in \mathcal{S}(5)$, the matrix of K_a is given by

$$\begin{bmatrix}
 & & & & & a_0^2 & & & & & \\
 & & & & a_1.a_0 & 0 & a_1.a_0 & & & & \\
 & & & a_2.a_0 & 0 & a_1^2 & 0 & a_2.a_0 & & & \\
 & & a_3.a_0 & 0 & a_1.a_2 & 0 & a_1.a_2 & 0 & a_3.a_0 & & \\
 & a_4.a_0 & 0 & a_1.a_3 & 0 & a_2^2 & 0 & a_1.a_3 & 0 & a_4.a_0 & \\
a_5.a_0 & 0 & a_1.a_4 & 0 & a_2.a_3 & 0 & a_2.a_3 & 0 & a_1.a_4 & 0 & a_5.a_0 \\
 & a_1.a_5 & 0 & a_2.a_4 & 0 & a_3^2 & 0 & a_2.a_4 & 0 & a_1.a_5 & \\
 & & a_2.a_5 & 0 & a_3.a_4 & 0 & a_3.a_4 & 0 & a_2.a_5 & & \\
 & & & a_3.a_5 & 0 & a_4^2 & 0 & a_3.a_5 & & & \\
 & & & & a_4.a_5 & 0 & a_4.a_5 & & & & \\
 & & & & & a_5^2 & & & & &
\end{bmatrix}$$

(non-written elements of that matrix are 0).

We shall make use of the *Kronecker product* of matrices, which for A and $B = [b_{i,j}]_{1\leqslant i,j\leqslant n}$ is the matrix defined by blocks as

$$A \otimes B = \begin{bmatrix} Ab_{1,1} & Ab_{1,2} & \dots & Ab_{1,n} \\ Ab_{2,1} & Ab_{2,2} & \dots & Ab_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ Ab_{n,1} & Ab_{n,2} & \dots & Ab_{n,n} \end{bmatrix}.$$

This product has the following elementary properties:

- $(A \otimes B)^* = A^* \otimes B^*$,
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

We shall compute the Kronecker product of two ambiguity matrices K_a and K_b and show that it corresponds to the ambiguity matrix of a new sequence c produced by a certain product rule involving a and b. This turns out to produce many natural examples of sequences with strange partners.

For this, it is convenient to change the way to enumerate the entries of such matrices, by introducing the following "lattice coordinates": let $\gamma = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\Gamma = \gamma\mathbb{Z}^2$ be a sub-lattice of $\mathbb{Z}^2$. Given $N \geqslant 1$ we consider the subset of entries $\Gamma_N = \{\left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]\colon 0 \leqslant m,\ l \leqslant N\}$. If $a = (a_0, a_1, \dots, a_N) \in \mathbb{C}^{N+1}$, then K_a is supported in Γ_N and

$$(K_a)_{i,j} = a_m a_\ell \quad \text{if } \begin{bmatrix} l\,i \\ j \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ \ell \end{bmatrix}\colon 0 \leqslant m,\ \ell \leqslant N.$$

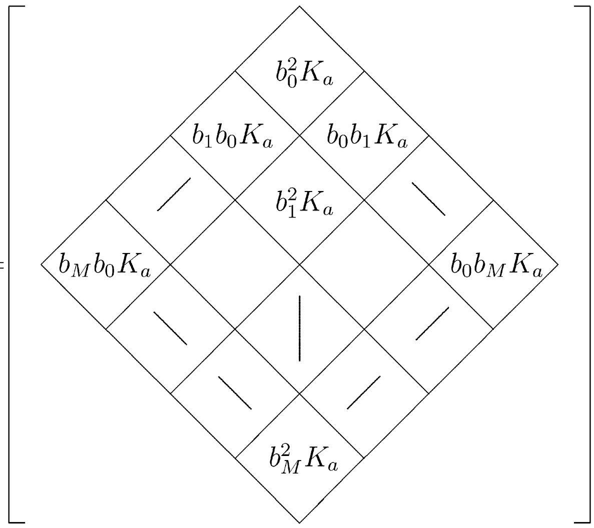
Thus, K_a is completely determined by the matrix $\widetilde{K}_a[m,\ell] := (K_a)_{i,j}$ when $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] = \gamma\left[\begin{smallmatrix} m \\ \ell \end{smallmatrix}\right]$.

Lemma 4.6. *Let $a = (a_0, \dots, a_N)$ and $b = (b_0, \dots, b_M)$ be two finite sequences with associated polynomials $P(z) = \sum_{k=0}^N a_k z^k$, $Q(z) = \sum_{k=0}^M b_k z^k$. Consider the polynomial $P(z)Q(z^{N+1}) = \sum_{k=0}^K c_k z^k$ and let $c = (c_0, \dots, c_K)$. Then the ambiguity matrix K_c is supported in $\Gamma_N + (N+1)\Gamma_M$ and satisfies*

$$\widetilde{K}_c\big[i + (N+1)m,\ j + (N+1)\ell\big] = b_m b_\ell a_i a_j. \tag{51}$$

In particular, $\widetilde{K}_c = \widetilde{K}_a \otimes \widetilde{K}_b$ *and the matrix* K_c *can be drawn as*

$K_c =$



Proof. As noted before

$$\widetilde{K}_c\big[i + (N+1)m, j + (N+1)\ell\big] = c_{i+(N+1)m} c_{j+(N+1)\ell}.$$

Now by construction of c, the only non-null coefficients are $c_{i+(N+1)m} = a_i b_m$ for $0 \leqslant i \leqslant N$ and $0 \leqslant m \leqslant M$. This gives (51). To justify the drawing observe that the submatrix with coordinates in $\Gamma_N + (N+1)\gamma\big[\begin{smallmatrix} m \\ \ell \end{smallmatrix}\big]$ is precisely $b_m b_\ell K_a|_{\Gamma_N}$, which as m and ℓ moves fills each of the parallelograms in the picture. $\square$

Lemma 4.7. *Let* $a = (a_0, \ldots, a_N)$ *and* $b = (b_0, \ldots, b_M)$ *be two finite sequences with associated polynomials* $P(z) = \sum_{k=0}^{N} a_k z^k$, $Q(z) = \sum_{k=0}^{M} b_k z^k$. *Consider this time the polynomial* $P(z)Q(z^{2N+1}) = \sum_{k=0}^{K} c_k z^k$ *and let* $c = (c_0, \ldots, c_K)$. *Then the ambiguity matrix of* c *is* $K_c = K_a \otimes K_b$.

Proof. Applying the previous lemma to $\widetilde{P}(z) = \sum_{k=0}^{2N} \widetilde{a}_k z^k$ where

$$\begin{cases} \widetilde{a}_i = a_i & \text{if } 0 \leqslant i \leqslant N, \\ \widetilde{a}_i = 0 & \text{if } N \leqslant i \leqslant 2N, \end{cases}$$

we see that

$$\operatorname{supp} K_c \subset \operatorname{supp} K_{\widetilde{a}} + (2N+1) \operatorname{supp} K_b.$$

Since $K_{\widetilde{a}}$ vanishes in $\Gamma_{2N} \setminus \Gamma_N$, we actually have

$$\operatorname{supp} K_c \subset \operatorname{supp} K_a + (2N+1) \operatorname{supp} K_b.$$

If we regard K_a as a square $(2N+1)$-matrix, this implies that K_c can be written as a collection of disjoint consecutive square blocks $\{K_a + (2N+1)\big[\begin{smallmatrix} i \\ j \end{smallmatrix}\big]\colon \big[\begin{smallmatrix} i \\ j \end{smallmatrix}\big] \in \operatorname{supp} K_b\}$. Next, if we take $\big[\begin{smallmatrix} i \\ j \end{smallmatrix}\big] = \gamma\big[\begin{smallmatrix} m \\ \ell \end{smallmatrix}\big] \in \operatorname{supp} K_b \subset \Gamma_M$, then by the previous lemma the value of K_c in the corresponding block is precisely

$$b_m b_\ell K_a|_{\Gamma_N}.$$

This shows $K_c = K_a \otimes K_b$ as asserted. $\square$

A sequence c constructed from a and b as in the statement of Lemma 4.7 will be denoted by $c = a \otimes b$. Recall also that $a \simeq b$ means that a and b are ambiguity partners as in (49).

Corollary 4.8. *Let* a, b, a', b' *be four finite sequences. If* $a \simeq a'$ *and* $b \simeq b'$, *then* $a \otimes b \simeq a' \otimes b'$.

Proof. From the previous lemma and elementary properties of the Kronecker product we see that

$$\begin{aligned} K^*_{a'\otimes b'}K_{a'\otimes b'} &= (K_{a'}\otimes K_{b'})^*(K_{a'}\otimes K_{b'}) \\ &= (K^*_{a'}\otimes K^*_{b'})(K_{a'}\otimes K_{b'}) = (K^*_{a'}K_{a'})\otimes(K^*_{b'}K_{b'}) \\ &= (K^*_a K_a)\otimes(K^*_b K_b) = K^*_{a\otimes b}K_{a\otimes b}. \end{aligned}$$

Thus, $a\otimes b \simeq a'\otimes b'$ as asserted. □

This corollary enables us to construct sequences $a\in\mathcal{S}(N)$ with strange partners, as soon as $N\geqslant 4$.

Example. Let $a=(1,2)$, $b=(1,2)$ and $b'=(2,1)$, then $a\otimes b\simeq a\otimes b'$. But

$$a\otimes b=(1,2,0,2,4)\quad \text{whereas } a\otimes b'=(2,4,0,1,2)$$

so that $a\otimes b$ and $a\otimes b'$ are not trivial partners and $a\otimes b\in\mathcal{E}(4)$. Moreover, applying the above construction to $a=(1,2,0,\dots,0)$ regarded as sequence in $\mathbb{C}^{N+1}$, we obtain a sequence $a\otimes b\in\mathcal{S}(2N+2)$, which shows that $\mathcal{E}(2N+2)\neq\emptyset$ for all $N\geqslant 1$.

Example. Other examples can be produced by iterating this process. For instance, consider the sequence c associated with the polynomial

$$R(z)=\prod_{j=0}^{J}\bigl(\alpha_j+\beta_j z^{3^j}\bigr).$$

Non-trivial ambiguity partners can be obtained by selecting a collection of j's and replacing the corresponding factors in the polynomial by $\alpha_j+c_j\beta_j z^{3^j}$ or $\beta_j+c_j\alpha_j z^{3^j}$, with $|c_j|=1$. It is possible to show (although harder) that these are all the possible ambiguity partners of c. Observe finally that these kind of examples are of a different nature than those in Proposition 3.4.

As an application we obtain the following remarkable result.

Corollary 4.9. *The set of all functions $u\in L^2(\mathbb{R})$ having strange ambiguity partners in the sense of* (3) *is dense in $L^2(\mathbb{R})$.*

Proof. Let $f\in L^2(\mathbb{R})$, which we may assume with $\|f\|\leqslant 1$. Given $0<\varepsilon<1$ we can find f_c with compact support such that $\|f-f_c\|<\varepsilon$. Suppose that supp $f_c\subset[-\frac{R}{4},\frac{R}{4}]$.

Further, taking $a=(1,\varepsilon)$, $b=(1,\varepsilon)$ and $b'=(\varepsilon,1)$, then $a\otimes b\simeq a\otimes b'$. But

$$a\otimes b=\bigl(1,\varepsilon,0,\varepsilon,\varepsilon^2\bigr)\quad \text{whereas } a\otimes b'=\bigl(\varepsilon,\varepsilon^2,0,1,\varepsilon\bigr)$$

so that the pulse type signals

$$u(t)=\sum(a\otimes b)_j f_c(t-Rj)\quad\text{and}\quad v(t)=\sum(c\otimes b)_j f_c(t-Rj)$$

are non-trivial ambiguity partners and

$$\|f-u\|\leqslant\|f-f_c\|+\bigl(2\varepsilon+\varepsilon^2\bigr)\|f_c\|\leqslant 7\varepsilon. \qquad \square$$

5. Conclusion

The radar ambiguity problem is a difficult and still widely open problem. In this paper we have concentrated in the most common classes of signals (Gaussian and rectangular pulses), and shown how to tackle such cases with real and complex analysis methods, and also with algebraic approaches. We are still unable to say much about the general case, but the originality of our methods may be useful when studying similar problems in the phase retrieval literature.

For Hermite functions, we rediscover a conjecture from the 70s which is stronger than the uncertainty principle for ambiguity functions in Section 2.1. We are almost certain that Hermite functions must have only trivial partners.

Indeed, we have only used a small part of the relations between partners to conclude in the generic case. On the other side, our proof becomes technically very complicate when dealing with other cases, and new ideas may be necessary.

In the case of pulse type signals, we have both the rareness of functions with strange partners, some criteria to have only trivial solutions (see [9]) and various ways to construct functions that have strange partners. On the other hand, we are unable to attack the discrete problem, that is Problem (8), for general sequences with infinite length. We know that sequences with strange partners are dense (as well as those with only trivial partners), but it seems likely to us that they must be "small" in a suitable sense (such as Baire category), although we still lack of evidence for this.

Note also that more general classes would be of interest for instance compactly supported functions (see [12] for some results) and functions of the form $P(x)e^{-x^2/2}$ with P an entire function of order <1. For the later, note that our techniques do not allow to say anything since we always start with the highest order coefficient of P when P is a polynomial (it may be shown that every ambiguity partner is of the same form).

Let us conclude by mentioning that this paper only deals with the *narrow band* case, so that some physical restrictions are assumed on the signal. If these assumptions are lifted, as is the case, e.g., for a sonar, then the radar does no longer measure $A(u)$, but the so-called *wide-band ambiguity function* which is linked to the wavelet transform in the same way as the narrow-band ambiguity function is linked to the short-time Fourier transform. We refer to [1,12,14, 18] for a presentation of the physics and mathematics of the wide-band ambiguity function. Of course, measurement is again phase-less so that there is also a "wide-band radar ambiguity problem." This problem is still widely open, but [12] contains some results for what may be called "logarithmic pulse-type signals." It is likely that some of the techniques developed in this paper may also be useful for the wide-band case.

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