THE BEHAVIOUR AT THE ORIGIN OF A CLASS OF BAND-LIMITED WAVELETS

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Abstract. The paper investigates the special role of the origin in the theory of orthonormal wavelets.

§1. Introduction and Presentation of the Results.

Each of the four authors has been involved in the construction of certain wavelets in order to exhibit particular properties of these functions. Standing by itself, each example is not sufficiently important to warrant its publication; however, the collection of these results has a unity that provides us with an understanding of the role played by the origin for the Fourier transform of wavelets. We will also see that the values $\frac{8}{3}\pi$ and 4π have special significance for these Fourier transforms. Since these values also depend (in a trivial way) on the choice of the Fourier transform we make, let us agree that

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx \tag{1.1}$$

are values of the Fourier transform, $\hat{f}(\xi)$, when $f \in L^1(\mathbb{R})$.

An *(orthonormal) wavelet* is a function ψ such that the family $\{\psi_{j,k}\colon j,k \in \mathbb{Z}\} = \{2^{\frac{j}{2}}\psi(2^j x - k)\colon j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. All wavelets can be characterized by two simple equations

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1, \tag{1.2}$$

$$t_q(\xi) := \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0, \quad \text{for} \quad q \in 2\mathbb{Z}+1 \tag{1.3}$$

and the condition $\|\psi\|_2 \geq 1$. In reality these equations are true almost everywhere; but here and in the sequel, in order to avoid repeating "for a.e. ξ" almost everywhere, we omit stating this explicitly. We refer the reader to Chapter 7 of [HW] for a proof of this fact.

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We assume that the reader is familiar with the concept of a *multiresolution analysis* MRA and what is meant by the statement "ψ is an MRA wavelet". We also remind the reader that ψ is an MRA wavelet if and only if

$$D_\psi(\xi) := \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 = 1 \tag{1.4}$$

(see Chapter 7 of [HW] for a proof of this fact).

A simple consequence of equation (1.2) is that $\hat{\psi}(0) = 0$ when $\hat{\psi}$ is continuous at 0. A deeper result asserts that for a band-limited wavelet ψ with $|\hat{\psi}|$ continuous at the origin, we have $\hat{\psi}(\xi) = 0$ for all ξ in a neighborhood of the origin (see [HW] page 108). We see, therefore, that the origin does, indeed, play a special role in the domain of the Fourier transform of wavelets; for many such ψ, in fact, we encounter this property that the support of $\hat{\psi}$ has "a hole in the middle" (a term that we shall use often to describe the fact that this support has an empty intersection with an open interval about 0).

If we do not assume that $|\hat{\psi}|$ is continuous at the origin the result we just announced is no longer true. In fact, we will construct band-limited wavelets such that $(\operatorname{supp}\hat{\psi})\cap(-\epsilon,\epsilon)$ has a positive measure for all $\epsilon > 0$. There is, however, the following probably well-known result, that gives further credence to the principle that $\hat{\psi}(0)$ "should have value 0":

Theorem (1.5). *If ψ is a wavelet then*

$$\lim_{\epsilon+\epsilon'\to 0} \frac{1}{\epsilon+\epsilon'} \int_{-\epsilon'}^{\epsilon} |\hat{\psi}(\xi)| d\xi = 0.$$

In other words, this result asserts that if we set $\hat{\psi}(0) = 0$ then the origin is a Lebesgue point of the function $\hat{\psi}$. The proof of (1.5) is an easy application of equality (1.2) and shall be presented in the following sections. This theorem should be compared with a result of D. Speegle [S] which shows that the origin is the only point in $\mathbb{R}$ that plays such a special role for the Fourier transform of a wavelet:

Theorem (1.6). *(Speegle). Suppose $\xi_0 \neq 0$ then there exists a neighborhood U of ξ_0, that does not contain the origin, with the property that if b is any measurable function on U with $|b| \leq 1$, then we can find a wavelet ψ such that $\hat{\psi}(\xi) = b(\xi)$ for all $\xi \in U$.*

This interesting extension property generalizes some parts of the results we shall present. We shall make appropriate comments about this when we announce the relevant results.

Some of the facts we shall announce can be stated more simply for *minimally supported frequency* (MSF) wavelets. These are those wavelets ψ such that $|\hat{\psi}|$ is the characteristic function, χ_W, of a set $W \subset \mathbb{R}$ whose measure, $|W|$, is 2π. As explained in the second section of Chapter 7 in [HW] all such sets, which we call *wavelet sets*, can be constructed from the *Shannon set* $[-2\pi, -\pi) \cup [\pi, 2\pi)$, though such constructions can be somewhat intricate.

When ψ is an MSF wavelet corresponding to the wavelet set W, Theorem (1.5) asserts that

$$\lim_{r\to 0^+} \frac{|W \cap [-r, r]|}{r} = 0. \tag{1.7}$$

This is a statement about the density of W around the origin. It is natural to inquire whether this measure of density (or sparseness) is "best possible". We will provide the following result that gives us an answer to this question:

Theorem (1.8). *If $\delta > 0$ there exist constants $c_1, c_2 > 0$ and an MSF, MRA wavelet ψ_δ such that $|\widehat{\psi_\delta}| = \chi_{W_\delta}$ and*

$$c_1 \le \frac{|W_\delta \cap [-r,r]|}{r^{1+\delta}} \le c_2,$$

for $0 < r \le \frac{\pi}{2}$.

As we shall see, it is of particular interest to us that the wavelet set W_δ we construct in Theorem (1.8) is contained in the interval $[-4\pi, 4\pi]$. We shall show that this interval is a "sort of limiting" case for MRA wavelets. In fact, the following result shows that if $\operatorname{supp}\hat{\psi}$ is contained in any smaller interval, then there is "a hole in the middle". More precisely:

Theorem (1.9). *Suppose ψ is an MRA wavelet and $\operatorname{supp}\hat{\psi} \subset [-4\pi + 2b, 4\pi - 2a]$ where $a, b \ge 0$, $0 < a + b \le 2\pi$, then $\hat{\psi}(\xi) = 0$ for $\xi \in (-a, b)$.*

Among other things, this result tells us that the length of the interval about the origin on which $\hat{\psi}$ is 0 can be as large as 2π. We show that for all orthonormal wavelets there is a natural boundary for the size of such intervals (a, b):

Theorem (1.10). *If ψ is an orthonormal wavelet such that $\hat{\psi}(\xi) = 0$ for $\xi \in (-\pi, \pi)$ then $|\hat{\psi}| = \chi_S$, where S is the Shannon wavelet set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.*

These facts show that in the MRA case the number 4π plays a special role. If we assume a yet smaller band for the support of $\hat{\psi}$ then the MRA property is automatic. We cite two results that give a precise formulation of our assertion. The first one is Theorem (4.1) in the third chapter of [HW] that illustrates the important features of this theorem. A brief accounting of these features is the following: Suppose ψ is a wavelet such that $\operatorname{supp}\hat{\psi} \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi]$. Then $\hat{\psi}(\xi) = 0$ for $\xi \in (-\frac{2}{3}\pi, \frac{2}{3}\pi)$. Furthermore, the values of $|\hat{\psi}(\xi)|$ for $\xi \in [\frac{2}{3}\pi, \frac{4}{3}\pi)$ completely determine all the values $|\hat{\psi}(\xi)|$ on $\mathbb{R}$; in fact, on $[\frac{2}{3}\pi, \frac{4}{3}\pi)$ the function $|\hat{\psi}|$ can be chosen to be any measurable function b with range in $[0, 1]$. Furthermore, ψ must be an MRA wavelet. The reader should compare this with the extension property obtained by Speegle in Theorem (1.6). A precise statement that includes a complete description of the phase of $\hat{\psi}$ is

Theorem (1.11). *Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$. Then ψ is an orthonormal wavelet if and only if it is an MRA wavelet and*

(i) $b(\xi) = 0 \quad$ *for* $\xi \in [-\frac{2}{3}\pi, \frac{2}{3}\pi]$
(ii) $b^2(\xi) + b^2(\frac{\xi}{2}) = 1 \quad$ *for* $\xi \in [\frac{4}{3}\pi, \frac{8}{3}\pi]$
(iii) $b^2(\xi) + b^2(\xi + 2\pi) = 1 \quad$ *for* $\xi \in [-\frac{4}{3}\pi, -\frac{2}{3}\pi]$
(iv) $b(\xi) = b(\frac{\xi}{2} + 2\pi) \quad$ *for* $\xi \in [-\frac{8}{3}\pi, \frac{4}{3}\pi]$
(v) $\hat{\psi}(\xi) = e^{i\alpha(\xi)} b(\xi)$,

where α satisfies $\alpha(\xi) + \alpha(2(\xi - 2\pi)) - \alpha(2\xi) - \alpha(\xi - 2\pi) = (2m(\xi) + 1)\pi$ for some $m(\xi) \in \mathbb{Z}$, for $\xi \in [\frac{2}{3}\pi, \frac{4}{3}\pi] \cap \operatorname{supp} b \cap (\frac{1}{2}\operatorname{supp} b)$.

This result was extended in [HWW] to the case where $\hat{\psi}$ has support contained in $[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, $0 < \alpha \leq \pi$ (see Theorem 2.1 and the accompanying Figure 2 on page 333). We see, therefore , that, again, 4π plays a special role in the domain of $\hat{\psi}$; but we also encounter the number $\frac{8}{3}\pi$ and its significance with respect to the MRA property. This leads us to wonder if there are non-MRA wavelets with Fourier transform supported in bands that lie strictly between the intervals $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ and $[-4\pi, 4\pi]$. We show that by enlarging the band ever so slightly more than this first interval we lose the MRA property:

Theorem (1.12). *If $0 < \epsilon < \frac{2}{3}\pi$, there exists a wavelet set $W = W_\epsilon \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi + \epsilon]$ such that $|W \cap (-\delta, \delta)| > 0$ for each $\delta > 0$.*

Observe that if such a wavelet were MRA, then, by (1.9), with $b = \frac{2}{3}\pi$ and $a = \frac{2}{3}\pi - \frac{\epsilon}{2}$ we must have $\hat{\psi}(\xi) = 0$ if $\xi \in (-\frac{2}{3}\pi + \frac{\epsilon}{2}, \frac{2}{3}\pi)$ and this is inconsistent with (1.12). We also remark that this last theorem provides us with examples of non-MRA wavelets having a Fourier transform supported in bands that are "significantly" smaller than those associated with examples found in the existing literature such as the Journé wavelet which is an MSF wavelet corresponding to the wavelet set $[-\frac{32}{7}\pi, -4\pi] \cup [-\pi, -\frac{4}{7}\pi] \cup [\frac{4}{7}\pi, \pi] \cup [4\pi, \frac{32}{7}\pi]$ (see page 64 in [HW]).

Finally, we see that if ψ is an MRA wavelet and the support of $\hat{\psi}$ extends all the way to 4π, the most we can hope is a "one sided hole " for the values $\hat{\psi}(\xi)$:

Theorem (1.13). *Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in $[-a, 4\pi]$, where $0 < a < \pi$. Then ψ is an MRA wavelet if and only if the function b, whose values lie in $[0, 1]$ satisfies*

(i) $b(\xi) = 0$ *for* $\xi \in [0, 2\pi - \frac{a}{2})$
(ii) $b(\xi) = 1$ *for* $\xi \in [2\pi, 4\pi - a)$
(iii) $b^2(\xi) + b^2(\frac{\xi}{2}) = 1$ *for* $\xi \in [4\pi - a, 4\pi)$
(iv) $b(\xi) = b(\frac{\xi}{2} + 2\pi)$ *for* $\xi \in [-a, -\frac{a}{2})$
(v) $b^2(\xi) + b^2(\xi + 2\pi) = b^2(\frac{\xi}{2} + 2\pi)$ *for* $\xi \in [-\frac{a}{2}, 0)$
(vi) $\lim_{j\to\infty} b(2^{-j}\xi + 2\pi) = 1$ *for* $\xi < 0$
(vii) $b^2(\xi)(1 - b^2(\frac{\xi}{2} + \pi)) = 0$ *for* $\xi \in [2\pi - \frac{a}{2}, 2\pi]$

and $\hat{\psi}(\xi) = e^{i\alpha(\xi)}b(\xi)$, where the measurable function α satisfies

$$\alpha(\xi) + \alpha(2(\xi + 2\pi)) - \alpha(2\xi) - \alpha(\xi + 2\pi) = (2m(\xi) + 1)\pi$$

for some measurable $m(\xi) \in \mathbb{Z}$ and $\xi \in [-\frac{a}{2}, 0) \cap \operatorname{supp} |\hat{\psi}| \cap (\operatorname{supp} |\hat{\psi}| - 2\pi)$.

Observe that the values of b on the interval $[2\pi - \frac{a}{2}, 2\pi)$ determine the remaining values of b. Furthermore, condition (v) requires that $b^2(\frac{\xi}{2} + 2\pi) - b^2(\xi + 2\pi) \geq 0$ if $\xi \in [-\frac{a}{2}, 0)$. Observe that from (vii) we have, if $\eta = \xi + 2\pi$, that $b^2(\eta)(1 - b^2(\frac{\eta}{2} + \pi)) = 0$; thus, this requirement is fulfilled because either $b^2(\eta) = 0$ or $1 - b^2(\frac{\eta}{2} + \pi) = 0$ and this is consistent with (v).

We present the proofs of these results in the following sections. The reader can amuse himself by deriving other consequences of this material. For example, Theorem (1.10) can be considered to be a characterization of the Shannon wavelet set. An easier characterization that follows from (1.9) and (1.11) is that all wavelets ψ with $\operatorname{supp} \hat{\psi} \subset [-2\pi, 2\pi]$ are such that $\hat{\psi}(\xi) = \eta(\xi)\chi_S(\xi)$, where η unimodular and S is the Shannon set.

§**2. The Proof of Theorem (1.5).** By (1.2) we obtain

$$\log 2=\int_1^2\sum_{j\in\mathbb{Z}}|\hat\psi(2^j\xi)|^2\frac{d\xi}{\xi}=\sum_{j\in\mathbb{Z}}\int_{2^j}^{2^{j+1}}|\hat\psi(\xi)|^2\frac{d\xi}{\xi}=\int_0^\infty|\hat\psi(\xi)|^2\frac{d\xi}{\xi}. \tag{2.1}$$

On the other hand by the Schwarz inequality we have

$$\int_0^\epsilon|\hat\psi(\xi)|d\xi\le\left(\int_0^\epsilon|\hat\psi(\xi)|^2\frac{d\xi}{\xi}\right)^{\frac12}\left(\int_0^\epsilon\xi\,d\xi\right)^{\frac12}\le\epsilon\left(\int_0^\epsilon|\hat\psi(\xi)|^2\frac{d\xi}{\xi}\right)^{\frac12};$$

thus,

$$\frac{1}{\epsilon}\int_0^\epsilon|\hat\psi(\xi)|d\xi\le\left(\int_0^\epsilon|\hat\psi(\xi)|^2\frac{d\xi}{\xi}\right)^{\frac12}.$$

But (2.1) implies that

$$\lim_{\epsilon\to0}\int_0^\epsilon|\hat\psi(\xi)|^2\frac{d\xi}{\xi}=0,$$

and, therefore,

$$\lim_{\epsilon\to0}\frac{1}{\epsilon}\int_0^\epsilon|\hat\psi(\xi)|d\xi=0$$

as well. To end our proof it is enough to notice that we can do the same calculations for $-\epsilon'$ using

$$\int_{-\infty}^0|\hat\psi(\xi)|^2\frac{d\xi}{\xi}=-\log 2, \tag{2.2}$$

instead of (2.1).

§**3. The Proof of Theorem (1.8).** Let us start our construction of W_δ by proving two lemmas

Lemma A. *Let* $S=\bigcap_{j=1}^{\infty}2^jE$*, where* $E\subset\mathbb{R}$ *satisfies*

$$E+\pi=E^c \tag{3.1}$$

and

$$E\cap[-\pi,\pi]\subset 2E. \tag{3.2}$$

Then $S=2\,(E\cap[-\pi,\pi])$ *and* $|S|=2\pi$*. Moreover* $\sum_{k\in\mathbb{Z}}\chi_S(\xi+2k\pi)=1$ *a.e.*

Proof. From (3.2) follows that $E\cap[-\pi,\pi]\subset 2^jE$ for every $j\ge0$. Hence

$$S=\bigcap_{j=1}^{\infty}2^jE=2\bigcap_{j=0}^{\infty}2^jE\supseteq 2\,(E\cap[-\pi,\pi])\,.$$

By (3.1) χ_E is 2π-periodic and $\chi_E(x-\pi) = 1-\chi_E(x)$. Since

$$|E\cap[-\pi,\pi]| = \int_{-\pi}^{\pi}\chi_E(x)dx = \int_0^{2\pi}\chi_E(x-\pi)dx =$$

$$\int_0^{2\pi}[1-\chi_E(x)]\,dx = 2\pi - \int_{-\pi}^{\pi}\chi_E(x)dx = 2\pi - |E\cap[-\pi,\pi]|$$

we have $|E\cap[-\pi,\pi]| = \pi$ and therefore $|S|\geq 2\pi$.

Let now $k\in\mathbb{Z}$, $k\neq 0$, $k=2^r q$ with q odd. Observe that

$$S = 2E\cap 2S = 2E\cap 4E\cap\ldots\cap 2^{r+1}E\cap 2^{r+1}S$$

and so $S+2k\pi = (2E+2k\pi)\cap\ldots\cap 2^{r+1}(E+q\pi)\cap 2^{r+1}(S+q\pi)$. Since $E+q\pi = E^c$ we deduce that $S+2k\pi\cap S=\emptyset$ and therefore that $\sum_{k\in\mathbb{Z}}\chi_S(\xi+2k\pi)\leq 1$. Hence

$$2\pi \geq \int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\chi_S(\xi+2k\pi)d\xi = \int_{\mathbb{R}}\chi_S = |S|\,.$$

This implies $|S| = 2\pi$, $S = 2\left(E\cap[-\pi,\pi]\right)$ and

$$\sum_{k\in\mathbb{Z}}\chi_S(\xi+2k\pi) = 1. \quad \blacksquare$$

Lemma B. *Let E and S be as in Lemma A. Assume that for a.e. $\xi\in\mathbb{R}$ and for j sufficiently large $2^{-j}\xi\in S$. Define $\widehat{\varphi}=\chi_S$. Then φ is a scaling function for an MRA and χ_E is the associated low pass filter.*

Proof. By Theorem 5.2, Chapter 7 of [HW] we only need to show that

$$\lim_{j\to\infty}\left|\widehat{\varphi}(2^{-j}\xi)\right| = 1 \quad \text{a.e. } \xi\in\mathbb{R}^n \tag{3.3}$$

and that

$$\chi_S(2\xi) = \chi_E(\xi)\chi_S(\xi). \tag{3.4}$$

Since $2^{-j}\xi\in S$ for j sufficiently large (3.3) is obvious while (3.4) is equivalent to $S = 2E\cap 2S$ which follows from the definition of S. $\blacksquare$

We shall try to construct a low pass filter by means of the previous two lemmas. Let m_o be the filter associated with the Shannon wavelet; that is,

$$m_o(\xi) = \chi_{E_o}(\xi)$$

and $E_o = \bigcup_{k\in\mathbb{Z}}\left[k-\frac{\pi}{2},k+\frac{\pi}{2}\right)$. The idea is to modify the set E_o without violating conditions (3.1) and (3.2) in Lemma A. Since the low pass filter is a periodic function we shall work on the torus that we identify with the interval $[-\pi,\pi)$.

We start with the set

$$A_o = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and we choose a real number $\frac{\pi}{4} < a_1 \leq \frac{\pi}{3}$. We consider the subinterval $\left[\frac{\pi}{4}, a_1\right]$ where $\frac{\pi}{4} < a_1 \leq \frac{\pi}{3}$ (the reason for the condition $a_1 \leq \frac{\pi}{3}$ will became clear later) and we translate it left by π (this does not violate condition (3.1)); we also translate the subinterval $\left[-a_1, -\frac{\pi}{4}\right]$ right by π. In this way we obtain the set

$$A_1 = \left[-\frac{3}{4}\pi, a_1 - \pi\right] \cup \left[-\frac{\pi}{2}, -a_1\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[a_1, \frac{\pi}{2}\right] \cup \left[\pi - a_1, \frac{3}{4}\pi\right]$$

Observe that by the assumption $a_1 \leq \frac{\pi}{3}$ if $\xi \in A_1$ then $\xi/2 \in A_1$. We now iterate this procedure choosing a_j, $j \geq 2$, satisfying $\frac{\pi}{2^{j+1}} < a_j < \frac{1}{2}a_{j-1}$ and translate the interval $\left[\frac{\pi}{2^{j+1}}, a_j\right]$ left by π and the interval $\left[-a_j, -\frac{\pi}{2^{j+1}}\right]$ right by π. In this way we obtain the set

$$A = \bigcup_{j=1}^{\infty} \left[a_j, \frac{\pi}{2^j}\right] \cup \bigcup_{j=1}^{\infty} \left[\pi - a_j, \pi - \frac{\pi}{2^{j+1}}\right] \cup \bigcup_{j=1}^{\infty} \left[-\frac{\pi}{2^j}, -a_j\right] \cup \bigcup_{j=1}^{\infty} \left[\frac{\pi}{2^{j+1}} - \pi, a_j - \pi\right].$$

If we define

$$E = \bigcup_{k \in \mathbb{Z}} (2k\pi + A)$$

we obtain a set satisfying condition (3.1) and (3.2). To apply Lemma B we have to check that the set

$$S = 2(E \cap [-\pi, \pi]) = 2A$$

is such that for almost every $\xi \in \mathbb{R}$ and for j sufficiently large $\xi \in 2^j S$. Since S is symmetric with respect to the origin we can assume $\xi > 0$. By condition (3.2) we obtain that $S \subset 2S$; therefore, we can also assume $\xi \in (\pi, 2\pi)$. Since $\left[2a_j, \frac{\pi}{2^{j-1}}\right] \subset S$ we have $\left[2^{j+1}a_j, 2\pi\right] \subset 2^j S$. If we assume that $2^{j+1}a_j \to \pi$ as $j \to \infty$ we have $\xi \in 2^j S$ for j sufficiently large.

We have proved the following

Proposition C. *Let* $\{a_j\}_{j=1}^{\infty}$ *be a sequence of real numbers satisfying the following conditions*

(i) $\frac{\pi}{4} < a_1 \leq \frac{\pi}{3}$

(ii) $\frac{\pi}{2^{j+1}} < a_j < \frac{1}{2}a_{j-1}$

(iii) $\lim_{j\to\infty} 2^j a_j = \frac{\pi}{2}$

and let m *be a* 2π *periodic function defined by*

$$m(\xi) = \sum_{j=1}^{\infty} \chi_{\left[a_j, \frac{\pi}{2^j}\right]}(\xi) + \sum_{j=1}^{\infty} \chi_{\left[\pi - a_j, \pi - \frac{\pi}{2^{j+1}}\right]}(\xi)$$

on the interval $[0, \pi]$ *and by* $m(\xi) = m(-\xi)$ *on the interval* $[-\pi, 0]$. *Then* m *is a low pass filter associated to an MRA whose scaling function is an even function defined by*

$$\widehat{\varphi}(\xi) = \sum_{j=1}^{\infty} \chi_{\left[2a_j, \frac{\pi}{2^{j-1}}\right]}(\xi) + \sum_{j=1}^{\infty} \chi_{\left[2\pi - 2a_j, 2\pi - \frac{\pi}{2^j}\right]}(\xi)$$

on $[0,\infty)$.

We can now construct a wavelet associated to the MRA. By Proposition 2.13 Chapter 2 of [HW]

$$\widehat{\psi}(2\xi) = e^{i\xi}\overline{m(\xi+\pi)}\widehat{\varphi}(\xi)$$

defines such a wavelet. Therefore we obtain

$$\widehat{\psi}(\xi) = e^{i\xi/2}\chi_W(\xi)$$

where

$$W = 2((E+\pi)\cap S) = 2(E^c \cap S) = 2(S\setminus(E\cap S)) = 2S\setminus S = 4A\setminus 2A.$$

Hence, W is symmetric with respect to the origin and

$$W\cap[0,\pi] = \bigcup_{j=1}^{\infty}[4a_{j+1}, 2a_j]. \tag{3.5}$$

As a concrete example we choose, for any $\delta > 0$, the sequence

$$a_j = \frac{\pi}{2^{j+1}}\left(1+\frac{1}{3(2^{\delta j})}\right).$$

Since the conditions in Proposition C are satisfied we are able to construct a wavelet ψ_δ such that $|\hat{\psi}_\delta| = \chi_{W_\delta}$, where W_δ satisfies (3.5). We there exist positive constants c_1, c_2 such that

$$c_1 \le \frac{|W_\delta\cap[-r,r]|}{r^{1+\delta}} \le c_2. \tag{3.6}$$

Indeed for $r \le \frac{\pi}{2}$

$$\bigcup_{j=N_r}^{\infty}[4a_{j+1}, 2a_j] \subset W_\delta\cap[0,r] \subset \bigcup_{j=N_r-1}^{\infty}[4a_{j+1}, 2a_j]$$

where N_r is the smallest integer so that $2a_{N_r} < r$. A simple computation then shows that

$$\frac{c}{2^{N_r(1+\delta)}} \le |W_\delta\cap[0,r]| \le \frac{c}{2^{(N_r-1)(1+\delta)}}$$

and since $\log_2\frac{\pi}{r} \le N_r \le \log_2\frac{4\pi}{r}$ this shows that

$$c_1' r^{1+\delta} \le |W_\delta\cap[0,r]| \le c_2' r^{1+\delta}$$

for suitable positive constants c_1' and c_2'. Since W_δ is symmetric with respect to the origin we obtain that (3.6) is satisfied and this ends the proof of Theorem (1.8).

§**4. The Proof of Theorem (1.9).** If ψ is an MRA wavelet then its dimension function D_ψ defined in (1.4) must be equal to 1 almost everywhere. By the assumption on the support of $\hat{\psi}$ we have $\hat{\psi}(2^j(\xi+2k\pi))=0$ for $\xi \in [-a,b]$ and $j \geq 1$, $k \in \mathbb{Z}$. Therefore, using (1.2) we see that for $\xi \in [-a,b]$

$$1 = D_\psi(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 = 1 - \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2,$$

and this implies that $\hat{\psi}(\xi)=0$ on $[-a,b]$.

§**5. The Proof of Theorem (1.10).** Let $E=[\pi,2\pi]$. By (2.1) and (2.2) we have

$$2\log 2 = \int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = \sum_{k\in\mathbb{Z}} \int_{(E\cup -E)+2k\pi} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} =$$

$$\sum_{k\in\mathbb{Z}} \int_E |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi+2k\pi|} + \sum_{k\in\mathbb{Z}} \int_{-E} |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi+2k\pi|}.$$

If $\xi \in E$ then

$$\frac{1}{|\xi+2k\pi|} \leq \frac{1}{|\xi|} \qquad \text{for} \quad k \in \mathbb{Z}\setminus\{-1\}, \tag{5.1}$$

but since $\hat{\psi}=0$ on $[-\pi,\pi]$ we have $\int_E |\hat{\psi}(\xi-2\pi)|^2 \frac{d\xi}{|\xi-2\pi|}=0$. By Proposition 1.11 of Chapter 2 [HW] we have $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1$; therefore,

$$\sum_{k\in\mathbb{Z}} \int_E |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi+2k\pi|} \leq \sum_{k\in\mathbb{Z}} \int_E |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi|} = \int_E \frac{d\xi}{|\xi|} = \log 2.$$

Using a similar reasoning for $\xi \in -E$; that is,

$$\frac{1}{|\xi+2k\pi|} \leq \frac{1}{|\xi|} \qquad \text{for} \quad k \in \mathbb{Z}\setminus\{1\}, \tag{5.2}$$

and $\int_{-E} |\hat{\psi}(\xi+2\pi)|^2 \frac{d\xi}{|\xi+2\pi|}=0$, we obtain

$$\sum_{k\in\mathbb{Z}} \int_{-E} |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi+2k\pi|} \leq \sum_{k\in\mathbb{Z}} \int_{-E} |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi|} = \log 2.$$

Therefore, we proved that

$$2\log 2 = \sum_{k\in\mathbb{Z}} \int_{E\cup -E} |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi+2k\pi|} \leq \sum_{k\in\mathbb{Z}} \int_{E\cup -E} |\hat{\psi}(\xi+2k\pi)|^2 \frac{d\xi}{|\xi|} = 2\log 2,$$

so the inequality in the above expression is just an equality. Looking back at (5.1) and (5.2) we discover that for $k \neq 0$ and $\xi \in E \cup -E$ we must have $\hat{\psi}(\xi+2k\pi)=0$, because (5.1) and (5.2) become equalities only for $k=0$. This proves that $\operatorname{supp}\hat{\psi} \subset \chi_{E\cup -E}$, so $|\hat{\psi}| = \chi_{E\cup -E}$.

§6. The Proof of Theorem (1.12). Let $L=[-\frac{8\pi}{3},-\frac{4\pi}{3}]$ and $R=[\frac{2\pi}{3},\frac{4\pi}{3}]$. We can use Corollary 2.4 of Chapter 7 [HW] to check that $L\cup R$ is a wavelet set. Now let

$$A_0=\left[-\frac{5\pi}{3}+\frac{\epsilon}{8},-\frac{5\pi}{3}+\frac{\epsilon}{2}\right], \qquad B_0=\frac{1}{4}A_0,$$

$$A_n=B_{n-1}-2\pi, \qquad B_n=2^{-(n+2)}A_n.$$

Then the wavelet set W defined by

$$W=(L\setminus\bigcup_{n=0}^{\infty}A_n)\cup\bigcup_{n=0}^{\infty}B_n\cup\left[\frac{\pi}{3}+\frac{\epsilon}{8},\frac{\pi}{3}+\frac{\epsilon}{2}\right]\cup\left[\frac{2\pi}{3}+\epsilon,\frac{4\pi}{3}\right]\cup\left[\frac{8\pi}{3},\frac{8\pi}{3}+\epsilon\right]$$

gives us an MSF wavelet with the desired properties.

First let us prove that W is in fact a wavelet set. It is enough to show that W is dilation and translation equivalent to the wavelet set $L\cup R$. For $0\le\epsilon\le\frac{2\pi}{3}$ we have

$$2\left[\frac{\pi}{3}+\frac{\epsilon}{8},\frac{\pi}{3}+\frac{\epsilon}{2}\right]\cup\left[\frac{2\pi}{3}+\epsilon,\frac{4\pi}{3}\right]\cup\frac{1}{4}\left[\frac{8\pi}{3},\frac{8\pi}{3}+\epsilon\right]=R.$$

Moreover, by induction, $A_n\subset L$ for all $n\in\mathbb{N}\cup\{0\}$; therefore,

$$(L\setminus\bigcup_{n=0}^{\infty}A_n)\cup\bigcup_{n=0}^{\infty}2^{n+2}B_n=(L\setminus\bigcup_{n=0}^{\infty}A_n)\cup\bigcup_{n=0}^{\infty}A_n=L.$$

Since the sets A_n, $n\in\mathbb{N}\cup\{0\}$ are disjoint and the same is true for the sets B_n, $n\in\mathbb{N}\cup\{0\}$, we proved that W is dilation equivalent to $L\cup R$. To check the translation equivalence let us observe that

$$\left[\frac{2\pi}{3}+\epsilon,\frac{4\pi}{3}\right]\cup\left(\left[\frac{8\pi}{3},\frac{8\pi}{3}+\epsilon\right]-2\pi\right)=R$$

and

$$(L\setminus\bigcup_{n=0}^{\infty}A_n)\cup\bigcup_{n=0}^{\infty}(B_n-2\pi)\cup\left(\left[\frac{\pi}{3}+\frac{\epsilon}{8},\frac{\pi}{3}+\frac{\epsilon}{2}\right]-2\pi\right)=(L\setminus\bigcup_{n=0}^{\infty}A_n)\cup\bigcup_{n=0}^{\infty}A_{n+1}\cup A_0=L.$$

To prove that there is no "hole in the middle" let us notice that since $A_n\subset L$ for all $n\in\mathbb{N}\cup\{0\}$ we get

$$B_n=2^{-(n+2)}A_n\subset 2^{-(n+2)}L=2^{-n}\left[-\frac{2\pi}{3},-\frac{\pi}{3}\right]$$

so the wavelet set W intersects with every neighborhood of the origin in a set of positive measure.

§7. The Proof of Theorem (1.13). Let us first assume that ψ is an MRA wavelet with supp $\hat{\psi}\subset[-a,4\pi]$. By Theorem (1.9) we obtain $\hat{\psi}(\xi)=0$ for $\xi\in[0,2\pi-\frac{a}{2}]$, which is consistent with (i), moreover it implies together with the equation $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1$ (see Proposition 1.11 of Chapter 2 [HW]) that $b(\xi) = 1$ for $\xi \in [2\pi, 4\pi - a]$, which proves (ii).

For $\xi \in [4\pi - a, 4\pi]$ we have by (1.2) that

$$1 = \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = |\hat{\psi}(\frac{\xi}{2})|^2 + |\hat{\psi}(\xi)|^2,$$

i.e. $1 = b^2(\frac{\xi}{2}) + b(\xi)^2$, which agrees with (iii).

For $\xi \in [-a, -\frac{a}{2})$ we have

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |\hat{\psi}(\xi)|^2 + |\hat{\psi}(\xi + 4\pi)|^2,$$

and using (iii) we obtain $b(\xi)^2 = b^2(\frac{\xi}{2} + 2\pi)$, what is consistent with (iv).

For $\xi \in [-\frac{a}{2}, 0)$ we have

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |\hat{\psi}(\xi)|^2 + |\hat{\psi}(\xi + 2\pi)|^2 + |\hat{\psi}(\xi + 4\pi)|^2,$$

so using (iii) again, we get $b(\xi)^2 + b^2(\xi + 2\pi) = b^2(\frac{\xi}{2} + 2\pi)$, what gives us (v).

To prove condition (vi) let us note that for $\xi \in 2^n[-a, -\frac{a}{2}]$, $n \in \mathbb{Z}$ we have

$$1 = \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = \sum_{j=n}^{\infty} |\hat{\psi}(2^{-j}\xi)|^2 = b^2(2^{-n-1}\xi + 2\pi) +$$

$$+ \sum_{j=n+1}^{\infty} \big(b^2(2^{-j-1}\xi + 2\pi) - b^2(2^{-j}\xi + 2\pi)\big) = \lim_{j\to\infty} b^2(2^{-j}\xi + 2\pi).$$

The last condition and the equation for the phase follow from considering the function t_q from (1.3) with $q = 1$. In fact, for $\xi \in [2\pi - \frac{a}{2}, 2\pi]$ we have

$$0 = t_1(\xi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2\pi))} = \hat{\psi}(\xi)\overline{\hat{\psi}(\xi + 2\pi)},$$

and that is how we get (vii). On the other hand if $\xi \in [-\frac{a}{2}, 0)$ we obtain

$$0 = t_1(\xi) = \hat{\psi}(\xi)\overline{\hat{\psi}(\xi + 2\pi)} + \hat{\psi}(2\xi)\overline{\hat{\psi}(2(\xi + 2\pi))},$$

i.e.

$$e^{i(\alpha(\xi)-\alpha(\xi+\pi))}|\hat{\psi}(\xi)\hat{\psi}(\xi + 2\pi)| = -e^{i(\alpha(2\xi)-\alpha(2(\xi+2\pi)))}|\hat{\psi}(2\xi)\hat{\psi}(2(\xi + 2\pi))|, \tag{7.1}$$

in this way we obtain the needed equation for α, what ends the first part of the proof.

In the second part we assume the above condition about the function ψ and we want to conclude that ψ is an MRA wavelet. First it is easy to check that (i)-(iii) together with

(vi) imply $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$. Similarly (i)-(ii) and (iv)-(v) give us $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1$, which implies $\|\psi\|_2 = 1$. Therefore,"' to see that ψ is a wavelet all we have to prove is

$$t_q(\xi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} = 0,$$

for $q \in 2\mathbb{Z}+1$. It is obvious that it is enough to consider only positive q. Moreover, since $\operatorname{diam}(\operatorname{supp}\hat{\psi}) \leq 5\pi$, we obtain $t_q = 0$ for $q \geq 3$. To see that $t_1 = 0$ as well we can restrict ourselves to $\xi \in [-\frac{a}{2}, 0) \cup [2\pi - \frac{a}{2}, 2\pi)$ by (i). For $\xi \in [2\pi - \frac{a}{2}, 2\pi)$ we have $t_1(\xi) = 0$ by (vii). The case $\xi \in [-\frac{a}{2}, 0)$ will be proved if we show that $|LHS|$ and $|RHS|$ of (7.1) are equal, since we already assumed that $\alpha(\xi) + \alpha(2(\xi+2\pi)) - \alpha(2\xi) - \alpha(\xi+2\pi) = (2m(\xi)+1)\pi$.

If $\xi \in [-\frac{a}{2}, -\frac{a}{4}]$ then

$$|LHS| = \sqrt{b^2(\xi/2+2\pi) - b^2(\xi+2\pi)}\, b(\xi+2\pi),$$
$$|RHS| = b(\xi+2\pi)\sqrt{1-b^2(\xi+2\pi)}.$$

Now if $b(\xi+2\pi) = 0$, then everything is clear. But if this is not the case then, by (vii), we have $b(\xi/2+2\pi) = 1$, so $|LHS| = |RHS|$ as well.

If $\xi \in 2^{-j}[-a, -\frac{a}{2})$ for $j \geq 2$, then

$$|LHS| = \sqrt{b^2(\xi/2+2\pi) - b^2(\xi+2\pi)}\, b(\xi+2\pi),$$
$$|RHS| = \sqrt{b^2(\xi+2\pi) - b^2(2\xi+2\pi)}\sqrt{1-b^2(\xi+2\pi)}.$$

Now $b(\xi+2\pi) = 0$ implies, by (vii), that $b(2\xi+2\pi) = 0$, so we have zeros on both sides. But if $b(\xi+2\pi) \neq 0$ then, by (vii), again we have $b(\xi/2+2\pi) = 1$, which also gives us $|LHS| = |RHS|$; since it is clear if $b(\xi+2\pi) = 1$ and for $b(\xi+2\pi) \neq 1$ we have $b(2\xi+2\pi) = 0$ by (vii).

We will end our proof by showing that ψ is an MRA wavelet; that is, we will prove that $D_\psi(\xi) := \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2$ is equal to 1 almost everywhere (see (1.4)). If $\xi \in [0, 2\pi - a)$ then by (i)-(iii)

$$D_\psi(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 = 1.$$

For $\xi \in [-a, -\frac{a}{2})$ it follows from (ii) and (iv) that

$$D_\psi(\xi) = |\hat{\psi}(2(\xi+2\pi))|^2 = 1.$$

Finally for $\xi \in 2^{-l}[-a, -\frac{a}{2})$, where $l \geq 1$ we have

$$D_\psi(\xi) = \sum_{j=1}^{l} |\hat{\psi}(2^j\xi)|^2 + |\hat{\psi}(2(\xi+2\pi))|^2.$$

Therefore, for $l = 1$ we obtain

$$D_\psi(\xi) = b^2(\xi+2\pi) + 1 - b^2(\xi+2\pi) = 1$$

and for $l \geq 2$

$$D_\psi(\xi) = \sum_{j=1}^{l}\{b^2(2^{j-1}\xi+2\pi) - b^2(2^j\xi+2\pi)\} + 1 - b^2(\xi+2\pi) = 1,$$

i.e. $D_\psi(\xi) = 1$ on $[-a, 2\pi - a)$ and since D_ψ is 2π-periodic we obtain $D_\psi(\xi) = 1$ almost everywhere.

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