L^p-L^q estimates for Bergman projections in bounded symmetric domains of tube type

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1 Introduction

Let D be an (irreducible) *bounded symmetric domain* of tube type in $\mathbb{C}^n$. That is, D is conformally equivalent to a tube domain $T_\Omega = \mathbb{R}^n + i\Omega$ over a symmetric cone Ω in $\mathbb{R}^n$. Irreducible symmetric cones are completely classified (see eg [10]), being either light-cones

$$\Lambda_n = \left\{(y_1, y') \in \mathbb{R}^n \ : \ y_1 > |y'|\right\}, \quad n \geq 3,$$

or cones of positive-definite symmetric or hermitian matrices, namely

$$\mathrm{Sym}_+(r, \mathbb{R}), \quad \mathrm{Her}_+(r, \mathbb{C}), \quad \mathrm{Her}_+(r, \mathbb{H}), \quad \mathrm{Her}_+(3, \mathbb{O}). \tag{1.1}$$

We write r for the *rank* of the cone (which in light-cones is $r = 2$), and Δ for the associated determinant function (which in light-cones is the Lorentz form $\Delta(y) = y_1^2 - |y'|^2$).

An important open question in these domains, D and T_Ω, concerns the L^p boundedness of the associated Bergman projections, that is, the orthogonal projection P mapping L^2 into the subspace of holomorphic functions A^2. In contrast with Cauchy-Szegő projections (which are not bounded in L^p for any $p \neq 2$, if $n > 1$), the L^p-boundedness of Bergman projections has been conjectured in a small interval around $p = 2$, namely

$$1 + \frac{n-r}{2n} < p < 1 + \frac{2n}{n-r}.$$

At the moment, positive results are only known to hold in a proper subinterval

$$1 + \frac{n-r}{2n-r} < p < 1 + \frac{2n-r}{n-r},$$

with a small improvement over this range in the case of light-cones. We refer to [3, 8, 6, 13, 12] for recent work in this topic.

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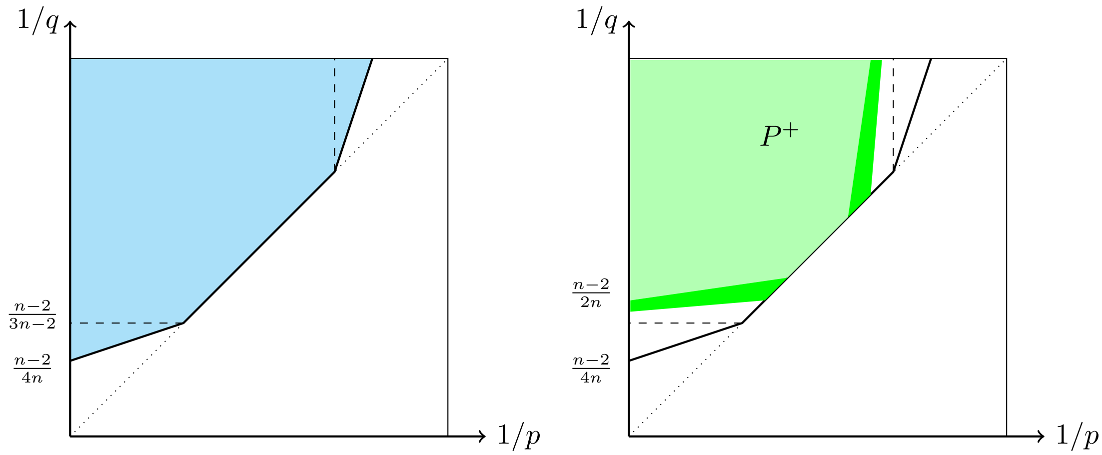

Figure 1.1: Conjectured and known results for boundedness of $P : L^p \to L^q$ when $r = 2$.

In this paper we shall be interested in the boundedness of P from L^p into L^q. When $p \neq q$, this question only makes sense in the bounded domain D. In this case, by Hölder's inequality, the L^{p_0}-boundedness of P immediately implies $L^p \to L^q$ bounds for all $1 \leq q \leq p_0 \leq p \leq \infty$. What may seem surprising is that the conjectured range for $L^p \to L^q$ boundedness of P is actually larger than what can be obtained from the diagonal case. For instance, if $r = 2$ (ie, in the light-cone setting) then $L^\infty \to L^q$ bounds may be conjectured for all $q < 4n/(n-2)$; see Figure 1.1.

This problem was first investigated in [3] in the case $r = 2$. In that paper, $L^\infty \to L^q$ boundedness for P was proved when $1 \leq q < 2n/(n-2)$, and this range was shown to be optimal for the *positive* operator P^+, for which the Bergman kernel has been replaced by its absolute value. By interpolation with diagonal results this range can be slightly extended for P, but seems still far from the conjectured region (see Figure 1.1 above).

The main contribution of the present paper is to extend these results to the higher rank setting $r \geq 3$. That is, we find the optimal range of boundedness for $P^+ : L^\infty \to L^q$, and derive from here new $L^p \to L^q$ inequalities for P. Unfortunately this is not a simple task, as the proof for $r = 2$ is mostly computational and does not easily generalize to the higher rank situation. We are able to set an induction method on r, which allows us to prove a crucial integral estimate (Theorem 3.3) from which most results are subsequently obtained.

Since our results are new only when $r \geq 3$, we shall assume that Ω is one of the cones of matrices mentioned in (1.1). We denote by d the dimension of the matrix entries (that is, 1,2,4 or 8 if they are real, complex, quaternion or octonions, respectively), so that $n = r + r(r-1)d/2$. Our results remain valid when $r = 2$ (ie when $\Omega = \Lambda_n$), in which case

$d = n - 2$. Below, we denote by $\lfloor s \rfloor$ and $\lceil s \rceil$, respectively, the integer parts from below and above of a real number s.

THEOREM 1.2 *The operator P^+ is bounded from $L^\infty(D) \to L^q(D)$ if and only if $1 \le q < q_+$, where*

$$q_+ := \min\left\{ \frac{2n}{\lfloor r/2 \rfloor \lceil r/2 \rceil \, d}, 2 + \tfrac{4}{d} \right\} = \begin{cases} 4 + \frac{2}{\lfloor r/2 \rfloor} & \text{if } d = 1 \\ 2 + \frac{4}{d} & \text{if } d \ge 2. \end{cases} \tag{1.3}$$

Our second result slightly improves these indices for the original operator P, using an interpolation argument with diagonal mixed-norm estimates (see section 6 below).

THEOREM 1.4 *The Bergman projection P is bounded from $L^\infty(D) \to L^q(D)$ when*

$$1 \le q < q_+ + 2\big(1 - \frac{q_+}{q_0}\big),$$

where $q_0 = 4 + \frac{4}{(r-1)d}$.

Observe that $q_0 > q_+$ (hence the improvement above is strict), except for the cones $\mathrm{Sym}_+(r, \mathbb{R})$ with r odd, for which $q_0 = q_+$. In this special case we do not know whether the exponent q_+ can be improved for the operator P.

Finally we state some necessary conditions for the boundedness of $P : L^\infty \to L^q$, which are also new when $r \ge 3$.

THEOREM 1.5 *The Bergman projection P is unbounded from $L^\infty(D) \to L^q(D)$ when*

$$q > \bar{q} := \frac{4n}{\lfloor r/2 \rfloor \lceil r/2 \rceil \, d} \quad .$$

Versions of these results also hold in the more general context of Bergman projections P_ν associated with weighted Bergman spaces A^p_ν. In fact, this setting arises naturally in our proofs, so we shall state and prove in the next sections the corresponding more general results (see Corollaries 3.7 and 6.4). We also state in sections 6 and 7 the $L^p \to L^q$ estimates which can be obtained by interpolation from the above results, as well the corresponding necessary conditions.

Finally we remark that, as an application of these results, one obtains new embeddings for the Bloch space $\mathcal{B}(D)$ (as defined in [1] or [11]) of the type

$$\mathcal{B}(D) \hookrightarrow L^q(D). \tag{1.6}$$

Such an embedding is equivalent to the boundedness of $P : L^\infty(D) \to L^q(D)$ (see eg [7]), so it holds at least in the range of Theorem 1.4. Finding the largest q so that (1.6) holds seems then to be a difficult question when $r > 1$.

2 The transference principle

As mentioned above, our results in this paper concern the bounded symmetric domain D. We find easier to work in the unbounded domain $T_\Omega = \mathbb{R}^n + i\Omega$, where techniques of symmetric cones are more directly applicable, so we shall use the transfer principle described in [3]. Roughly speaking, this principle states that the $L^p - L^q$ continuity of an operator in D is equivalent to the "*local continuity*" of the transferred operator in the unbounded domain T_Ω. A general version of this principle is stated below in more detail.

Throughout this section Φ denotes the *Cayley transform*, mapping D conformally onto T_Ω, and explicitly given by

$$w \longmapsto \Phi(w) = i(\mathbf{e}+w)(\mathbf{e}-w)^{-1} \ ;$$

see eg [10, p. 190]. We write $J_\Phi(w)$ for its complex jacobian determinant, which is an analytic non-vanishing function in the open set Dom $\Phi = \{w \in \mathbb{C}^n \ : \ \Delta(\mathbf{e}-w) \neq 0\}$.

2.1 A general framework

The transference principle can be applied to general operators of the form

$$f \longmapsto Tf(z) = \int_D T(z,w)f(w)\,d\mu(w), \quad z \in D,$$

say with continuous kernel $T(z,w)$ and bounded in $L^2(D,d\mu)$. We shall assume the following rotation invariance

$$T(e^{i\theta}z, e^{i\theta}w) = T(z,w) \quad \text{and} \quad d\mu(e^{i\theta}w) = d\mu(w), \quad \forall\, \theta \in \mathbb{R}. \tag{2.1}$$

Given T and a set $H \subset \mathbb{C}^n$ we define a new operator T^H, acting on functions in D by

$$f \longmapsto T^H f(z) = \chi_H(z) T\big(f\chi_H\big)(z), \quad z \in D.$$

The next proposition gives a first criterion for the boundedness of T.

Proposition 2.2 *Let $1 \leq p,q \leq \infty$ and T and μ be as in (2.1). If $T^H \in \mathcal{B}(L^p(\mu), L^q(\mu))$ for all compact sets $H \subset$ Dom Φ, then $T \in \mathcal{B}(L^p(\mu), L^q(\mu))$.*

This follows essentially from the arguments in [3, §3], but we sketch the proof below for completeness. We shall use the following elementary lemmas.

Lemma 2.3 *For every $(z,w) \in \bar{D} \times \bar{D}$ there exists $\theta = \theta(z,w) \in \mathbb{R}$ such that $e^{i\theta}z, e^{i\theta}w \in$ Dom Φ.*

PROOF: Define the polynomials

$$p(\lambda) = \Delta(\mathbf{e} - \lambda z) \quad \text{and} \quad q(\lambda) = \Delta(\mathbf{e} - \lambda w), \quad \lambda \in \mathbb{C}.$$

Arguing by contradiction, assume that $p(\lambda)q(\lambda) = 0$ for all $|\lambda| = 1$. Then necessarily $pq \equiv 0$, which is not possible since $p(0) = q(0) = 1$.

□

LEMMA 2.4 *There exist compact sets $H_j, K_j \subset \bar{D}$ and numbers $\theta_j \in \mathbb{R}$, $j = 1, \dots, J$ such that*

(a) $\bar{D} \times \bar{D} = \cup_{j=1}^{J} H_j \times K_j$

(b) $e^{i\theta_j} H_j \,\cup\, e^{i\theta_j} K_j \subset \text{Dom}\ \Phi, \ \forall\, j = 1, \dots, J.$

PROOF: For each $(z,w) \in \bar{D} \times \bar{D}$ we can find, by the previous lemma, open neighborhoods $\mathcal{U}(e^{i\theta(z,w)}z), \mathcal{V}(e^{i\theta(z,w)}w) \Subset \text{Dom}\ \Phi$. This gives a covering of the form

$$\bar{D} \times \bar{D} \subset \bigcup_{(z,w) \in \bar{D} \times \bar{D}} e^{-i\theta(z,w)} \big[\mathcal{U}(e^{i\theta(z,w)}z) \times \mathcal{V}(e^{i\theta(z,w)}w)\big].$$

By compactness there is a subcovering indexed by a finite set $\Lambda \subset \bar{D} \times \bar{D}$. For each such index $\lambda = (z,w) \in \Lambda$ define

$$H_\lambda = \big[e^{-i\theta(z,w)} \overline{\mathcal{U}(e^{i\theta(z,w)}z)}\big] \cap \bar{D} \quad \text{and} \quad K_\lambda = \big[e^{-i\theta(z,w)} \overline{\mathcal{V}(e^{i\theta(z,w)}w)}\big] \cap \bar{D}.$$

These sets clearly satisfy $\bar{D} \times \bar{D} = \cup_{\lambda \in \Lambda} H_\lambda \times K_\lambda$ and $e^{i\theta(\lambda)} H_\lambda, e^{i\theta(\lambda)} K_\lambda \subset \text{Dom}\ \Phi$.

□

PROOF of Proposition 2.2: Let $f, g \in C_c(D)$. By the previous lemma, a change of variables and (2.1) we can write

$$\begin{aligned} \int_D \int_D f(w) T(z,w) g(z)\, d\mu(w) d\mu(z) &= \sum_{j=1}^{J} \int_{H_j} \int_{K_j} \dots \\ &= \sum_{j=1}^{J} \int_{\widetilde{H}_j} \int_{\widetilde{K}_j} f(e^{-i\theta_j}\eta) T(\xi,\eta) g(e^{-i\theta_j}\xi)\, d\mu(\eta) d\mu(\xi) \\ &= \sum_{j=1}^{J} \big\langle \chi_{\widetilde{H}_j} T\big[\chi_{\widetilde{K}_j} f(e^{-i\theta_j}\cdot)\big], g(e^{-i\theta_j}\cdot)\big\rangle_{d\mu}, \end{aligned}$$

where we have set $\widetilde{H}_j = e^{i\theta_j} H_j$ and $\widetilde{K}_j = e^{i\theta_j} K_j$. Then

$$\begin{aligned} \Big| \int_D Tf(z)\, g(z)\, d\mu(z) \Big| &\leq \sum_{j=1}^{J} \Big\| T^{\widetilde{H}_j \cup \widetilde{K}_j} \big[\chi_{\widetilde{K}_j} f(e^{-i\theta_j}\cdot)\big] \Big\|_{L^q(\mu)} \big\| g(e^{-i\theta_j}\cdot) \big\|_{L^{q'}(\mu)} \\ &\leq C \big\| f \big\|_{L^p(\mu)} \big\| g \big\|_{L^{q'}(\mu)} \end{aligned}$$

where in the last step we have used the assumption $T^H \in \mathcal{B}(L^p, L^q)$ when $H \subset \text{Dom}\,\Phi$. This proves that $T \in \mathcal{B}(L^p, L^q)$.

$\square$

Next, we consider operators $\mathcal{T}$, acting on functions in T_Ω, of the form

$$g \longmapsto \mathcal{T}g(\zeta) = \int_{T_\Omega} \mathcal{T}(\zeta,\eta) g(\eta)\, d\lambda(\eta), \quad \zeta \in T_\Omega.$$

We shall assume that $\mathcal{T}(\zeta,\eta)$ and $d\lambda$ are homogeneous, in the sense that for some $\alpha, \beta \in \mathbb{R}$

$$\mathcal{T}(R\zeta, R\eta) = R^\alpha \mathcal{T}(\zeta,\eta) \quad \text{and} \quad d\lambda(R\eta) = R^\beta d\lambda(\eta), \quad \forall\, R > 0. \tag{2.5}$$

Finally we say that $\mathcal{T} \sim T$ if the corresponding kernels and measures are related by

$$\mathcal{T}\big(\Phi(z), \Phi(w)\big) = T(z,w)\psi_1\big(J_\Phi(z)\big)\psi_2\big(J_\Phi(w)\big), \quad z, w \in D, \tag{2.6}$$

and

$$d\big(\lambda \circ \Phi\big)(w) = \psi_3\big(J_\Phi(w)\big)\, d\mu(w). \tag{2.7}$$

for some continuous non-vanishing functions ψ_1, ψ_2, ψ_3 in $\mathbb{C} \setminus (-\infty, 0]$.

The transference principle is then summarized in the following proposition. As before, for $K \subset \mathbb{C}^n$, we denote by $\mathcal{T}^K$ the operator acting on functions in T_Ω by

$$g \longmapsto \mathcal{T}^K g(\zeta) = \chi_K(\zeta)\, T\big(g\,\chi_K\big)(\zeta), \quad \zeta \in T_\Omega.$$

Also $\mathbb{B}$ denotes the closed unit ball of $\mathbb{C}^n$, and $\mathbb{B}_R$ the ball of radius R (centered at 0).

Proposition 2.8 *Let T and $\mathcal{T}$ be operators as above satisfying the properties* (2.1), (2.5), (2.6), (2.7). *Then, for every $p, q \in [1,\infty]$ the following are equivalent*

(a) T is bounded from $L^p(D, d\mu) \to L^q(D, d\mu)$;

(b) T^H is bounded from $L^p(D, d\mu) \to L^q(D, d\mu)$, for all compact $H \subset$ Dom Φ;

(c) $\mathcal{T}^K$ is bounded from $L^p(T_\Omega, d\lambda) \to L^q(T_\Omega, d\lambda)$, for all compact $K \subset \mathbb{C}^n$;

(d) $\mathcal{T}^{\mathbb{B}}$ is bounded from $L^p(T_\Omega, d\lambda) \to L^q(T_\Omega, d\lambda)$.

Proof: We have already proved the equivalence "(a)⇔(b)" in Proposition 2.2. The equivalence "(b)⇔(c)" is just a change of variables with the Cayley transform Φ, using the assumptions (2.6) and (2.7), and the fact that if $H \Subset \text{Dom}\,\Phi$ then

$$\frac{1}{c} \le \big|\psi_j\big(J_\Phi(w)\big)\big| \le c, \quad \forall\, w \in H \cap D, \quad j = 1, 2, 3,$$

for some constant $c = c(H) > 0$. Since "(c)⇒(d)" is trivial, it remains to show "(d)⇒(c)". Pick $R > 0$ such that $K \subset \mathbb{B}_R$. For each $f \in L^p(T_\Omega)$ with $\operatorname{Supp} f \subset K$ define $\tilde{f}(z) = f(z/R)$, which is supported in $\mathbb{B}$. By (d) we have

$$\Big[\int_{T_\Omega \cap \mathbb{B}} \Big| \int_{T_\Omega \cap \mathbb{B}} \mathcal{T}(z,w) \tilde{f}(w)\, d\lambda(w) \Big|^q d\lambda(z) \Big]^{\frac{1}{q}} \leq C \Big[\int_{T_\Omega \cap \mathbb{B}} \big| \tilde{f}(z) \big|^p d\lambda(z) \Big]^{\frac{1}{p}}.$$

Changing variables $\zeta = Rz$ and $\eta = Rw$ and using the homogeneity in (2.5) we see that

$$\Big[\int_{T_\Omega \cap \mathbb{B}_R} \Big| \int_{T_\Omega \cap \mathbb{B}_R} \mathcal{T}(\zeta,\eta) f(\eta)\, d\lambda(\eta) \Big|^q d\lambda(\zeta) \Big]^{\frac{1}{q}} \leq C\, R^{\alpha+\beta(1+\frac{1}{q}-\frac{1}{p})} \Big[\int_{T_\Omega \cap \mathbb{B}_R} \big| f(\zeta) \big|^p d\lambda(\zeta) \Big]^{\frac{1}{p}}.$$

This gives

$$\big\| \mathcal{T}^K f \big\|_{L^q(\lambda)} \leq \big\| \mathcal{T}^{\mathbb{B}_R} f \big\|_{L^q(\lambda)} \leq C_R \, \|f\|_{L^p(\lambda)}, \tag{2.9}$$

as we wished to prove.

□

2.2 Bergman projections

We are interested in applying the transference principle to the family of weighted Bergman projections in D and T_Ω. Using the notation in the text [10, Chapter XIII], these operators are defined for $\nu > \frac{2n}{r} - 1$ by

$$P_\nu f(z) = \int_D B^D_\nu(z,w)\, f(w)\, d\mu_\nu(w), \quad z \in D,$$

and

$$\mathcal{P}_\nu g(\zeta) = \int_{T_\Omega} B^{T_\Omega}_\nu(\zeta,\eta)\, g(\eta)\, d\lambda_\nu(\eta), \quad \zeta \in T_\Omega,$$

where the Bergman kernels and their associated measures have the explicit expressions

$$B^D_\nu(z,w) = c_\nu\, h(z,w)^{-\nu}, \qquad d\mu_\nu(w) = h(w)^{\nu - \frac{2n}{r}}\, dw$$

and

$$B^{T_\Omega}_\nu(\zeta,\eta) = c'_\nu\, \Delta(\zeta - \bar{\eta})^{-\nu}, \qquad d\lambda_\nu(\eta) = \Delta(\Im m\, \eta)^{\nu - \frac{2n}{r}}\, d\eta$$

for certain constants c_ν, c'_ν. Here $h(z,w)$ denotes the unique polynomial (holomorphic in z and antiholomorphic in w) such that $h(z) := h(z,z)$ is U-invariant and $h(x) = \Delta(\mathbf{e} - x^2)$, $x \in \mathbb{R}^n$. The unweighted case discussed in the introduction corresponds to $\nu = 2n/r$.

The Bergman kernels in these two domains are related by the formula

$$B^D_\nu(z,w) = \tilde{c}_\nu\, B^{T_\Omega}_\nu(\Phi(z), \Phi(w)) \Big(J_\Phi(z)\, \overline{J_\Phi(w)} \Big)^{\frac{\nu}{2n/r}}$$

(see [10, p.264]), while the measures satisfy*

$$d\big(\lambda_\nu \circ \Phi\big)(w) \;=\; 2^{2n-r\nu}\,\big|J_\Phi(w)\big|^{\frac{\nu}{n/r}}\,d\mu_\nu(w).$$

From here it is easily seen that the operators $T = P_\nu$ and $\mathcal{T} = \mathcal{P}_\nu$ satisfy the properties in (2.1), (2.5), (2.6) and (2.7). Thus, from Proposition 2.8 we obtain the following.

Corollary 2.10 *Let $\nu > \frac{2n}{r} - 1$ and $p, q \in [1, \infty]$. Then, the following are equivalent*

(a) P_ν is bounded from $L^p(D, d\mu_\nu) \to L^q(D, d\mu_\nu)$;

(b) $\mathcal{P}_\nu^{\mathbb{B}}$ is bounded from $L^p(T_\Omega, d\lambda_\nu) \to L^q(T_\Omega, d\lambda_\nu)$.

Remark 2.11 The statements in the corollary can only hold if $p \geq q$. Indeed, if we assume (b), from (2.9) and the homogeneity of the Bergman kernel it follows that, for all $R \geq 1$ and $f \in C_c^\infty(T_\Omega)$

$$\big\|\mathcal{P}_\nu f\big\|_{L^q(\mathbb{B}_R \cap T_\Omega, d\lambda_\nu)} \;\leq\; C\,R^{\nu r(\frac{1}{q}-\frac{1}{p})}\,\big\|f\big\|_{L^p(d\lambda_\nu)}.$$

Letting $R \to \infty$ we see that this is not possible if $p < q$ (unless $f \equiv 0$). When $p = q$ this argument also shows that the "local continuity" in (b) is actually equivalent to the boundedness of $\mathcal{P}_\nu$ in $L^p(T_\Omega, d\lambda_\nu)$.

2.3 Positive Bergman projections

The transference principle also applies to the positive operators

$$P_\nu^+ f(z) = \int_D |B_\nu^D(z,w)|\,f(w)\,d\mu_\nu(w) \quad \text{and} \quad \mathcal{P}_\nu^+ g(\zeta) = \int_{T_\Omega} |B_\nu^{T_\Omega}(\zeta,\eta)|\,g(\eta)\,d\lambda_\nu(\eta).$$

In this case we can even state a stronger result. We consider a new operator, acting on functions in Ω by

$$f \longmapsto \mathcal{Q}_\nu f(y) = \chi_{\mathbb{B}}(y) \int_{\Omega\cap\mathbb{B}} \frac{f(u)}{\Delta(y+u)^{\nu-\frac{n}{r}}}\,d\lambda_\nu(u), \quad y \in \Omega.$$

Here, with a slight abuse of notation, we still write $d\lambda_\nu$ for the measure $\Delta^{\nu-\frac{2n}{r}}(u)\,du$ in Ω and $\mathbb{B}$ for the closed unit ball in $\mathbb{R}^n$. We write $L_\nu^{\tilde{p},p}(T_\Omega) = L_y^p(\Omega, \lambda_\nu; L_x^{\tilde{p}}(\mathbb{R}^n))$, ie the space with mixed norm given by

$$\big\|F\big\|_{L_\nu^{\tilde{p},p}} \;=\; \Big(\int_\Omega \Big[\int_{\mathbb{R}^n} |F(x+iy)|^{\tilde{p}}\,dx\Big]^{p/\tilde{p}}\,d\lambda_\nu(y)\Big)^{\frac{1}{p}}.$$

*This formula is not explicit in [10], but is easily derived from the identity $\Delta\big(\Im m\,\Phi(w)\big) \;=\; |\Delta(\mathbf{e} - w)|^{-2}\,h(w)$ in [10, p. 263] and the fact $J_\Phi(w) = (2i)^n\Delta(e-w)^{-2n/r}$ in [10, p. 202].

PROPOSITION 2.12 *Let $\nu > \frac{2n}{r} - 1$ and $1 \leq q \leq p \leq \infty$. The following are equivalent*

(a) P_ν^+ is bounded from $L^p(D, d\mu_\nu) \to L^q(D, d\mu_\nu)$;

(b) $\mathcal{P}_\nu^{+,\mathbb{B}}$ is bounded from $L^p(T_\Omega, d\lambda_\nu) \to L^q(T_\Omega, d\lambda_\nu)$;

(c) $\mathcal{P}_\nu^{+,\mathbb{B}}$ is bounded from $L_\nu^{\tilde{p},p}(T_\Omega) \to L_\nu^{\tilde{q},q}(T_\Omega)$ for all $1 \leq \tilde{q} \leq \tilde{p} \leq \infty$;

(d) $\mathcal{Q}_\nu$ is bounded from $L^p(\Omega, d\lambda_\nu) \to L^q(\Omega, d\lambda_\nu)$.

In the proof we shall use the following fact (see [6, Lemma 2.18]): if $\alpha > \frac{2n}{r} - 1$ then

$$\int_{\mathbb{R}^n} \frac{dx}{|\Delta(x+iy)|^\alpha} = \frac{c_\alpha}{\Delta(y)^{\alpha - \frac{n}{r}}}, \quad \forall\, y \in \Omega\,. \tag{2.13}$$

We also need a local converse (see eg [5, Lemma 4.11]).

LEMMA 2.14 *Let $\alpha \in \mathbb{R}$ and $\gamma > 0$. Then there is a constant $C_{\alpha,\gamma} > 0$ such that*

$$\int_{\mathbb{R}^n \cap \mathbb{B}} \frac{dx}{|\Delta(x+iy)|^\alpha} \geq \frac{C_{\alpha,\gamma}}{\Delta(y)^{\alpha - \frac{n}{r}}}, \quad \forall\, y \in \Omega \text{ with } |y| \leq \gamma. \tag{2.15}$$

PROOF: We sketch the proof for completeness. Since $y \in \Omega$ has $|y| \leq \gamma$ it must hold that the invariant ball $B_1(cy/\gamma) \subset \mathbb{B}$, for a sufficiently small constant $c > 0$ (depending only on Ω; see eg [6, Lemma 2.9]). Writing $y = g\mathbf{e}$ with $g \in G$, and changing variables $x = gu$ we see that

$$\int_{\mathbb{R}^n \cap \mathbb{B}} \frac{dx}{|\Delta(x+iy)|^\alpha} \geq \int_{B_1(cy/\gamma)} \frac{dx}{|\Delta(x+iy)|^\alpha} = \Delta(y)^{\frac{n}{r} - \alpha} \int_{B_1(c\mathbf{e}/\gamma)} \frac{du}{|\Delta(u+i\mathbf{e})|^\alpha},$$

and the integral on the right is a positive constant $C_{\alpha,\gamma}$.

□

PROOF of Proposition 2.12: We already know "(a)⇔(b)", while "(c)⇒(b)" is trivial. To prove "(b)⇒(d)" it is enough to test with $F(u+iv) = \chi_{\mathbb{B}}(u)\,|f(v)|$. Indeed, Lemma 2.14 then easily gives

$$\mathcal{P}_\nu^{+,\mathbb{B}} F(x+iy) \geq \int_{\Omega \cap \mathbb{B}} |f(v)| \int_{\mathbb{B}_{1/2}} \frac{c'_\nu\, du}{|\Delta(u + i(y+v))|^\nu}\, \Delta^{\nu - \frac{2n}{r}}(v)\, dv \geq c'\, \mathcal{Q}_\nu |f|(y),$$

for $|x| \leq 1/2$ and $y \in \Omega \cap \mathbb{B}$. Therefore, averaging in x and using (b)

$$\|\mathcal{Q}_\nu f\|_{L^q(\Omega,\lambda_\nu)} \lesssim \|\mathcal{P}_\nu^{+,\mathbb{B}} F\|_{L^q(\Omega,\lambda_\nu; L^q_x(\mathbb{B}))} \lesssim \|\chi_{\mathbb{B}}(x) f(y)\|_{L^p_y(\lambda_\nu; L^p_x)} = c\,\|f\|_{L^p(\Omega,\lambda_\nu)}.$$

It remains to show that "(d)⇒(c)". We write $f_v = f(\cdot + iv)$. Then, for every fixed $y \in \Omega$ Minkowski's integral inequality gives

$$\begin{aligned}\left\|\mathcal{P}_\nu^{+,\mathbb{B}} f(\cdot + iy)\right\|_{L^{\tilde q}(\mathbb{R}^n)} &\le \int_{\Omega\cap\mathbb{B}}\int_{\mathbb{R}^n} \frac{c'_\nu \,\|f_v \chi_{\mathbb{B}}\|_{L^{\tilde q}(\mathbb{R}^n)}}{|\Delta(u+i(y+v))|^\nu}\,du\,d\lambda_\nu(v)\\ &= c''\int_{\Omega\cap\mathbb{B}} \frac{\|f_v\|_{L^{\tilde q}(\mathbb{B})}}{\Delta(y+v)^{\nu-\frac{n}{r}}}\,d\lambda_\nu(v) \;=\; c''\,\mathcal{Q}_\nu\big(\|f_v\|_{L^{\tilde q}(\mathbb{B})}\big)(y),\end{aligned}$$

where in the second step we have used (2.13). Thus, since $\tilde q \le \tilde p$ we conclude that

$$\left\|\mathcal{P}_\nu^{+,\mathbb{B}} f\right\|_{L_\nu^{\tilde q,q}(T_\Omega)} \;\lesssim\; \left\|\mathcal{Q}_\nu\big(\|f_v\|_{L^{\tilde p}(\mathbb{B})}\big)\right\|_{L^q(\Omega,\lambda_\nu)} \lesssim \left\| \,\|f_v\|_{L^{\tilde p}}\, \right\|_{L^p(\Omega,\lambda_\nu)} \;=\; \|f\|_{L_\nu^{\tilde p,p}}.$$

□

Finally, in the special case $p = \infty$ we can state the following.

Corollary 2.16 *Let $\nu > \frac{2n}{r} - 1$ and $q \in [1,\infty]$. Then P_ν^+ is bounded from $L^\infty(D, d\mu_\nu) \to L^q(D, d\mu_\nu)$ if and only if*

$$\int_{\Omega\cap\mathbb{B}} \Big| \int_{\Omega\cap\mathbb{B}} \frac{d\lambda_\nu(u)}{\Delta(y+u)^{\nu-\frac{n}{r}}}\Big|^q d\lambda_\nu(y) \;<\; \infty. \tag{2.17}$$

Proof: Since $\mathcal{Q}_\nu$ has a positive kernel, part (d) in Proposition 2.12 for $p = \infty$ is equivalent to $\mathcal{Q}_\nu \mathbf{1} \in L^q(\Omega, \lambda_\nu)$. This last condition is the same as (2.17).

□

3 The boundedness of $P_\nu^+ : L^\infty \to L_\nu^q$

3.1 The key integral estimate

By Corollary 2.16 we have reduced matters to show the integral estimate (2.17). For fixed $\gamma > \frac{n}{r} - 1$ we shall write:

$$I^\gamma(\mathbf{t}) \;:=\; \int_{\Omega\cap\mathbb{B}} \Delta^{-\gamma}(y+\mathbf{t})\,\Delta^{\gamma-\frac{n}{r}}(y)\,dy, \quad \mathbf{t} \in \Omega\cap\mathbb{B}, \tag{3.1}$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^n$. Our goal is to find precise estimates for this integral so that we can determine for what q's we have

$$\int_{\Omega\cap\mathbb{B}} \left|I^\gamma(t)\right|^q \Delta^{\gamma-\frac{n}{r}}(t)\,dt \;<\; \infty. \tag{3.2}$$

Setting $\gamma = \nu - \frac{n}{r}$ we shall then deduce results about $\mathcal{P}_\nu^+$.

Unfortunately, (3.1) does not seem easily computable, but we can estimate it quite precisely modulo logarithmic factors of $\mathbf{t} \in \Omega\cap\mathbb{B}$. We shall use the notation

$$A \lessapprox B \quad \text{if} \quad A \;\le\; c\,\big(1 + \log_+ \tfrac{1}{\Delta(\mathbf{t})}\big)^{c'} B, \quad \forall\, \mathbf{t} \in \Omega\cap\mathbb{B},$$

with constants c, c' which may depend on γ, r, n but are independent of $\mathbf{t}$. As usual, $A \lesssim B$ means $A \le cB$, with c a constant as before, while $A \approx B$ means $\frac{1}{c}B \le A \le cB$. We shall also use the notation

$$A \simeq B \quad \text{if} \quad B \lesssim A \lessapprox B.$$

Finally, and following the notation in the text [10], since the function in (3.1) is K-invariant, we may assume that $\mathbf{t}$ is a diagonal element of Ω, that is

$$\mathbf{t} = t_1 c_1 + \cdots + t_r c_r$$

where $\{c_1, \ldots, c_r\}$ is a fixed Jordan frame, and say $0 < t_1 \le t_2 \le \ldots \le t_r \le 1$. Our two main results can then be stated as follows.

Theorem 3.3 *Let $\gamma > \frac{n}{r} - 1$. Then, for all $\mathbf{t} = t_1 c_1 + \cdots + t_r c_r \in \Omega \cap \mathbb{B}$ with $0 < t_1 \le t_2 \le \ldots \le t_r \le 1$ we have*

$$I^\gamma(\mathbf{t}) \simeq \begin{cases} \left(\frac{1}{t_2 t_3}\right)^{\frac{d}{2}} \left(\frac{1}{t_4 t_5}\right)^{2\frac{d}{2}} \cdots \left(\frac{1}{t_{r-1} t_r}\right)^{\frac{r-1}{2}\frac{d}{2}}, & \text{if } r \text{ is odd} \\ \left(\frac{1}{t_2 t_3}\right)^{\frac{d}{2}} \left(\frac{1}{t_4 t_5}\right)^{2\frac{d}{2}} \cdots \left(\frac{1}{t_{r-2} t_{r-1}}\right)^{\frac{r-2}{2}\frac{d}{2}} \left(\frac{1}{t_r}\right)^{\frac{r}{2}\frac{d}{2}}, & \text{if } r \text{ is even.} \end{cases} \tag{3.4}$$

For another expression of (3.4) in terms of symmetric polynomials, see Remark 4.9 below.

Theorem 3.5 *Let $\gamma > \frac{n}{r} - 1$. Then, the integral in (3.2) is finite if and only if $q < Q_\gamma$, where*

$$Q_\gamma := \begin{cases} 2 + \frac{4}{d}(\gamma - \frac{n}{r} + 1) & \text{if } \frac{(r-1)d}{2} < \gamma \le \frac{rd}{2} \\ 8(\gamma - \frac{d}{2})/[(r-1)d] & \text{if } \frac{rd}{2} < \gamma < \lceil \frac{r}{2} \rceil d \\ 2r\gamma / \big(\lfloor r/2 \rfloor \lceil r/2 \rceil \, d\big) & \text{if } \gamma \ge \lceil \frac{r}{2} \rceil d. \end{cases} \tag{3.6}$$

As a consequence we obtain the following generalization of Theorem 1.2. We denote

$$q_{+,\nu} = Q_{\nu - n/r}.$$

Corollary 3.7 *Let $\nu > \frac{2n}{r} - 1$. Then P_ν^+ is bounded from $L^\infty(D) \to L^q(D, d\mu_\nu)$ if and only if $1 \le q < q_{+,\nu}$.*

Remark 3.8 The unweighted case $\nu = \frac{2n}{r}$ corresponds to $\gamma = \frac{n}{r} = 1 + (r-1)\frac{d}{2}$ in (3.6). That is, $q_{+,2n/r} = q_+$ as defined in (1.3), hence establishing Theorem 1.2.

3.2 A more general integral estimate

To prove Theorem 3.3 we shall use an induction process in the rank which requires computing the following more general integrals:

$$I_r^{\mu,\gamma}(\mathbf{t}) \;=\; \int_{\Omega\cap\mathbb{B}} \Delta^{-\mu}(y+\mathbf{t})\,\Delta^{\gamma-\frac{n}{r}}(y)\,dy, \quad \mathbf{t}\in\Omega\cap\mathbb{B}, \tag{3.9}$$

where as before $\gamma > \frac{n}{r}-1$ is fixed, and the parameter μ can be any real number (although we shall focus below in the case $\mu \geq \gamma$). Since $I_r^{\mu,\gamma}(\mathbf{t})$ is K-invariant, we shall assume throughout that

$$\mathbf{t} = t_1c_1 + \ldots + t_rc_r \quad \text{with} \quad 0 < t_1 \leq t_2 \leq \ldots \leq t_r \leq 1. \tag{3.10}$$

Optimal bounds for these integrals can be easily obtained when μ is sufficiently large (or small) compared to γ; namely

$$I_r^{\mu,\gamma}(\mathbf{t}) \;\approx\; \begin{cases} \left(\frac{1}{\Delta(\mathbf{t})}\right)^{\mu-\gamma} & , \text{ when } \mu > \gamma + \frac{n}{r} - 1 \\ \quad 1 & , \text{ when } \mu < \gamma - (\frac{n}{r}-1). \end{cases} \tag{3.11}$$

Indeed, each of the inequalities "$\lesssim$" follow easily from the trivial majorizations

$$I_r^{\mu,\gamma}(\mathbf{t}) \leq \int_{\Omega} \Delta^{-\mu}(y+\mathbf{t})\,\Delta^{\gamma-\frac{n}{r}}(y)\,dy \quad \text{and} \quad I_r^{\mu,\gamma}(\mathbf{t}) \leq I_r^{\mu,\gamma}(\mathbf{0}).$$

Conversely, the inequalities "$\gtrsim$" can be obtained restricting the respective domains of integration to the invariant balls $B_1(\mathbf{t})$ and $B_1(\mathbf{e}/r)$. We leave details to the reader.

We are interested in obtaining optimal estimates (modulos log's) for $I_r^{\mu,\gamma}(\mathbf{t})$ in the remaining cases, that is when

$$\gamma - \tfrac{(r-1)d}{2} \;\leq\; \mu \;\leq\; \gamma + \tfrac{(r-1)d}{2},$$

and more specifically, when $\mu = \gamma$. As we shall see, the behavior of the integral changes as the parameter μ belongs to each of the subintervals

$$\Big[\gamma + (\ell-1)\tfrac{d}{2}\,,\; \gamma + \ell\tfrac{d}{2}\Big), \quad \ell = -(r-2), \ldots, r-1.$$

For instance, when $r = 2$ an explicit computation in the lines of [3] gives the following (which is also a particular instance of Theorem 3.12 below):

$$I_2^{\mu,\gamma}(\mathbf{t}) \simeq \begin{cases} \left(\frac{1}{t_1t_2}\right)^{\mu-\gamma} & , \text{ if } \quad \mu \geq \gamma + \frac{d}{2} \\ \left(\frac{1}{t_1}\right)^{\mu-\gamma}\left(\frac{1}{t_2}\right)^{d/2} & , \text{ if } \quad \gamma \leq \mu < \gamma + \frac{d}{2} \\ \left(\frac{1}{t_2}\right)^{\mu-\gamma+\frac{d}{2}} & , \text{ if } \quad \gamma - \frac{d}{2} \leq \mu < \gamma \\ \quad 1 & , \text{ if } \quad \mu < \gamma - \frac{d}{2}. \end{cases}$$

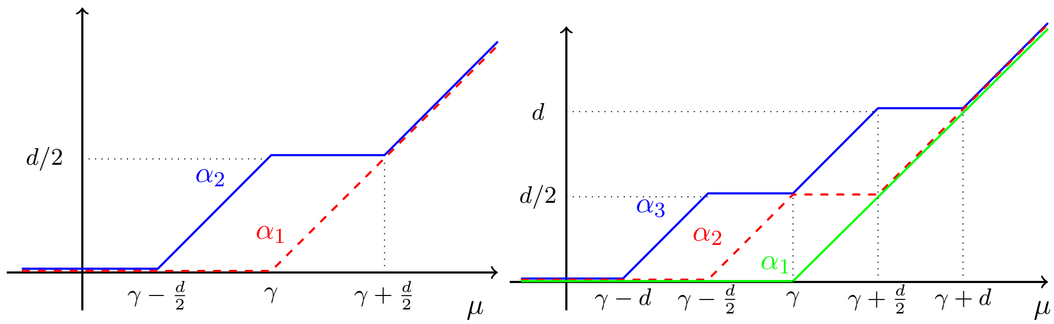

Figure 3.1: Graph of the powers $\alpha_i(\mu)$ as μ varies, for the cases $r = 2$ and $r = 3$.

It is possible (but quite tedious) to carry out similar explicit computations when $r = 3$. If we write $I_r^{\mu,\gamma}(\mathbf{t}) \simeq \prod_{i=1}^r (1/t_i)^{\alpha_i}$, then Figure 2.1 shows the graph of the powers $\alpha_i = \alpha_i(\mu)$ as the parameter μ varies, when $r = 2, 3$. Our next theorem will prove that the pattern that one can see in these examples continues to hold in higher ranks (at least when $\mu \geq \gamma$).

Theorem 3.12 *Let $\gamma > (r-1)d/2$. Assume that*

$$\mu \in \Big[\gamma + (\ell - 1)\tfrac{d}{2}\,,\ \gamma + \ell\tfrac{d}{2}\Big), \qquad \text{for some } \ell = 1, 2, \ldots, r.$$

Then, the following holds for all $\mathbf{t} = t_1 c_1 + \cdots + t_r c_r$ with $0 < t_1 \leq t_2 \leq \ldots \leq t_r \leq 1$:

(a) If $r - \ell$ is even, then

$$I_r^{\mu,\gamma}(\mathbf{t}) \simeq \left(\frac{1}{t_1 \cdots t_\ell}\right)^{\mu-\gamma} \prod_{i=1}^{\frac{r-\ell}{2}} \left[\left(\frac{1}{t_{\ell+2i-1}}\right)^{(\ell+i-1)\frac{d}{2}} \left(\frac{1}{t_{\ell+2i}}\right)^{\mu-\gamma+i\frac{d}{2}}\right]. \tag{3.13}$$

(b) If $r - \ell$ is odd, then

$$I_r^{\mu,\gamma}(\mathbf{t}) \simeq \left(\frac{1}{t_1 \cdots t_\ell}\right)^{\mu-\gamma} \prod_{i=1}^{\lfloor\frac{r-\ell}{2}\rfloor} \left[\left(\frac{1}{t_{\ell+2i-1}}\right)^{(\ell+i-1)\frac{d}{2}} \left(\frac{1}{t_{\ell+2i}}\right)^{\mu-\gamma+i\frac{d}{2}}\right] \left(\frac{1}{t_r}\right)^{(\ell+\lfloor\frac{r-\ell}{2}\rfloor)\frac{d}{2}}. \tag{3.14}$$

Remark 3.15 Theorem 3.3 corresponds to the special case $\mu = \gamma$ of the above theorem. It is possible to find similar expressions for $I_r^{\mu,\gamma}(\mathbf{t})$ when $\ell = 0, -1, -2, \ldots$, in accordance with Figure 3.1, but we shall not make use of these cases here.

4 Proof of Theorem 3.12

The theorem will be proved by induction in the rank. With our scheme we shall establish the result for the ℓ-th interval in cones of rank r, from the hypothesis in the $(\ell+1)$-th interval in cones of rank $r-1$. This explains why we must consider the full range of μ to get to the case $\mu = \gamma$, and also the distinction between the cases (a) and (b).

We also observe that for $\ell = r$ the result is already known, since then $\mu \geq \gamma + (r-1)d/2$ and the identity (3.13) is just one of the easy cases described in (3.11). The only comment concerns the endpoint $\mu = \gamma + (r-1)d/2$, but it is not difficult to see that (3.11) remains valid also in this case if we replace "$\approx$" by "$\simeq$" (see eg [6, Lemma 2.20]).

4.1 The recursive lemma

We follow the notation in the text [10]. The ambient space $V = \mathbb{R}^n$ is endowed with the structure of Euclidean Jordan algebra induced by the symmetric cone Ω, and $\{c_1, c_2, \dots, c_r\}$ is a fixed Jordan frame in V. Consider the Peirce decomposition with respect to c_r (see [10, Ch. IV]), that is

$$V = \mathbb{R}c_r \oplus V(c_r, \tfrac{1}{2}) \oplus V(c_r, 0).$$

We shall write $V' = V(c_r, 0)$, which is a Jordan algebra of rank $r-1$, and denote by Ω' the associated symmetric cone (the interior of the cone of squares; see [10, Ch. 3]). In general, if $\mathbf{x} \in V$ we write $\mathbf{x}'$ for the orthogonal projection of $\mathbf{x}$ onto V'. In particular, if $\mathbf{t} = t_1c_1 + \ldots + t_rc_r \in \Omega$, then $\mathbf{t}' = t_1c_1 + \ldots + t_{r-1}c_{r-1} \in \Omega'$. Finally recall that $V(c_r, 1/2)$ is a vector space of dimension $(r-1)d$, which we sometimes identify with $\mathbb{R}^{(r-1)d}$.

The next lemma establishes the iteration procedure which will be used in the proof of Theorem 3.12.

Lemma 4.1 *For all $\mu, \gamma \in \mathbb{R}$ and $\mathbf{t}$ as in* (3.10) *we have*

$$I_r^{\mu,\gamma}(\mathbf{t}) \approx \frac{1}{t_r^{\mu-\gamma}} \int_0^{1/t_r} \frac{s^\gamma}{(1+s)^\mu} \int_{\mathbf{v} \in V(c_r, \frac{1}{2}) :\, |\mathbf{v}| \leq 1} I_{r-1}^{\mu, \gamma - \frac{d}{2}} \Big(\mathbf{t}' + \tfrac{(\mathbf{v}^2)'}{1+s} \Big) \, d\mathbf{v} \, \frac{ds}{s}. \tag{4.2}$$

Proof: The proof follows from a change of variables as in [10, p. 139], which we reproduce here for completeness. Using the Peirce decomposition of an element $\mathbf{y} \in \Omega$,

$$\mathbf{y} = u\, c_r + \mathbf{y}_{1/2} + \mathbf{y}' \in \mathbb{R}^+ c_r \oplus V(c_r, 1/2) \oplus V', \tag{4.3}$$

the determinant Δ can be factored as follows:

$$\Delta(\mathbf{y}) = u\, \Delta' \big(\mathbf{y}' - \tfrac{1}{u} (\mathbf{y}_{1/2}^2)' \big) ,$$

where Δ' denotes the determinant associated with the algebra V' (see [10, pp. 107,114]). In particular we can also write

$$\Delta(\mathbf{t}+\mathbf{y}) \;=\; (t_r+u)\,\Delta'\Big(\mathbf{t}'+\mathbf{y}'-\tfrac{1}{t_r+u}(\mathbf{y}_{1/2}^2)'\Big)\,.$$

As shown in [10, Prop. VI.3.2], the element $\mathbf{w}=\mathbf{y}'-\frac{1}{u}(\mathbf{y}_{1/2}^2)'$ belongs to the cone Ω' if and only if $\mathbf{y}\in\Omega$. Now we perform the following change of variables

$$\mathbf{y}_{1/2}=\sqrt{u}\,\mathbf{v},\quad \mathbf{y}'=\mathbf{w}+(\mathbf{v}^2)', \tag{4.4}$$

where $\mathbf{v}\in V(c_r,1/2)$ and $\mathbf{w}\in\Omega'$. This leads to

$$\begin{aligned} I_r^{\mu,\gamma}(\mathbf{t}) &= \int_{\Omega\cap\mathbb{B}} \Delta^{-\mu}(\mathbf{t}+\mathbf{y})\,\Delta^{\gamma-\frac{n}{r}}(\mathbf{y})\,d\mathbf{y} \qquad (4.5)\\ &= \int_R \frac{u^{\gamma-\frac{n}{r}}}{(u+t_r)^{\mu}}\,\Delta'\Big(\mathbf{t}'+\mathbf{w}+\tfrac{t_r}{u+t_r}(\mathbf{v}^2)'\Big)^{-\mu}\,\Delta'(\mathbf{w})^{\gamma-\frac{n}{r}}\,\sqrt{u}^{(r-1)d}\,du\,d\mathbf{v}\,d\mathbf{w}, \end{aligned}$$

where the region of integration $R\subset\Omega$ is specified below. Recall first that $|\xi|\approx(\xi|\mathbf{e})$ when $\xi\in\Omega$ (see eg [6, Lemma 2.9]), and observe from (4.4) that

$$(\mathbf{y}'|\mathbf{e}) \;=\; (\mathbf{w}|\mathbf{e})+(\mathbf{v}|\mathbf{v}\mathbf{e}') \;=\; (\mathbf{w}|\mathbf{e})+\tfrac12|\mathbf{v}|^2.$$

Since $\mathbf{y}\in\Omega\cap\mathbb{B}$, then necessarily the region R is contained in

$$\big\{(u,\mathbf{v},\mathbf{w})\in\mathbb{R}^+\times V(c_r,\tfrac12)\times\Omega' \;:\; 0<u\le\gamma_1,\ |\mathbf{v}|\le\gamma_2,\ |\mathbf{w}|\le\gamma_3\big\}, \tag{4.6}$$

for some constants $\gamma_1,\gamma_2,\gamma_3>0$. Conversely, we claim that if $\gamma_1,\gamma_2,\gamma_3$ are chosen sufficiently small, then the set in (4.6) must also be contained in R. Indeed, for each $(u,\mathbf{v},\mathbf{w})$ in that set the corresponding vector $\mathbf{y}$ (defined by (4.3) and (4.4)) belongs to Ω and has norm

$$|\mathbf{y}| \;\le\; |u|+|\mathbf{y}_{1/2}|+|\mathbf{y}'| \;\le\; u+\sqrt{u}\,|\mathbf{v}|+|\mathbf{w}|+\tfrac12|\mathbf{v}|^2.$$

So if $\gamma_1,\gamma_2,\gamma_3$ are sufficiently small then $|\mathbf{y}|\le1$, and hence $(u,\mathbf{v},\mathbf{w})\in R$. Going back to (4.5), and using $\frac{n}{r}=1+(r-1)\frac{d}{2}$ we see that

$$I_r^{\mu,\gamma}(\mathbf{t}) \approx \int_{0<u\lesssim1} \frac{u^{\gamma}}{(u+t_r)^{\mu}} \int_{\mathbf{v}\in V(c_r,\frac12):\,|\mathbf{v}|\lesssim1} I_{r-1}^{\mu,\gamma-\frac{d}{2}}\Big(\mathbf{t}'+\tfrac{t_r}{u+t_r}(\mathbf{v}^2)'\Big)\,d\mathbf{v}\,\frac{du}{u}.$$

From here, the change of variables $u=t_rs$ (incorporating constants if necessary) easily leads to the expression stated in the lemma.

□

4.2 Facts about determinants

We denote by $a_i(\mathbf{x})$, $i = 0, 1, \dots, r$, the *symmetric polynomials* in V, defined by

$$\Delta(\lambda\mathbf{e} - \mathbf{x}) \;=\; \sum_{i=0}^{r} a_i(\mathbf{x})(-1)^i \lambda^{r-i}$$

(see [10, p. 28]). The above expression is K-invariant (in $\mathbf{x}$), so the polynomials $a_i(\mathbf{x})$ only depend on the eigenvalues of $\mathbf{x}$, say $t_1, \dots, t_r$, and are symmetric in these variables. Observe that $a_i(t_1, \dots, t_r)$ are also given by the formula

$$\prod_{j=1}^{r}(\lambda - t_j) \;=\; \sum_{i=0}^{r} a_i(t_1, \dots, t_r)(-1)^i \lambda^{r-i}\,,$$

from which we obtain the usual expressions

$$a_1 = t_1 + \ldots + t_r, \quad a_2 = t_1 t_2 + \ldots + t_{r-1} t_r, \qquad \ldots \quad , \quad a_r = t_1 \cdots t_r.$$

In fact, we have the following

Lemma 4.7 *If $\mathbf{t} \in \Omega$ has eigenvalues $\{t_1, \dots, t_r\}$ with $0 < t_1 \le t_2 \le \ldots \le t_r$, then*

$$a_j(\mathbf{t}) \;\approx\; t_{r-j+1} \cdots t_r, \quad j = 1, 2, \dots, r.$$

Using symmetric polynomials we can rewrite the outcomes in Theorem 3.12 as follows.

Proposition 4.8 *Let $\mu, \gamma \in \mathbb{R}$ and $\ell \in \{1, 2, \dots, r\}$. Define*

$$\alpha \equiv \gamma + (\ell - 1)\,\tfrac{d}{2} \qquad \textit{and} \qquad \beta \equiv \gamma + \ell\,\tfrac{d}{2}.$$

For $\mathbf{t}$ as in (3.10) we have the following:

(i) If $r - \ell = 2j$, then the right hand side of (3.13) is $\approx$ to

$$\left(\tfrac{1}{a_r(\mathbf{t})}\right)^{\mu-\gamma}\left(\tfrac{1}{a_{2j}(\mathbf{t})}\right)^{\beta-\mu}\left(\tfrac{1}{a_{2j-1}(\mathbf{t})}\right)^{\mu-\alpha}\cdots\left(\tfrac{1}{a_2(\mathbf{t})}\right)^{\beta-\mu}\left(\tfrac{1}{a_1(\mathbf{t})}\right)^{\mu-\alpha}.$$

(ii) If $r - \ell = 2j - 1$, then the right hand side of (3.14) is $\approx$ to

$$\left(\tfrac{1}{a_r(\mathbf{t})}\right)^{\mu-\gamma}\left(\tfrac{1}{a_{2j-1}(\mathbf{t})}\right)^{\beta-\mu}\left(\tfrac{1}{a_{2j-2}(\mathbf{t})}\right)^{\mu-\alpha}\cdots\left(\tfrac{1}{a_3(\mathbf{t})}\right)^{\beta-\mu}\left(\tfrac{1}{a_2(\mathbf{t})}\right)^{\mu-\alpha}\left(\tfrac{1}{a_1(\mathbf{t})}\right)^{\beta-\mu}.$$

Remark 4.9 Similarly, the expression for $I^\gamma(\mathbf{t})$ in (3.4) takes the simpler form

$$I^\gamma(\mathbf{t}) \simeq \begin{cases} \big[a_2(\mathbf{t})a_4(\mathbf{t})\cdots a_{r-1}(\mathbf{t})\big]^{-d/2} & \text{if } r \text{ is odd} \\ \big[a_1(\mathbf{t})a_3(\mathbf{t})\cdots a_{r-1}(\mathbf{t})\big]^{-d/2} & \text{if } r \text{ is even.} \end{cases}$$

In view of formula (4.2), we need to find estimates for the expressions $a_i'(\mathbf{t}'+(\mathbf{v}^2)')$, where a_i' denotes the symmetric polynomial with respect to the algebra V'. This is done in the next lemma.

Lemma 4.10 *Let* $\mathbf{v}\in V(c_r,1/2)$*, which we write as* $\mathbf{v}=\sum_{i=1}^{r-1}\mathbf{v}_i$ *with* $\mathbf{v}_i\in V_{ir}$*. Let* $\mathbf{s}=s_1c_1+\ldots+s_{r-1}c_{r-1}\in\Omega'$*. Then*

$$\Delta'\big(\mathbf{s}+(\mathbf{v}^2)'\big)\;=\;s_1\cdots s_{r-1}\left(1+\sum_{i=1}^{r-1}\frac{|\mathbf{v}_i|^2}{s_i}\right). \tag{4.11}$$

Moreover, when $\mathbf{t}$ *is as in* (3.10) *and* $j=1,2,\ldots,r-1$ *we have*

$$a_j'\big(\mathbf{t}'+(\mathbf{v}^2)'\big)\;=\;\sum_{1\le\ell_1<\ldots<\ell_j\le r-1}t_{\ell_1}\cdots t_{\ell_j}\left(1+\sum_{k\in\{\ell_1,\ldots,\ell_j\}}\frac{|\mathbf{v}_k|^2}{t_k}\right). \tag{4.12}$$

Proof: Observe first that by definition of the a_j''s

$$\Delta'\Big((\mathbf{v}^2)'+\mathbf{t}'-\lambda\mathbf{e}'\Big)\;=\;\sum_{i=0}^{r-1}a_i'\big(\mathbf{t}'+(\mathbf{v}^2)'\big)\,(-\lambda)^{r-1-i}$$

Thus, using (4.11) (with $\mathbf{s}=\mathbf{t}'-\lambda\mathbf{e}'$) one can easily compute the left hand side to obtain the formula stated in (4.12). Therefore, we must only prove the formula in (4.11).

At the moment we do not know a direct proof of (4.11), so we shall establish the formula separately for each type of Jordan algebra. Assume first that $V=\mathrm{Her}(r,\mathbb{C})$. Then $\mathbf{v}\in V(c_r,1/2)$ can be identified with the hermitian matrix

$$\mathbf{v}\;=\;\begin{pmatrix} & & & v_1\\ & \mathbf{0} & & \vdots\\ & & & v_{r-1}\\ \bar v_1 & \cdots & \bar v_{r-1} & 0\end{pmatrix}, \tag{4.13}$$

where $\overrightarrow{v}=(v_1,\ldots,v_{r-1})\in\mathbb{C}^{r-1}$, and therefore

$$(\mathbf{v}^2)'=\mathrm{row}\,(\bar v_1\overrightarrow{v},\ldots,\bar v_{r-1}\overrightarrow{v}). \tag{4.14}$$

Since Δ' coincides with the usual determinant of complex matrices, we can use multilinearity to obtain

$$\begin{aligned}\Delta'\big(\mathbf{s}+(\mathbf{v}^2)'\big)\;&=\;\mathrm{Det}\big(s_1\overrightarrow{e}_1+\bar v_1\overrightarrow{v},\ldots,s_{r-1}\overrightarrow{e}_{r-1}+\bar v_{r-1}\overrightarrow{v}\big)\\ &=\;s_1\cdots s_{r-1}\;+\;\sum_{j=1}^{r-1}\mathrm{Det}\big(s_1\overrightarrow{e}_1,\ldots,\bar v_j\overrightarrow{v},\ldots,s_{r-1}\overrightarrow{e}_{r-1}\big)\;+\;0\\ &=\;s_1\cdots s_{r-1}\;+\;\sum_{j=1}^{r-1}s_1\cdots|v_j|^2\cdots s_{r-1},\end{aligned} \tag{4.15}$$

which is the same as (4.11). This proof can also be applied when $V = \mathrm{Sym}(r,\mathbb{R})$ (replacing the complex entries in $\mathbf{v}$ by real numbers) and even when $V = \mathrm{Her}(r,\mathbb{H})$. In this last case some care is needed with quaternionic determinants (which in general are not multilinear), so we describe the argument in some more detail in the appendix.

We now prove (4.11) when $V = \mathbb{R}^n$ is a Jordan algebra of rank 2 (ie it is associated with a light-cone). In this case we let $c_1 = (\frac{1}{2}, \frac{1}{2}, \mathbf{0})$ and $c_2 = (\frac{1}{2}, -\frac{1}{2}, \mathbf{0})$, so that $V(c_2, 1/2) = V_{12} = \{\mathbf{v} = (0, 0, \overrightarrow{v}) \;:\; \overrightarrow{v} \in \mathbb{R}^{n-2}\}$. An easy computation using the Jordan product in V (see [10, Ch. II]) shows that

$$(\mathbf{v}^2)' \;=\; |\overrightarrow{v}|^2\, c_1.$$

Thus,

$$\Delta'\big(s_1 c_1 + (\mathbf{v}^2)'\big) \;=\; s_1 + |\overrightarrow{v}|^2 \tag{4.16}$$

as we wished to prove.

Finally we verify the exceptional Jordan algebra $V = \mathrm{Her}(3,\mathbb{O})$, which has rank 3. In this case, $\mathbf{v} \in V(c_3, 1/2)$ can also be identified with a matrix as in (4.13) (with $r = 3$), where v_1, v_2 are octonions. Then (4.14) remains valid, and we have

$$\begin{aligned} \Delta'\big(\mathbf{s} + (\mathbf{v}^2)'\big) \;&=\; \det_{V'} \begin{pmatrix} s_1 + |v_1|^2 & \bar{v}_1 v_2 \\ \bar{v}_2 v_1 & s_2 + |v_2|^2 \end{pmatrix} \\ &=\; (s_1 + |v_1|^2)(s_2 + |v_2|^2) - |v_1 v_2|^2 \;=\; s_1 s_2 \left(1 + \tfrac{|v_1|^2}{s_1} + \tfrac{|v_2|^2}{s_2}\right), \end{aligned}$$

where in the second line we use the determinant of the 2-rank algebra V', and the property of octonions $|v_1 v_2| = |v_1||v_2|$.

□

An immediate consequence of (4.12) is the following.

Corollary 4.17 *Let* $\mathbf{v} = \sum_{i=1}^{r-1} \mathbf{v}_i \in V(c_r, 1/2) = \oplus_{1 \le i < r} V_{ir}$*, and let* $\mathbf{t}$ *be as in* (3.10)*. Then, if* $j = 1, 2, \ldots, r-1$ *we have*

$$a_j'\big(\mathbf{t}' + (\mathbf{v}^2)'\big) \;\ge\; t_{r-j} \cdots t_{r-1} \left(1 + \tfrac{|\mathbf{v}_{r-j}|^2}{t_{r-j}} + \ldots + \tfrac{|\mathbf{v}_{r-1}|^2}{t_{r-1}}\right). \tag{4.18}$$

Conversely, if $|\mathbf{v}| \le 1$*, and* $|\mathbf{v}_i| \ge 1/2$ *for* $r - j \le i \le r - 1$*, then*

$$a_j'\big(\mathbf{t}' + (\mathbf{v}^2)'\big) \;\lesssim\; t_{r-j} \cdots t_{r-1} \left(1 + \tfrac{|\mathbf{v}_{r-j}|^2}{t_{r-j}} + \ldots + \tfrac{|\mathbf{v}_{r-1}|^2}{t_{r-1}}\right). \tag{4.19}$$

4.3 Proof of Theorem 3.12, part (a)

When $r = 2$ there is nothing to prove, since then $\ell = 2$ and we are in one of easy cases described in (3.11).

Assume by induction that part (a) of the theorem holds with r replaced by $r-1$, and let us show how to obtain from here the case of rank r. Since $\ell = r$ is already known, we choose $\ell \in \{1, \dots, r-2\}$ as in (a), and for simplicity write $r - \ell = 2j$ for some $j \in \big\{1, \dots, \lfloor \frac{r-1}{2} \rfloor\big\}$. For such fixed ℓ (and j), and for μ in the interval

$$\gamma + (\ell - 1)\, \tfrac{d}{2} \,\leq\, \mu \,<\, \gamma + \ell\, \tfrac{d}{2}\,, \tag{4.20}$$

we must evaluate $I_r^{\mu,\gamma}(\mathbf{t})$. By the recursive lemma 4.1 we have

$$I_r^{\mu,\gamma}(\mathbf{t}) \,\approx\, \frac{1}{t_r^{\mu-\gamma}} \int_0^{1/t_r} \frac{s^\gamma}{(1+s)^\mu} \int_{\mathbf{v}\in V(c_r,\frac12):\,|\mathbf{v}|\leq 1} I_{r-1}^{\mu,\gamma-\frac{d}{2}} \Big(\mathbf{t}' + \tfrac{(\mathbf{v}^2)'}{1+s}\Big)\, d\mathbf{v}\, \frac{ds}{s}. \tag{4.21}$$

We shall use the induction hypothesis to estimate the factor $I_{r-1}^{\mu,\gamma-\frac{d}{2}}$. To do so, observe that (4.20) is the same as

$$\big(\gamma - \tfrac{d}{2}\big) \,+\, \ell\, \tfrac{d}{2} \,\leq\, \mu \,<\, \big(\gamma - \tfrac{d}{2}\big) \,+\, \big(\ell + 1\big)\, \tfrac{d}{2}, \tag{4.22}$$

with $\ell + 1 \in \{2, \dots, r-1\}$, so we are indeed in the situation of part (a) of the theorem for the cone Ω' of rank $r-1$. Call $\mathbf{y} = \mathbf{t}' + (\mathbf{v}^2)'/(1+s)$. Then the induction hypothesis, formulated as in Proposition 4.8, gives

$$I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y}) \,\simeq\, \Big(\tfrac{1}{a'_{r-1}(\mathbf{y})}\Big)^{\mu-(\gamma-d/2)} \Big(\tfrac{1}{a'_{2j-2}(\mathbf{y})}\Big)^{\beta-\mu} \Big(\tfrac{1}{a'_{2j-3}(\mathbf{y})}\Big)^{\mu-\alpha} \cdots\, \Big(\tfrac{1}{a'_{2}(\mathbf{y})}\Big)^{\beta-\mu} \Big(\tfrac{1}{a'_{1}(\mathbf{y})}\Big)^{\mu-\alpha} \tag{4.23}$$

where α and β denote respectively the left and right endpoints in (4.22) (which coincide with the left and right endpoints in (4.20)). Observe that the logarithmic terms hidden in "$\simeq$" of (4.23) are actually controlled by $(1 + \log 1/\Delta(\mathbf{t}))^c$, independently of s and $\mathbf{v}$, since $\Delta'(\mathbf{y}) \geq \Delta'(\mathbf{t}') \geq \Delta(\mathbf{t})$ if $t_r \leq 1$.

To deal with the expression in (4.23) we need to use the estimates in (4.18). For simplicity, we change variables $\mathbf{v}_i = \sqrt{t_i(1+s)}\; z_i$, $i = 1, \dots, r-1$, and identify $\mathbf{z} = (z_1, \dots, z_{r-1})$ with an element in $\mathbb{R}^{d(r-1)}$. Then (4.23) and (4.18) give

$$I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y}) \,\lessapprox\, A(\mathbf{t}) \,\times\, B(\mathbf{z}), \tag{4.24}$$

where

$$A(\mathbf{t}) = \Big(\tfrac{1}{t_1 \cdots t_{r-1}}\Big)^{\mu-(\gamma-\frac{d}{2})} \Big(\tfrac{1}{t_{\ell+2} \cdots t_{r-1}}\Big)^{\beta-\mu} \Big(\tfrac{1}{t_{\ell+3} \cdots t_{r-1}}\Big)^{\mu-\alpha} \cdots\, \Big(\tfrac{1}{t_{r-2} t_{r-1}}\Big)^{\beta-\mu} \Big(\tfrac{1}{t_{r-1}}\Big)^{\mu-\alpha}, \tag{4.25}$$

and

$$B(\mathbf{z}) \;=\; \Big((1+|\mathbf{z}|^2)^{\mu-\gamma+\frac{d}{2}} \prod_{k=\ell+2}^{r-1} (1+|z_k|^2+\ldots+|z_{r-1}|^2)^{\gamma_k}\Big)^{-1}, \tag{4.26}$$

where for simplicity we have written

$$\gamma_k \;=\; \begin{cases} \beta-\mu & \text{if } \ell-k \text{ is even} \\ \mu-\alpha & \text{if } \ell-k \text{ is odd.} \end{cases} \tag{4.27}$$

Inserting the resulting expression in (4.21) (with the corresponding jacobian of change of variables in place of $d\mathbf{v}$) we obtain

$$I_r^{\mu,\gamma}(\mathbf{t}) \;\lessapprox\; \left(\tfrac{1}{t_r}\right)^{\mu-\gamma} A(\mathbf{t})\,(t_1\,\cdots\,t_{r-1})^{\frac{d}{2}} \int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^\mu} \int_{\substack{\mathbf{z}\in\mathbb{R}^{d(r-1)} \\ |z_i|\le \frac{1}{\sqrt{(1+s)t_i}}}} B(\mathbf{z})\,d\mathbf{z}\,\frac{ds}{s}. \tag{4.28}$$

To compute the integral in $d\mathbf{z}$ we shall need the following elementary lemma.

Lemma 4.29 *Let $R \ge A \ge 1$, then*

$$\int_{z\in\mathbb{R}^d:\,|z|\le R} (A+|z|)^{-\alpha}\,dz \;\approx\; \begin{cases} R^{d-\alpha} & \text{if } 0\le\alpha<d \\ 1+\log(R/A) & \text{if } \alpha=d \\ A^{-(\alpha-d)} & \text{if } \alpha>d \end{cases} \tag{4.30}$$

Proof: Changing variables $z = Au$, the result is a simple calculation. □

Going back to (4.28), we shall compute the first part of the integral in $d\mathbf{z}$. We write $R_i = 1/\sqrt{(1+s)t_i}$, so that $R_1 \ge R_2 \ge \ldots \ge R_{r-1}$. Applying $(\ell+1)$-times the previous lemma we obtain

$$\int_{|z_{\ell+1}|\le R_{\ell+1}} \cdots \int_{|z_1|\le R_1} \frac{dz_1\,\cdots\,dz_{\ell+1}}{(1+|z_1|+\ldots+|z_{r-1}|)^{2(\mu-\gamma)+d}} \;\simeq\; R_{\ell+1}^{\ell d-2(\mu-\gamma)} = R_{\ell+1}^{2(\beta-\mu)}, \tag{4.31}$$

since by (4.20) we know that $\ell d \le 2(\mu-\gamma)+d < (\ell+1)d$. Thus

$$\begin{aligned} \int B(\mathbf{z})\,d\mathbf{z} &\simeq R_{\ell+1}^{2(\beta-\mu)} \int_{|z_{r-1}|\le R_{r-1}} \cdots \int_{|z_{\ell+2}|\le R_{\ell+2}} \frac{dz_{\ell+2}\,\cdots\,dz_{r-1}}{\prod_{k=\ell+2}^{r-1}(1+|z_k|+\ldots+|z_{r-1}|)^{2\gamma_k}} \\ &\simeq R_{\ell+1}^{2(\beta-\mu)} \prod_{k=\ell+2}^{r-1} R_k^{d-2\gamma_k} \\ &= \left(\tfrac{1}{1+s}\right)^{\beta-\mu+\frac{r-\ell-2}{2}\frac{d}{2}} \left(\tfrac{1}{t_{\ell+1}}\right)^{\beta-\mu} \left(\tfrac{1}{t_{\ell+2}}\right)^{\mu-\alpha} \left(\tfrac{1}{t_{\ell+3}}\right)^{\beta-\mu} \cdots \left(\tfrac{1}{t_{r-2}}\right)^{\mu-\alpha} \left(\tfrac{1}{t_{r-1}}\right)^{\beta-\mu}, \end{aligned} \tag{4.32}$$

where the second step follows from (4.30) since all the exponents $2\gamma_k \in [0,d]$, and the last equality is a straightforward computation from the definition of the exponents.

Inserting this estimate into (4.28) we obtain

$$I_r^{\mu,\gamma}(\mathbf{t}) \lessapprox A_1(\mathbf{t}) \times \int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^{\beta+(r-\ell-2)\frac{d}{4}}}\,\frac{ds}{s}, \tag{4.33}$$

where

$$\begin{aligned} A_1(\mathbf{t}) &= \left(\tfrac{1}{t_r}\right)^{\mu-\gamma}\left(\tfrac{1}{t_{\ell+1}}\right)^{\beta-\mu}\left(\tfrac{1}{t_{\ell+2}}\right)^{\mu-\alpha}\cdots\left(\tfrac{1}{t_{r-2}}\right)^{\mu-\alpha}\left(\tfrac{1}{t_{r-1}}\right)^{\beta-\mu}(t_1\,\cdots\,t_{r-1})^{\frac{d}{2}}\,A(\mathbf{t}) \\ &= \left(\tfrac{1}{t_r}\right)^{\mu-\gamma}\left(\tfrac{1}{t_1\cdots t_\ell}\right)^{\mu-\gamma}\left(\tfrac{1}{t_{\ell+1}}\right)^{\beta-\gamma}\left(\tfrac{1}{t_{\ell+2}}\right)^{\mu-\gamma+\frac{d}{2}}\cdots\left(\tfrac{1}{t_{r-2}}\right)^{\mu-\gamma+\frac{r-\ell-2}{2}\frac{d}{2}}\left(\tfrac{1}{t_{r-1}}\right)^{\beta-\gamma+\frac{r-\ell-2}{2}\frac{d}{2}}, \end{aligned}$$

using in the last equality the definition of $A(\mathbf{t})$ in (4.25), and straightforward computations with the exponents. The integral on the right of (4.33) is easily calculated

$$\int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^{\gamma+\ell\,\frac{d}{2}+(r-\ell-2)\frac{d}{4}}}\,\frac{ds}{s} \approx \left(\tfrac{1}{t_r}\right)^{(r-\ell)\frac{d}{4}}.$$

Putting together in (4.33) the two expressions calculated above one obtains precisely the inequality "$\lessapprox$" of (3.13) that we wished to prove.

We now turn to the converse inequality "$\gtrsim$" of (3.13). The only difference with respect to the previous reasoning is that in (4.21) we shall restrict the range of integration to $\frac{1}{2} \leq |\mathbf{v}_i| \leq 1$ when $i = \ell+2, \dots, r-1$. In this way, by Lemma 4.17, we can reverse the estimates of $a_1'(\mathbf{y}), \dots, a_{2j-2}'(\mathbf{y})$ that we used to pass from (4.23) to (4.24). Observe that for the remaining factor we have an equality

$$a_{r-1}'(\mathbf{y}) = t_1 \cdots t_{r-1}\left(1+|\mathbf{z}|^2\right)$$

(by Lemma 4.10), so no restriction is needed in the variables $\mathbf{v}_1, \dots, \mathbf{v}_{\ell+1}$ (this will be important below). Thus we have shown that

$$I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y}) \gtrsim A(\mathbf{t}) \times B(\mathbf{z}),$$

with $A(\mathbf{t})$ and $B(\mathbf{z})$ defined exactly as before (see (4.25) and (4.26)). This gives a converse inequality to (4.28), namely

$$I_r^{\mu,\gamma}(\mathbf{t}) \gtrsim \left(\tfrac{1}{t_r}\right)^{\mu-\gamma} A(\mathbf{t})\,(t_1\,\cdots\,t_{r-1})^{\frac{d}{2}} \int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^\mu} \int B(\mathbf{z})\,d\mathbf{z}\,\tfrac{ds}{s}\,,$$

where $B(\mathbf{z})$ is integrated over the set of all $\mathbf{z} = (z_1, \dots, z_{r-1}) \in \mathbb{R}^{d(r-1)}$ such that

$$|z_i| \leq R_i \text{ if } i = 1, \dots, \ell+1, \quad \text{and} \quad \tfrac{R_i}{2} \leq |z_i| \leq R_i \text{ if } i = \ell+2, \dots, r-1,$$

with the notation $R_i = 1/\sqrt{(1+s)t_i}$ as before. The crucial point is that in this smaller range of integration the estimates for $\int B(\mathbf{z})\,d\mathbf{z}$ we did before continue to be valid. Indeed, (4.31) does not change, since we did not impose new restrictions on $z_1, \ldots, z_{\ell+1}$, while (4.32) holds even if we only integrate over $|z_i| \approx R_i$ when $i = \ell+2, \ldots, r-1$. Thus the inequality in (4.33) can be reversed to "$\gtrsim$", and therefore the same steps as before lead to a complete proof of the case (a) in Theorem 3.12.

4.4 Proof of part (b)

We first establish the case $r = 2$, which necessarily implies $\ell = 1$. By the recursion lemma 4.1 (and the explicit expression in (4.16)) we see that

$$I_2^{\mu,\gamma}(\mathbf{t}) \approx \frac{1}{t_2^{\mu-\gamma}} \int_0^{1/t_2} \frac{s^\gamma}{(1+s)^\mu} \int_{v\in\mathbb{R}^d:\,|v|\leq 1} \frac{dv}{\Big(t_1 + \frac{|v|^2}{1+s}\Big)^{\mu-\gamma+\frac{d}{2}}} \, \frac{ds}{s}.$$

Changing variables $v = \sqrt{t_1(1+s)}\,z$ and using that $d \leq 2(\mu-\gamma)+d < 2d$, the integral in dv becomes

$$\int \ldots\, dv \;\asymp\; \frac{(1+s)^{d/2}}{t_1^{\mu-\gamma}}.$$

Inserting this expression in the above formula we obtain

$$I_2^{\mu,\gamma}(\mathbf{t}) \;\asymp\; \Big(\frac{1}{t_1 t_2}\Big)^{\mu-\gamma} \int_0^{1/t_2} \frac{s^\gamma (1+s)^{\frac{d}{2}}}{(1+s)^\mu} \, \frac{ds}{s} \approx \Big(\frac{1}{t_1}\Big)^{\mu-\gamma} \Big(\frac{1}{t_2}\Big)^{\frac{d}{2}},$$

since $\mu < \gamma + d/2$. This proves (3.14) when $r = 2$ and $\ell = 1$.

The general case will be obtained with an entirely similar proof as in the previous subsection. Suppose by induction that part (b) of Theorem 3.9 holds with r replaced by $r-1$, and assume we are given a cone Ω of rank r and an integer $\ell \in \{1, \ldots, r-1\}$ as in (b), that is, $r - \ell = 2j - 1$ for some $j \in \{1, \ldots, \lfloor \frac{r}{2} \rfloor\}$. For μ in the interval

$$\gamma + (\ell-1)\,\tfrac{d}{2} \;\leq\; \mu \;<\; \gamma + \ell\,\tfrac{d}{2}\,,$$

we must evaluate $I_r^{\mu,\gamma}(\mathbf{t})$, for which we shall use the recursion formula in (4.23). As before, we are then led to estimate the factor $I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y})$, noticing that $\gamma - \frac{d}{2} > (r-2)\frac{d}{2}$ and

$$\big(\gamma - \tfrac{d}{2}\big) + \ell\,\tfrac{d}{2} \;\leq\; \mu \;<\; \big(\gamma - \tfrac{d}{2}\big) + \big(\ell+1\big)\,\tfrac{d}{2}.$$

The induction hypothesis, formulated as in Proposition 4.8, then gives

$$I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y}) \;\asymp\; \Big(\tfrac{1}{a'_{r-1}(\mathbf{y})}\Big)^{\mu-(\gamma-d/2)} \Big(\tfrac{1}{a'_{2j-3}(\mathbf{y})}\Big)^{\beta-\mu} \Big(\tfrac{1}{a'_{2j-4}(\mathbf{y})}\Big)^{\mu-\alpha} \cdots \Big(\tfrac{1}{a'_{2}(\mathbf{y})}\Big)^{\mu-\alpha} \Big(\tfrac{1}{a'_{1}(\mathbf{y})}\Big)^{\beta-\mu} \tag{4.34}$$

with the caution that when $\ell = r-1$ (ie $j=1$) the right hand side only involves the first factor (this is one of the easy cases described in (3.11)). We then estimate the $a_i'(\mathbf{y})$'s using Corollary 4.17. As before, changing variables $\mathbf{v}_i = \sqrt{t_i(1+s)}\, z_i$, $i=1,\dots,r-1$, the inequalities in (4.18) give

$$I_{r-1}^{\mu,\gamma-\frac{d}{2}}(\mathbf{y}) \lessapprox A(\mathbf{t}) \times B(\mathbf{z}), \tag{4.35}$$

where now

$$A(\mathbf{t}) = \Big(\tfrac{1}{t_1\cdots t_{r-1}}\Big)^{\mu-(\gamma-\frac{d}{2})}\Big(\tfrac{1}{t_{\ell+2}\cdots t_{r-1}}\Big)^{\beta-\mu}\Big(\tfrac{1}{t_{\ell+3}\cdots t_{r-1}}\Big)^{\mu-\alpha}\cdots\Big(\tfrac{1}{t_{r-2}t_{r-1}}\Big)^{\mu-\alpha}\Big(\tfrac{1}{t_{r-1}}\Big)^{\beta-\mu}, \tag{4.36}$$

and $B(\mathbf{z})$ is the same as in (4.26). From here the rest of the proof is exactly the same as before. In particular, the same reasonings as in (4.31) and (4.32) allow to compute

$$\int B(\mathbf{z})\,d\mathbf{z} \simeq \Big(\tfrac{1}{1+s}\Big)^{\frac{r-\ell-1}{2}\frac{d}{2}}\Big(\tfrac{1}{t_{\ell+1}}\Big)^{\beta-\mu}\Big(\tfrac{1}{t_{\ell+2}}\Big)^{\mu-\alpha}\cdots\Big(\tfrac{1}{t_{r-2}}\Big)^{\beta-\mu}\Big(\tfrac{1}{t_{r-1}}\Big)^{\mu-\alpha},$$

the result being slightly different only because of the parity. Thus we obtain

$$I_r^{\mu,\gamma}(\mathbf{t}) \lessapprox A_1(\mathbf{t}) \times \int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^{\mu+(r-\ell-1)\frac{d}{4}}}\,\frac{ds}{s}, \tag{4.37}$$

with

$$A_1(\mathbf{t}) = \Big(\tfrac{1}{t_r}\Big)^{\mu-\gamma}\Big(\tfrac{1}{t_1\cdots t_\ell}\Big)^{\mu-\gamma}\Big(\tfrac{1}{t_{\ell+1}}\Big)^{\beta-\gamma}\Big(\tfrac{1}{t_{\ell+2}}\Big)^{\mu-\gamma+\frac{d}{2}}\cdots\Big(\tfrac{1}{t_{r-2}}\Big)^{\beta-\gamma+\frac{r-\ell-3}{2}\frac{d}{2}}\Big(\tfrac{1}{t_{r-1}}\Big)^{\mu-\gamma+\frac{r-\ell-1}{2}\frac{d}{2}}.$$

The integral on the right of (4.37) now gives

$$\int_0^{1/t_r} \frac{s^\gamma\,(1+s)^{(r-1)\frac{d}{2}}}{(1+s)^{\mu+(r-\ell-1)\frac{d}{4}}}\,\frac{ds}{s} \approx \Big(\tfrac{1}{t_r}\Big)^{\gamma-\mu+(r+\ell-1)\frac{d}{4}},$$

since the exponent $\gamma-\mu+(r+\ell-1)\frac{d}{4} > 0$. Putting together in (4.37) the previous two expressions one obtains the inequality "$\lessapprox$" of (3.14) that we wished to prove. The converse inequality is proved with the same argument we used for part (a), namely, restricting the integration in $d\mathbf{v}$ in equation (4.21) to the smaller range $\frac{1}{2} \le |\mathbf{v}_i| \le 1$ when $i=\ell+2,\dots,r-1$, so that the inequality (4.35) can be reverted. This completes the proof of part (b), and hence establishes Theorem 3.12.

$\square$

5 The proof of Theorem 3.5

We must compute

$$I_q \equiv \int_{\Omega\cap\mathbb{B}} \big|I^\gamma(\mathbf{t})\big|^q\,\Delta^{\gamma-\frac{n}{r}}(t)\,d\mathbf{t}.$$

To do so we shall use polar coordinates in Ω, that is, if f is K-invariant then

$$\int_\Omega f(\mathbf{t})\,\frac{d\mathbf{t}}{\Delta(\mathbf{t})^{\frac{n}{r}}} \;=\; c\int\cdots\int_{-\infty<s_r<\ldots<s_1<\infty} f\big(e^{-s_1}c_1+\ldots+e^{-s_r}c_r\big)\prod_{1\le j<k\le r}\Big[\mathrm{sh}\,\big(\tfrac{s_j-s_k}{2}\big)\Big]^d ds_1\ldots ds_r$$

(see [10, Corollary VI.2.4]). Observe that sh $u\le e^u$ implies

$$\prod_{1\le j<k\le r}\Big[\mathrm{sh}\,\big(\tfrac{s_j-s_k}{2}\big)\Big]^d \;\le\; \exp\Big(\tfrac{d}{2}\sum_{1\le j<k\le r}[s_j-s_k]\Big) \;=\; \prod_{j=1}^r e^{(r-2j+1)\frac{d}{2}\,s_j}\;.$$

Conversely, since sh $u\ge e^u/4$ when $u\ge 1$, if we restrict the integration in the variables $s_1,\ldots,s_{r-1}$ to the range

$$s_r\;\le\;s_{r-1}-2\;\le s_{r-2}-4\;\le\;\ldots\;\le\;s_1-2(r-1) \tag{5.1}$$

we see that $(s_j-s_k)/2\ge 1$, $\forall k>j$, and therefore

$$\prod_{1\le j<k\le r}\Big[\mathrm{sh}\,\big(\tfrac{s_j-s_k}{2}\big)\Big]^d \;\approx\; \Big[e^{(r-1)s_1}\,e^{(r-3)s_2}\;\cdots\;e^{-(r-1)s_r}\Big]^{\frac{d}{2}}\;. \tag{5.2}$$

5.1 Case $r=$ odd

Let $\mathbf{t}=e^{-s_1}c_1+\ldots+e^{-s_r}c_r$ with $0\le s_r<\ldots<s_1$. Using the upper bound in (3.4) we have

$$I^\gamma(\mathbf{t})\;\lesssim\;(1+s_1)^c\,\Big[e^{s_2+s_3}\,e^{2(s_4+s_5)}\;\cdots\;e^{\frac{r-1}{2}(s_{r-1}+s_r)}\Big]^{\frac{d}{2}}\;. \tag{5.3}$$

Denote by $A(\mathbf{s})$ and $B(\mathbf{s})$, respectively, the expressions on the right hand side of (5.3) and (5.2), where $\mathbf{s}$ refers to the r-tuple $(s_1,\ldots,s_r)$. We also write

$$C(\mathbf{s})=\Delta(\mathbf{t})^\gamma=e^{-\gamma(s_1+\ldots+s_r)}.$$

Then, the formula of polar coordinates gives

$$I_q\;\lesssim\;\int_0^\infty\!\!\int_{s_r}^\infty\!\!\cdots\int_{s_2}^\infty A(\mathbf{s})^q\,B(\mathbf{s})\,C(\mathbf{s})\;\,ds_1\ldots ds_{r-1}\,ds_r. \tag{5.4}$$

The integral in ds_1 is always finite; in fact, since $\gamma>(r-1)d/2$,

$$\int_{s_2}^\infty(1+s_1)^{cq}\,e^{(r-1)\frac{d}{2}s_1}\,e^{-\gamma s_1}\,ds_1\;\approx\;(1+s_2)^{cq}\,e^{(r-1)\frac{d}{2}s_2}\,e^{-\gamma s_2}\;.$$

Thus, inserting this expression into (5.4) we obtain

$$I_q\;\lesssim\;\int_0^\infty\!\!\int_{s_r}^\infty\!\!\cdots\int_{s_3}^\infty A(\mathbf{s}_2)^q\,B(\mathbf{s}_2)\,C(\mathbf{s}_2)\;\,ds_2\ldots ds_{r-1}\,ds_r,$$

where we write

$$\mathbf{s}_\ell = (s_\ell, \ldots, s_\ell, s_{\ell+1}, \ldots, s_r), \quad \text{if} \quad \ell = 2, 3, \ldots, r.$$

The remaining integrals can be computed by iteration; at the ℓ-th step we start with

$$I_q \lesssim \int_0^\infty \int_{s_r}^\infty \cdots \int_{s_{\ell+1}}^\infty A(\mathbf{s}_\ell)^q \, B(\mathbf{s}_\ell) \, C(\mathbf{s}_\ell) \; ds_\ell \ldots ds_{r-1} \, ds_r.$$

The integral in ds_ℓ will be finite if and only if the integrand factor $e^{\alpha_\ell s_\ell}$, comprising all the products of exponentials in the s_ℓ-variable within $A(\mathbf{s}_\ell)^q B(\mathbf{s}_\ell) C(\mathbf{s}_\ell)$, has $\alpha_\ell < 0$. Now, a simple computation from the explicit expressions of $A(\mathbf{s}), B(\mathbf{s}), C(\mathbf{s})$ shows that

$$\alpha_\ell = \begin{cases} j^2 \, q \, \frac{d}{2} + 2j \big[(r-2j) \frac{d}{2} - \gamma \big] & \text{if} \quad \ell = 2j \\ \\ j(j+1) \, q \, \frac{d}{2} + (2j+1) \big[(r-2j-1) \frac{d}{2} - \gamma \big] & \text{if} \quad \ell = 2j+1. \end{cases}$$

Performing all the integrations we see that $I_q < \infty$ if

$$q < \min_{1 \le j \le \frac{r-1}{2}} \left\{ \frac{4}{jd} \big[\gamma - (r-2j) \tfrac{d}{2} \big] \, , \, \frac{2(2j+1)}{dj(j+1)} \big[\gamma - (r-2j-1) \tfrac{d}{2} \big] \right\}. \tag{5.5}$$

Conversely, using the reverse inequality in (5.3) with $c = 0$ (which follows from the lower bound in (3.4)), and restricting the integration to the range in (5.1), all the above computations can be carried out similarly with "$\gtrsim$" in place of "$\lesssim$". Hence we conclude that $I_q < \infty$ if and only if (5.5) holds.

Finally, to establish (3.6) we must minimize the expression on the right hand side of (5.5). This can be written as

$$Q_\gamma \equiv \min \Big\{ f(2), f(4), \ldots, f(r-1); \; g(3), g(5), \ldots, g(r) \Big\},$$

where

$$f(x) = \frac{4(a+x)}{x} \quad \text{and} \quad g(x) = \frac{4x(a+x)}{x^2 - 1}, \tag{5.6}$$

and we have set $a = (\gamma - r\frac{d}{2})/(\frac{d}{2})$, so that $a > -1$. The following properties are then easy to verify

(i) $f(x) \le g(x)$, $\forall \, x > 1$;

(ii) If $a < 0$, then f is increasing, and thus $Q_\gamma = f(2)$;

(iii) If $a \ge 0$, then both f and g are non-increasing, and thus $Q_\gamma = \min\{f(r-1), g(r)\}$.

Thus we have shown that

$$Q_\gamma = \min \Big\{ f(2), f(r-1), g(r) \Big\},$$

which is the same as (3.6) when r is odd.

5.2 Case $r =$ even

The proof is entirely analogous to the previous case, except that (5.3) must be replaced by

$$I^\gamma(\mathbf{t}) \lesssim (1+s_1)^c \left[e^{s_2+s_3}\, e^{2(s_4+s_5)} \cdots e^{\frac{r-2}{2}(s_{r-2}+s_{r-1})}\, e^{\frac{r}{2}s_r} \right]^{\frac{d}{2}}, \tag{5.7}$$

which follows from Theorem 3.3 when r is even. Thus, the integrals in (5.4) are computed iteratively as before, leading to $I_q < \infty$ if and only if

$$q < \min_{1 \le j \le \frac{r-2}{2}} \left\{ \frac{4}{jd}\big[\gamma - (r-2j)\tfrac{d}{2}\big],\ \frac{8\gamma}{rd},\ \frac{2(2j+1)}{dj(j+1)}\big[\gamma - (r-2j-1)\tfrac{d}{2}\big] \right\} \tag{5.8}$$

(the middle term now corresponds to the last integration in ds_r). Using the same functions $f(x)$ and $g(x)$ as in (5.6) one easily sees that (5.8) is equivalent to

$$q < \min\Big\{ f(2), f(r) \Big\}$$

(the term $g(r-1)$ is not necessary since it is in between these two). The right hand side coincides with the desired expression in (3.6) for r even, hence establishing Corollary 3.5. □

6 The boundedness of $P_\nu : L^\infty \to L^q_\nu$

We shall interpolate between the following two estimates

$$\mathcal{P}^{\mathbb{B}}_\nu : L^\infty(T_\Omega) \to L^{\infty,q}_\nu(T_\Omega) \quad \text{when } 1 \le q < q_{+,\nu} \tag{6.1}$$

and

$$\mathcal{P}^{\mathbb{B}}_\nu : L^\infty(T_\Omega) \to L^{2,q}_\nu(T_\Omega) \quad \text{when } 1 \le q < q_{0,\nu}. \tag{6.2}$$

The boundedness of (6.1) follows from Corollary 3.7 and Proposition 2.12. The boundedness of the local operator in (6.2) is a consequence of Hölder's inequality and the following result about the global operator.

Theorem 6.3 : see [4]. *Let $\nu > \frac{2n}{r} - 1$. Then $\mathcal{P}_\nu$ is bounded from $L^{2,q}_\nu(T_\Omega) \to L^{2,q}_\nu(T_\Omega)$ if and only if $q'_{0,\nu} < q < q_{0,\nu}$, where*

$$q_{0,\nu} := \frac{2(\nu-1)}{\frac{n}{r}-1} = 4 + \frac{4(\nu - \frac{2n}{r} + 1)}{(r-1)d}.$$

Interpolating (6.1) and (6.2) one easily obtains the following.

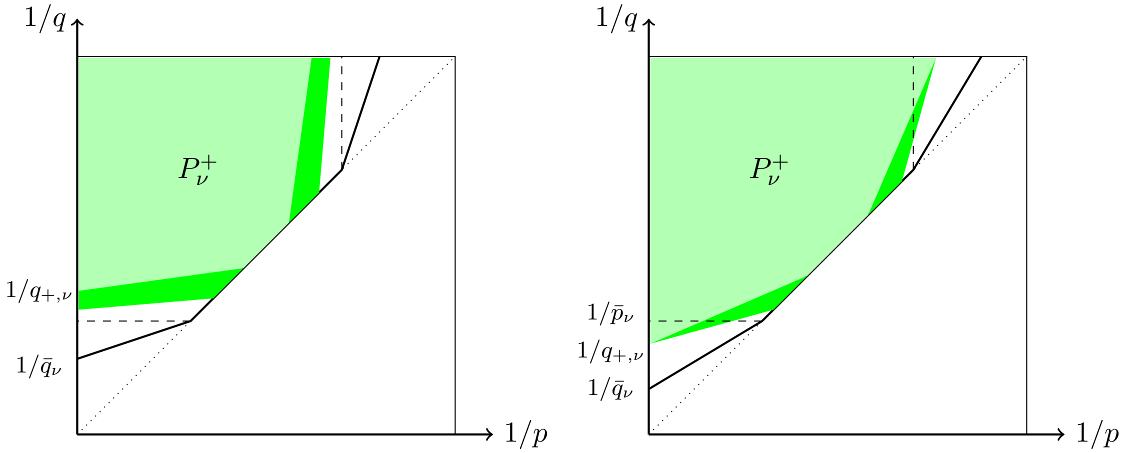

Figure 6.1: Regions of boundedness of $P_\nu : L^p_\nu \to L^q_\nu$, for ν small and ν large, respectively.

Corollary 6.4 *Let $\nu > \frac{2n}{r} - 1$. Then $\mathcal{P}^{\mathbb{B}}_\nu$ is bounded from $L^\infty(T_\Omega) \to L^q_\nu(T_\Omega)$ when*

$$1 \le q \, < \, q_{+,\nu} \, + \, 2\Big(1 - \frac{q_{+,\nu}}{q_{0,\nu}}\Big)_+ \, . \tag{6.5}$$

In particular, P_ν maps $L^\infty(D) \to L^q(D, \mu_\nu)$ in this same range.

Remark 6.6 Specializing to the unweighted case $\nu = \frac{2n}{r}$, one obtains Theorem 1.4.

Remark 6.7 Corollary 6.4 only produces new results (compared to Corollary 3.7) when $q_{+,\nu} < q_{0,\nu}$. An elementary computation shows that this happens precisely when

$$\nu \, < \, \tfrac{2n}{r} - 1 \, + \, \frac{\frac{n}{r} - 1}{\lceil \frac{r}{2} \rceil - 1}.$$

This is always the case when $r = 2$, and also when $r \ge 3$ and ν is sufficiently small. In particular, it holds in the unweighted situation $\nu = \frac{2n}{r}$, except when $d = 1$ and r odd. Observe also that, for ν large, the index $q_{+,\nu}$ is bigger than $\bar{p}_\nu = 1 + \nu/(\frac{n}{r} - 1)$ (the conjectured index in the diagonal situation). Thus the region of boundedness takes the form in Figure 6.1.

Remark 6.8 When $r = 2$ it is easily checked that (6.5) takes the form

$$1 \, \le \, q \, < \, q_{+,\nu} \, + \, \frac{n-2}{\nu - 1} \, , \tag{6.9}$$

with $q_{+,\nu} = 4(\nu - \frac{n}{2})/(n-2)$. We remark that this exponent can be slightly improved for certain ν's using the stronger boundedness results which are known for $\mathcal{P}_\nu$ in this case. For instance, using [13, Corol 2.5] and [6, Prop 5.5] we could replace (6.2) by

$$\mathcal{P}_\nu^{\mathbb{B}} : L^\infty(T_\Omega) \to L_\nu^{\hat{p},\hat{q}}(T_\Omega)$$

for $\hat{p} = 2(n+2)/n$ and a suitable $\hat{q} \in (q_{+,\nu}, q_{0,\nu})$. Interpolation now gives the validity of Corollary 6.4 when

$$1 \le q \; < \; q_{+,\nu} \; + \; \hat{p}\Big(1 - \frac{q_{+,\nu}}{\hat{q}}\Big)_+ \, ,$$

which produces a small improvement over (6.9), at least for $\nu = n$.

7 Necessary conditions

7.1 Necessary conditions for $P_\nu : L^\infty \to L^q_\nu$

Assume that $P_\nu : L^p(D, \mu_\nu) \to L^q(D, \mu_\nu)$. Corollary 2.10 and duality implies that

$$\big\|\mathcal{P}_\nu^{\mathbb{B}} f\big\|_{L_\nu^{p'}(T_\Omega)} \le C \, \|f\|_{L_\nu^{q'}(T_\Omega)}. \tag{7.1}$$

We test with $f(z) = \Delta(\Im m\, z)^{\frac{2n}{r} - \nu} \, \chi_{D_\delta}(z)$, where D_δ denotes the polydisk centered at $i\delta\mathbf{e}$ of radius $c\delta$, with $\delta \ll 1$ and $c > 0$ a small universal constant such that $D_\delta \Subset T_\Omega \cap \mathbb{B}$. By the mean value property,

$$\mathcal{P}_\nu^{\mathbb{B}} f(z) \; = \; c' \, \delta^{2n} \, B_\nu^{T_\Omega}(z, i\delta\mathbf{e}), \quad z \in T_\Omega \cap \mathbb{B}.$$

Thus, Lemma 2.14 gives

$$\big\|\mathcal{P}_\nu^{\mathbb{B}} f\big\|^{p'}_{L_\nu^{p'}(T_\Omega)} \gtrsim \delta^{2np'} \int_{\Omega \cap \mathbb{B}} \frac{d\lambda_\nu(y)}{\Delta(y + \delta\mathbf{e})^{\nu p' - \frac{n}{r}}} \; = \; \delta^{2np'} \, I^{p'\nu - \frac{n}{r}, \nu - \frac{n}{r}}(\delta\mathbf{e}),$$

with the notation in (3.9). On the other hand, an easy computation gives

$$\|f\|_{L_\nu^{q'}(T_\Omega)} \approx \delta^{2n - \frac{r\nu}{q}}.$$

Inserting these two estimates in (7.1) and simplifying a bit, we see that it must hold

$$I^{p'\nu - \frac{n}{r}, \nu - \frac{n}{r}}(\delta\mathbf{e}) \lesssim \delta^{-p' r\nu/q} \tag{7.2}$$

for all $\delta \ll 1$. When $p = \infty$ this reduces to

$$\delta^{-\frac{r\nu}{q}} \gtrsim I^{\nu - \frac{n}{r}, \nu - \frac{n}{r}}(\delta\mathbf{e}) \gtrsim \begin{cases} \delta^{-(1+2+\ldots+\frac{r-1}{2})d} & \text{if } r \text{ is odd} \\ \delta^{-(1+2+\ldots+\frac{r-2}{2})d - \frac{r}{2}\frac{d}{2}} & \text{if } r \text{ is even} \end{cases}$$

where in the last step we have used Theorem 3.3. Elementary algebra then shows that, necessarily

$$q \leq \bar{q}_\nu := \frac{2r\nu}{\lfloor \frac{r}{2} \rfloor \lceil \frac{r}{2} \rceil d}.$$

Thus, we have shown the following, which for $\nu = 2n/r$ gives Theorem 1.5.

PROPOSITION 7.3 *Let $\nu > \frac{2n}{r} - 1$. Then P_ν is not bounded from $L^\infty(D) \to L^q_\nu(D)$ when $q > \bar{q}_\nu$.*

7.2 Necessary conditions for $P_\nu : L^p_\nu \to L^q_\nu$ with $p < \infty$

When $p < \infty$ we can estimate the left hand side of (7.2) with Theorem 3.12, letting $\mu = p'\nu - \frac{n}{r}$ and $\gamma = \nu - \frac{n}{r}$. We must distinguish cases depending on the value of $\mu - \gamma = \nu/(p-1)$, which will lead to different necessary conditions. Namely, we obtain

$$\delta^{-p'r\nu/q} \gtrsim \delta^{-(r-j)(\frac{\nu}{p-1}+j\frac{d}{2})}, \qquad \text{if } \frac{\nu}{p-1} \in \Big([r-2j-1]\tfrac{d}{2}, [r-2j+1]\tfrac{d}{2}\Big), \tag{7.4}$$

for $j = 1, 2, \ldots, \lfloor \frac{r}{2} \rfloor$, with the agreement that if r is even and $j = \frac{r}{2}$ the interval on the right reduces to $(0, \frac{d}{2})$. The right hand side of (7.4) can also be read as

$$1 + \tfrac{2\nu}{(r-2j+1)d} \ < \ p \ < \ 1 + \tfrac{2\nu}{(r-2j-1)d}, \tag{7.5}$$

which as j varies gives a partition for the region $\bar{p}_\nu < p < \infty$. We denote the separating points in this partition by

$$\mathtt{p}_j = 1 + \tfrac{2\nu}{(r-2j-1)d}, \quad j = 0, 1, \ldots, \lfloor \tfrac{r}{2} \rfloor,$$

noticing that $\mathtt{p}_0 = \bar{p}_\nu$ and $\mathtt{p}_{\lfloor r/2 \rfloor} = \infty$. According to (7.4) when $p \in (\mathtt{p}_{j-1}, \mathtt{p}_j)$ the index q must necessarily satisfy

$$\frac{1}{q} \geq \frac{r-j}{rp} + \frac{(r-j)jd/2}{r\nu p'}. \tag{7.6}$$

We write $\mathtt{q}_j$ for the point paired with $\mathtt{p}_j$ according to this rule, that is setting $\frac{\nu}{p-1} = (r-2j-1)d/2$ in (7.6)

$$\mathtt{q}_j = \tfrac{r(r-2j-1)}{(r-j)(r-j-1)}\, \mathtt{p}_j, \quad j = 0, 1, \ldots, \lfloor \tfrac{r}{2} \rfloor,$$

with the agreement that $\mathtt{q}_0 = \mathtt{p}_0 = \bar{p}_\nu$ and $\mathtt{q}_{\lfloor r/2 \rfloor} = \bar{q}_\nu$. Finally, let

$$\mathtt{A}_j = \big(\tfrac{1}{\mathtt{p}_j}, \tfrac{1}{\mathtt{q}_j}\big) \quad \text{and} \quad \mathtt{A}'_j = \big(\tfrac{1}{\mathtt{q}'_j}, \tfrac{1}{\mathtt{p}'_j}\big), \quad j = 0, 1, \ldots, \lfloor \tfrac{r}{2} \rfloor,$$

and consider the closed convex region

$$\mathtt{K}_\nu = \overline{\mathrm{co}} \big\{ \mathtt{A}_0, \ldots, \mathtt{A}_{\lfloor r/2 \rfloor}, (0,1), \mathtt{A}'_{\lfloor r/2 \rfloor}, \ldots, \mathtt{A}'_0 \big\} \,;$$

see Figure 7.1. Then we have shown the following.

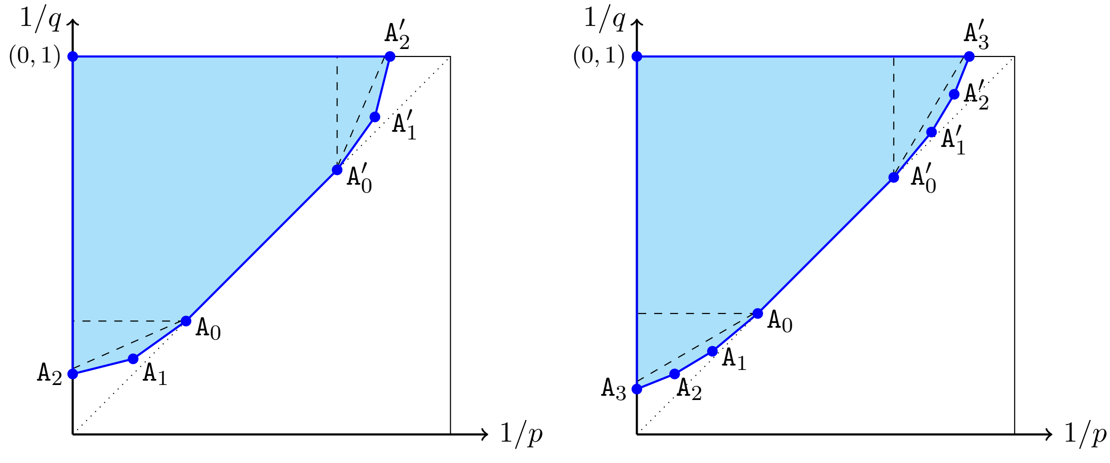

Figure 7.1: Regions of possible boundedness for P_ν when $r \in \{4,5\}$ and $r \in \{6,7\}$.

Proposition 7.7 *Let $\nu > \frac{2n}{r} - 1$. If P_ν is bounded from $L^p_\nu(D) \to L^q_\nu(D)$, then necessarily $(\frac{1}{p}, \frac{1}{q}) \in \mathtt{K}_\nu$.*

When $r \in \{2,3\}$ the region $\mathtt{K}_\nu$ has the shape of a pentagon, like on the left of Figure 1.1. For higher ranks, however, $\mathtt{K}_\nu$ is a more complicated N-gon, with $N = 2\lfloor \frac{r}{2} \rfloor + 1$, like those drawn in Figure 7.1. We do not know whether this region could be reduced to the smaller pentagon $\overline{\text{co}}\,\{\mathtt{A}_0, \mathtt{A}_{\lfloor r/2 \rfloor}, (0,1), \mathtt{A}_{\lfloor r/2 \rfloor}, \mathtt{A}_0\}$ with additional examples.

7.3 Necessary conditions for $P^+_\nu : L^p_\nu \to L^q_\nu$ with $p < \infty$

Proposition 7.8 *Let $\nu > \frac{2n}{r} - 1$. If P^+_ν is bounded from $L^p_\nu(D) \to L^q_\nu(D)$, then necessarily $q < q_{+,\nu}$ and $(\frac{1}{p}, \frac{1}{q}) \in \mathtt{K}_{\nu - \frac{n}{r}}$.*

Proof: The first condition is clear by Corollary 3.7, since $L^\infty \hookrightarrow L^p_\nu(D)$. For the second condition, by Proposition 2.12, the assertion about P^+_ν is equivalent to the boundedness of $\mathcal{Q}_\nu : L^{q'}_\nu(\Omega) \to L^{p'}_\nu(\Omega)$. We test with $f = \chi_{B(\delta\mathbf{e})}$, where $B(\delta\mathbf{e})$ is an invariant ball centered at $\delta\mathbf{e}$ and δ is sufficiently small so that $B(\delta\mathbf{e}) \Subset \Omega \cap \mathbb{B}$. Then

$$\mathcal{Q}_\nu f(y) = \int_{B(\delta\mathbf{e})} \frac{\Delta(v)^{\nu - \frac{2n}{r}}\, dv}{\Delta(y+v)^{\nu - \frac{n}{r}}} \approx \frac{\delta^{r\nu - n}}{\Delta(y + \delta\mathbf{e})^{\nu - \frac{n}{r}}}$$

(see eg [8, Cor 2.3]). This implies that

$$\big\|\mathcal{Q}_\nu f\big\|^{p'}_{L^{p'}_\nu(\Omega)} \approx \delta^{p'(r\nu - n)}\, I^{p'(\nu - \frac{n}{r}),\, \nu - \frac{n}{r}}(\delta\mathbf{e}) \lesssim \|f\|^{p'}_{L^{q'}_\nu} \approx \delta^{(r\nu - n)p'/q'},$$

which writing $\gamma = \nu - \frac{n}{r}$ and rearranging takes the form

$$I^{p'\gamma,\gamma}(\delta\mathbf{e}) \lesssim \delta^{-p'r\gamma/q}. \tag{7.9}$$

Now, by Theorem 3.12 the left hand side only depends on $p'\gamma - \gamma = \gamma/(p-1)$, so (7.9) coincides exactly with (7.2) with ν there replaced by γ. So the reasoning in the previous subsection gives that, necessarily $(\frac{1}{p}, \frac{1}{q}) \in \mathtt{K}_\gamma$, as we wished to prove.

□

Remark 7.10 Interpolating Corollary 3.7 with the known diagonal results for $\mathcal{P}_\nu^+$ (ie $p \in (p'_{+,\nu}, p_{+,\nu})$ with $p_{+,\nu} = (\nu-1)/(\frac{n}{r}-1) = \bar{p}_{\nu-n/r}$; see eg [5, Th. 4.3]), we see that $L^p_\nu \to L^q_\nu$ boundedness holds in the interior of the pentagon

$$\mathtt{K}_{+,\nu} = \overline{\mathrm{co}}\left\{(\tfrac{1}{p_{+,\nu}}, \tfrac{1}{p_{+,\nu}}), (0, \tfrac{1}{q_{+,\nu}}), (0,1), (\tfrac{1}{q'_{+,\nu}}, 1), (\tfrac{1}{p'_{+,\nu}}, \tfrac{1}{p'_{+,\nu}})\right\}.$$

When $r = 2$ we have $\mathtt{K}_{+,\nu} = \mathtt{K}_{\nu-\frac{n}{r}}$, so the region of $L^p_\nu \to L^q_\nu$ boundedness for P_ν^+ is optimal (except perhaps for the endpoints). This continues to hold when $r = 3$, at least for large ν; namely $\nu \geq \frac{n}{r} + \lceil \frac{r}{2} \rceil d = 1 + 3d$, so that $q_{+,\nu} = \bar{q}_{\nu-\frac{n}{r}}$. In other situations, however, there is a gap between necessary and sufficient conditions for P_ν^+; see eg Figure 7.2.

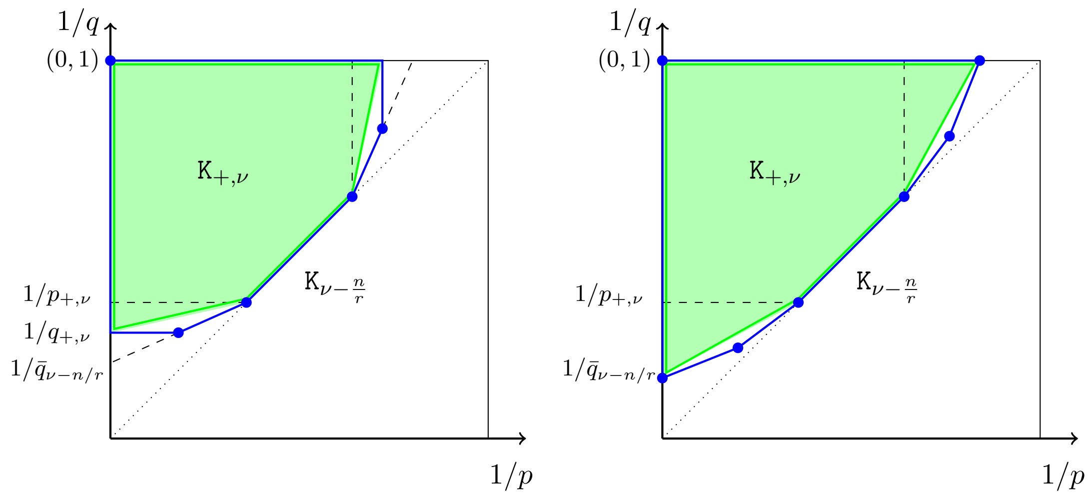

Figure 7.2: Regions of boundedness for P_ν^+ when $r = 3$ and ν small and $r = 4$ and ν large.

8 Appendix

8.1 A remark on quaternionic determinants

As pointed out in the proof of Lemma 4.10, the determinant of a matrix of quaternions (for which actually there are various definitions) is in general not multilinear; see eg [2, 9]. Thus, we briefly explain here what definition must be used in order to justify (4.15) in the case $V = \mathrm{Her}(r, \mathbb{H})$.

A matrix of quaternions $A = (a_{ij})_{i,j=1}^n \in M_{n\times n}(\mathbb{H})$ is called *almost-hermitian* if there exists (at most) one index $k \in \{1, \dots, n\}$ so that

$$a_{ij} = \bar{a}_{ji}, \quad \forall\, i, j \in \{1, \dots, n\} \setminus \{k\}.$$

For such matrices the *Moore determinant* is defined by

$$\mathrm{Mdet}\ (A) \;=\; \sum_{\ell=1}^{n} \varepsilon_{k\ell}\, a_{k\ell}\, \mathrm{Mdet}(A[k, \ell]),$$

where k is the index in the definition of almost hermitian, $A[k, \ell]$ is the matrix obtained from A by first interchanging the ℓ-th and k-th columns, and then deleting both the k-th row and k-th column, and $\varepsilon_{k\ell} = -1$ if $\ell \neq k$ and $\varepsilon_{kk} = 1$. We refer to [9, 14] for the consistency of this definition and various properties of such determinants. It is easy to verify from the definition that, if A, B, C are almost hermitian matrices with the same index k, and satisfying the linear relation

$$c_{kj} = a_{kj} + b_{kj} \quad \forall\, j, \quad \text{and} \quad c_{ij} = a_{ij} = b_{ij} \quad \forall\, i \neq k,\ \ \forall\, j,$$

then

$$\mathrm{Mdet}(C) = \mathrm{Mdet}(A) + \mathrm{Mdet}(B)$$

(see eg [9, Thm.2]). Using this property and the definition of Mdet, it is straightforward to justify the analogue of (4.15) for quaternionic matrices, ie

$$\mathrm{Mdet}\big(\mathbf{s} + (\mathbf{v}^2)'\big) \;=\; s_1 \cdots s_{r-1} \;+\; \sum_{j=1}^{r-1} s_1 \cdots |v_j|^2 \cdots s_{r-1},$$

when $s_j > 0$ and $v_j \in \mathbb{H}$.

Finally, we remark that the Jordan algebra determinant $\Delta(x)$ of $x \in \mathrm{Her}(r, \mathbb{H})$ (defined as the independent coefficient of the minimal polynomial of x; see [10, Ch II]) coincides with the *Moore determinant* $\mathrm{Mdet}(x)$ defined above. This follows for instance from the formulation of each of these determinants as pfaffians of matrices in $\mathrm{Skew}(2r, \mathbb{C})$ (see [10, p. 40] and [9, (4.7)]).

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