POINTWISE CONVERGENCE OF FRACTIONAL POWERS OF HERMITE TYPE OPERATORS

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Dedicated to the memory of Eleonor Pola *Harboure, whose human qualities and professional behavior will always be our guidance*

Abstract. When L is the Hermite or the Ornstein-Uhlenbeck operator, we find minimal integrability and smoothness conditions on a function f so that the fractional power $L^\sigma f(x_0)$ is well-defined at a given point x_0. We illustrate the optimality of the conditions with various examples. Finally, we obtain similar results for the fractional operators $(-\Delta + R)^\sigma$, with $R > 0$.

1. Introduction

Let L be a positive self-adjoint differential operator densely defined in a Hilbert space $L_2(\Omega, d\mu)$. Fractional powers L^σ, for $\sigma > 0$, can be defined in various (abstract) equivalent ways, one of the most standard being via spectral theory:

$$\langle L^\sigma f, g\rangle = \int_{\sigma(L)} \lambda^\sigma \, dE_{f,g}, \quad f \in \mathrm{Dom}(L^\sigma), \;\; g \in L_2(\Omega, d\mu), \tag{1.1}$$

where E denotes a resolution of the identity associated with L; see e.g. [13, Ch 13]. When the spectrum is discrete $\sigma(L) = \{\lambda_n\}_{n=0}^\infty$, and $\{\varphi_n\}_{n=0}^\infty$ is an orthonormal basis of eigenfunctions, then (1.1) takes the form

$$L^\sigma f = \sum_{n=0}^\infty \lambda_n^\sigma \, \langle f, \varphi_n\rangle \, \varphi_n,$$

say for $f \in \operatorname{span}\{\varphi_n\}$. Alternatively, it is also possible to express L^σ in terms of the contraction semigroup $\{e^{-tL}\}_{t>0}$. For instance, when $\sigma \in (0,1)$, via the Bochner formula

$$L^\sigma f = \tfrac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-tL} f - f) \, \frac{dt}{t^{1+\sigma}}; \tag{1.2}$$

see e.g. [17, Ch IX.11] and references therein.

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When L equals the classical Laplacian $-\Delta$ and $f \in \mathcal{S}(\mathbb{R}^d)$, both (1.1) and (1.2) lead to the explicit expression

$$(-\Delta)^\sigma f(x) \;=\; c_{d,\sigma}\,\mathrm{PV}\int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2\sigma}}\,dy, \quad x\in\mathbb{R}^d,\;\; \sigma\in(0,1), \tag{1.3}$$

with a suitable constant $c_{d,\sigma}>0$; see e.g. [9]. This pointwise formula does actually make sense for a larger class of functions, e.g. when $f\in L_1(dy/(1+|y|)^{d+2\sigma})$ and f is Hölder continuous of order $2\sigma+\varepsilon$ near the point x. For general operators L, however, explicit expressions such as (1.3) are not common, and the definition of $L^\sigma f$ as in (1.1)-(1.2) must necessarily be restricted to a suitable dense class of functions f.

Based on work of Caffarelli and Silvestre [2], Stinga and Torrea proposed in [14] to define $L^\sigma f$ as the Neumann boundary value associated with the elliptic PDE

$$\begin{cases} u_{tt}+\frac{1-2\sigma}{t}\,u_t \;=\; Lu, & t>0\\ u(0,x)=f(x). \end{cases} \tag{1.4}$$

More precisely, if $\sigma\in(0,1)$ and $\mathfrak{c}_\sigma = 2^{2\sigma-1}\Gamma(\sigma)/\Gamma(1-\sigma)$, they set

$$L^\sigma f(x) \;=\; -\mathfrak{c}_\sigma \lim_{t\to0^+} t^{1-2\sigma}\,\partial_t u(t,x), \tag{1.5}$$

where $u(t,x)$ is the solution of (1.4) given by the Poisson-like integral

$$u(t,x) = P_t^\sigma f(x) := \frac{(t/2)^{2\sigma}}{\Gamma(\sigma)}\int_0^\infty e^{-\frac{t^2}{4s}}\,e^{-sL}f(x)\,\frac{ds}{s^{1+\sigma}}; \tag{1.6}$$

see [14, Theorem 1.1]. Then the authors specialize to the Hermite operator

$$L=-\Delta+|x|^2 \quad \text{in } L_2(\mathbb{R}^d),$$

and prove that the *pointwise limit* in (1.5) exists at every $x\in\mathbb{R}^d$, and coincides with (1.2), whenever $f\in C^2(\mathbb{R}^d)\cap L_1(dy/(1+|y|)^N)$, for some $N>0$; see [14, Theorem 4.2], and [15, 12] for slightly less restrictive smoothness assumptions.

The purpose of this paper is to show the validity of (1.5) for a wider class of functions f, with optimal integrability assumptions and very mild smoothness at a given point $x_0\in\mathbb{R}^d$. We also consider the slightly more general family of operators

$$L=-\Delta+|x|^2+m, \quad \text{in } L_2(\mathbb{R}^d), \tag{1.7}$$

with a constant parameter $m\geq -d$ (so that L is positive). For $m=0$ this is the usual Hermite operator, while for $m=-d$ it can be transformed (after a change of variables) into the Ornstein-Uhlenbeck operator

$$\mathcal{O}=-\Delta+2x\cdot\nabla, \quad \text{in } L_2(\mathbb{R}^d, e^{-|x|^2}\,dx);$$

see §4 below.

To state our results, we now describe the integrability and smoothness conditions we shall use below. Given a weight $\Phi(y)\geq0$, we say that $f\in L_1(\Phi)$ when

$$\int_{\mathbb{R}^d} |f(y)|\,\Phi(y)\,dy<\infty. \tag{1.8}$$

Associated with $L = -\Delta + |x|^2 + m$ in (1.7), and $\sigma \in (0,1)$ we define the weight

$$\Phi_\sigma(y) = \begin{cases} \dfrac{e^{-|y|^2/2}}{(1+|y|)^{\frac{d+m}{2}}[\ln(e+|y|)]^{1+\sigma}} & \text{if } m > -d \\[2ex] \dfrac{e^{-|y|^2/2}}{[\ln(e+|y|)]^{\sigma}} & \text{if } m = -d. \end{cases} \tag{1.9}$$

Then, $f \in L_1(\Phi_\sigma)$ ensures that the Poisson-like integral $P_t^\sigma f(x)$ in (1.6) is a well-defined (smooth) function when $(t,x) \in (0,\infty) \times \mathbb{R}^d$, and this condition is actually best possible for that property; see [6, Theorem 1.1] (or [8, 5]).

Next, when $\alpha \in (0,2)$, we shall say that a (locally integrable) function f is *α-smooth at $x_0 \in \mathbb{R}^d$*, denoted $f \in \mathcal{D}^\alpha(x_0)$, if it satisfies the Dini-type condition

$$\int_{|h|\le\delta} \frac{|f(x_0+h) + f(x_0-h) - 2f(x_0)|}{|h|^{d+\alpha}}\, dh \,<\, \infty, \quad \text{for some } \delta > 0.$$

We say that f is *strictly α-smooth at x_0*, denoted $f \in \mathcal{D}^\alpha_{\rm st}(x_0)$, if

$$f \in \mathcal{D}^\alpha(x_0) \quad \text{and} \quad \int_{|h|\le\delta} \frac{|f(x_0+h) - f(x_0-h)|}{|h|^{d+\alpha-3}}\, dh \,<\, \infty.$$

Observe that this last condition is redundant if $d+\alpha-3 \le 0$; in particular, if $d=1$ and $\alpha \in (0,2)$, or $d=2$ and $\alpha \in (0,1]$. In other cases however, the classes are different and we shall need this distinction to obtain our results. We refer to §6 below for several examples of these notions, and their relation with local Lipschitz conditions at x_0.

Our first main result can now be stated as follows.

Theorem 1.1. *Let L be the Hermite operator in* (1.7)*, $\sigma \in (0,1)$ and $\Phi_\sigma(y)$ as in* (1.9)*. Suppose that*

$$f \in L_1(\Phi_\sigma), \quad \text{and} \quad f \in \mathcal{D}^{2\sigma}_{\rm st}(x_0) \quad \text{for some } x_0 \in \mathbb{R}^d.$$

Then, the number $L^\sigma f(x_0)$ exists both, in the limiting sense of (1.5) *and as the absolutely convergent integral in* (1.2)*, and both definitions agree.*

The optimality of the smoothness condition is discussed in §6 below. The corresponding version for the Ornstein-Uhlenbeck operator takes the following form.

Theorem 1.2. *Let $L = \mathcal{O} = -\Delta + 2x\cdot\nabla$ and $\sigma \in (0,1)$. Suppose that*

$$f \in L_1(e^{-|y|^2}/[\log(e+y)]^\sigma), \quad \text{and} \quad f \in \mathcal{D}^{2\sigma}_{\rm st}(x_0) \quad \text{for some } x_0 \in \mathbb{R}^d.$$

Then, the number $\mathcal{O}^\sigma f(x_0)$ exists both, in the limiting sense of (1.5) *and as the absolutely convergent integral in* (1.2)*, and both definitions agree.*

Below, we have attempted to present our results in sufficient generality so that many of the arguments can be applied to general operators L, while only a few steps require explicit estimates on the kernels. We illustrate this fact in §5, where with a minimal effort we obtain a version of the above theorems for the operator $\mathbb{L} = -\Delta + R$, with $R > 0$; see Theorem 5.2 below.

2. Preliminary results for general operators L

In this section we shall assume that L is the infinitesimal generator of a semigroup of operators $\{e^{-tL}\}_{t>0}$ in $L_2(\mathbb{R}^d)$, and that these are described by the integrals

$$e^{-tL}f(x) = \int_{\mathbb{R}^d} h_t(x,y)f(y)\,dy, \tag{2.1}$$

for a suitable positive kernel $h_t(x,y)$. For each $\sigma \in (0,1)$ we then consider the family of subordinated operators $\{P_t^\sigma = P_t^{\sigma,L}\}_{t>0}$, formally given by

$$P_t^\sigma f := \tfrac{(t/2)^{2\sigma}}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4s}}\, e^{-sL}(f)\, \frac{ds}{s^{1+\sigma}}.$$

More precisely, we let

$$P_t^\sigma f(x) = \int_{\mathbb{R}^d} p_t^\sigma(x,y)f(y)\,dy, \tag{2.2}$$

where the corresponding kernels $p_t^\sigma(x,y) = p_t^{\sigma,L}(x,y)$ are defined by

$$p_t^\sigma(x,y) = \tfrac{(t/2)^{2\sigma}}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4s}}\, h_s(x,y)\, \frac{ds}{s^{1+\sigma}}. \tag{2.3}$$

Observe that a crude estimate such as $0 < h_s(x,y) \lesssim s^{-d/2}$ (which will be satisfied by all the operators L we shall use) guarantees that the integral in (2.3) is absolutely convergent, and moreover

$$\partial_t\big[p_t^\sigma(x,y)\big] \;=\; \mathfrak{a}_\sigma\, t^{2\sigma-1} \int_0^\infty \Big(2\sigma - \frac{t^2}{2s}\Big) e^{-\frac{t^2}{4s}} h_s(x,y) \frac{ds}{s^{1+\sigma}}, \tag{2.4}$$

with $\mathfrak{a}_\sigma = 1/(4^\sigma\Gamma(\sigma))$. It is also not hard to show that

$$t\Big|\partial_t\big[p_t^\sigma(x,y)\big]\Big| \lesssim p_t^\sigma(x,y) + p^\sigma_{t/\sqrt{2}}(x,y), \tag{2.5}$$

using the fact that $\sup_{v>0} ve^{-v} < \infty$. However, in order to handle derivatives of the expression $P_t^\sigma f(x)$ in (2.2) we shall need more information on the decay of the kernel $p_t^\sigma(x,y)$.

In the case that L is a Hermite operator the decay is given by the following result from [6, Lemma 3.1], which also clarifies the optimal role of the functions $\Phi_\sigma(y)$ in (1.9).

Lemma 2.1. *Let L be the Hermite operator in* (1.7)*, $\sigma \in (0,1)$ and $\Phi_\sigma(y)$ as in* (1.9)*. Then, for every $t > 0$ and $x \in \mathbb{R}^d$, there exist finite numbers $c_1(t,x) > 0$ and $c_2(t,x) > 0$ such that*

$$c_1(t,x)\,\Phi_\sigma(y) \;\le\; p_t^\sigma(x,y) \;\le\; c_2(t,x)\,\Phi_\sigma(y)\;, \quad \forall\, y \in \mathbb{R}^d. \tag{2.6}$$

In this section we wish to keep the general setting for the operator L described above, but we shall *additionally* assume that, for each given $\sigma \in (0,1)$, the kernel $p_t^\sigma(x,y)$ satisfies

$$p_t^\sigma(x,y) \;\le\; c_2(t,x)\,\Phi(y)\;, \quad \forall\, y \in \mathbb{R}^d. \tag{2.7}$$

for some function $\Phi(y)$ and some $c_2(t,x)>0$. In all the results in this section we shall not need the explicit expression of $\Phi(y)$.

Our first result will establish a relation between the two *pointwise* definitions of $L^\sigma f(x)$ presented in (1.2) and (1.5) above. We first consider the following general definition.

Definition 2.2. Let $\sigma\in(0,1)$ and L be an operator such that (2.7) holds for some $\Phi(y)$. Given $f\in L_1(\Phi)$, we say that a point $x_0\in\mathbb{R}^d$ is L^σ-admissible for f, denoted $x_0\in\mathcal{A}_f(L^\sigma)$, if

$$\int_0^\infty \left|e^{-sL}f(x_0)-f(x_0)\right|\frac{ds}{s^{1+\sigma}}<\infty. \tag{2.8}$$

In that case we let

$$L^\sigma f(x_0):=\tfrac{1}{\Gamma(-\sigma)}\int_0^\infty\Big(e^{-sL}f(x_0)-f(x_0)\Big)\,\frac{ds}{s^{1+\sigma}}, \tag{2.9}$$

where $\Gamma(-\sigma)=\Gamma(1-\sigma)/(-\sigma)$.

The following result is partially based on the proof of [14, (4.6)].

Proposition 2.3. *Let $\sigma\in(0,1)$ and L be an operator such that (2.7) holds for some $\Phi(y)$. If $f\in L_1(\Phi)$ and $x\in\mathcal{A}_f(L^\sigma)$ then*

$$\lim_{t\to0^+}-\mathfrak{c}_\sigma\,t^{1-2\sigma}\,\partial_t\Big[P_t^\sigma f(x)\Big]=L^\sigma f(x)\,, \tag{2.10}$$

with $\mathfrak{c}_\sigma=2^{2\sigma-1}\Gamma(\sigma)/\Gamma(1-\sigma)$.

Proof. Using (2.2), (2.4), (2.5) and $f\in L_1(\Phi)$ one can justify that

$$t^{1-2\sigma}\,\partial_t\Big[P_t^\sigma f(x)\Big]=\mathfrak{a}_\sigma\int_{\mathbb{R}^d}\int_0^\infty\Big(2\sigma-\frac{t^2}{2s}\Big)e^{-\frac{t^2}{4s}}h_s(x,y)\,\frac{ds}{s^{1+\sigma}}\,f(y)\,dy,$$

with $\mathfrak{a}_\sigma=1/(4^\sigma\Gamma(\sigma))$. We claim that

$$I=\int_0^\infty\Big(2\sigma-\frac{t^2}{2s}\Big)e^{-\frac{t^2}{4s}}\,\frac{ds}{s^{1+\sigma}}=0.$$

In fact, using the change $z=t^2/4s$ we see that

$$I=\frac{2\cdot4^\sigma}{t^{2\sigma}}\int_0^\infty(\sigma-z)e^{-z}z^{\sigma-1}\,dz=\frac{2\cdot4^\sigma}{t^{2\sigma}}\big(\sigma\Gamma(\sigma)-\Gamma(\sigma+1)\big)=0.$$

Then

$$\begin{aligned}t^{1-2\sigma}\,\partial_t\Big[P_t^\sigma f(x)\Big]&=\mathfrak{a}_\sigma\int_0^\infty\Big(2\sigma-\frac{t^2}{2s}\Big)e^{-\frac{t^2}{4s}}\Big[\int_{\mathbb{R}^d}h_s(x,y)f(y)\,dy-f(x)\Big]\frac{ds}{s^{1+\sigma}}\\&=\mathfrak{a}_\sigma\int_0^\infty\Big(2\sigma-\frac{t^2}{2s}\Big)e^{-\frac{t^2}{4s}}\Big[e^{-sL}f(x)-f(x)\Big]\frac{ds}{s^{1+\sigma}}.\end{aligned}$$

Since we assume that $x\in\mathcal{A}_f(L^\sigma)$, by the Lebesgue dominated convergence theorem we can take limits as $t\to0^+$, and after adjusting the constants one easily obtains the result. □

In view of Proposition 2.3, we are interested in finding conditions on a function f which guarantee that a given point $x \in \mathcal{A}_f(L^\sigma)$. Our next observation shows that only one part of the integral in (2.8) must be checked.

Lemma 2.4. *Let $\sigma \in (0,1)$ and L be an operator such that (2.7) holds for some $\Phi(y)$. Then, for every $A>0$ and every $x \in \mathbb{R}^d$ there exists $c(x,A)>0$ such that*

$$\int_A^\infty h_s(x,y)\,\frac{ds}{s^{1+\sigma}} \leq c(x,A)\,\Phi(y), \quad y \in \mathbb{R}^d. \tag{2.11}$$

Moreover, if $f \in L_1(\Phi)$ and $|f(x)|<\infty$ then

$$\int_A^\infty |e^{-sL}f(x)-f(x)|\,\frac{ds}{s^{1+\sigma}} < \infty.$$

Proof. To prove (2.11), note that

$$\int_A^\infty h_s(x,y)\,\frac{ds}{s^{1+\sigma}} \leq e^{\frac{1}{4A}} \int_0^\infty e^{-\frac{1}{4s}}\,h_s(x,y)\,\frac{ds}{s^{1+\sigma}} = c\,p_1^\sigma(x,y) \leq c(x,A)\,\Phi(y).$$

For the last statement,

$$\begin{aligned} \int_A^\infty |e^{-sL}f(x)-f(x)|\frac{ds}{s^{1+\sigma}} &\leq \int_{\mathbb{R}^d}|f(y)|\int_A^\infty h_s(x,y)\,\frac{ds}{s^{1+\sigma}}\,dy + |f(x)|\int_A^\infty \frac{ds}{s^{1+\sigma}} \\ \text{by (2.11)} \quad &\lesssim c(x,A)\int_{\mathbb{R}^d}|f(y)|\,\Phi(y)\,dy + |f(x)| < \infty. \end{aligned}$$

□

In order to show that $x_0 \in \mathcal{A}_f(L^\sigma)$ we expect that some smoothness of f at the point x_0 must be required. Actually, the smoothness of f will only play a *local* role in the integrals defining the property $\mathcal{A}_f(L^\sigma)$. This motivates to consider a local notion of L^σ-admissibility.

Definition 2.5. Let $\sigma \in (0,1)$, and L an operator as above. Given a locally integrable function f, we say that a point $x_0 \in \mathbb{R}^d$ is *locally L^σ-admissible for f*, denoted $x_0 \in \mathcal{A}_f^{\rm loc}(L^\sigma)$, if there exists $\delta>0$ and $A>0$ such that the integrals

$$\mathcal{I}_\delta f(x_0,s) = \int_{|x_0-y|<\delta} h_s(x_0,y)\,\big[f(y)-f(x_0)\big]dy, \quad s\in(0,A) \tag{2.12}$$

satisfy the property

$$\int_0^A \big|\mathcal{I}_\delta f(x_0,s)\big|\,\frac{ds}{s^{1+\sigma}} \;<\; \infty. \tag{2.13}$$

The next lemma gives decay conditions on the kernel and smoothness of f at x_0 that guarantee the validity of the previous property.

Proposition 2.6. *Let $\sigma \in (0,1)$, and L an operator as above. Let $x_0 \in \mathbb{R}^d$ be fixed, and assume that the kernel $h_s(x_0,\cdot)$ in (2.1) satisfies, for some $\delta>0$ and $A\in(0,\infty]$, the estimates*

$$\int_0^A h_s(x_0,x_0+y)\frac{ds}{s^{1+\sigma}} \;\leq\; \frac{c(x_0)}{|y|^{d+2\sigma}}, \quad \textit{when} \quad |y|\leq\delta \tag{2.14}$$

and

$$\int_0^A \Big|h_s(x_0,x_0+y)-h_s(x_0,x_0-y)\Big|\frac{ds}{s^{1+\sigma}} \le \frac{c(x_0)}{|y|^{(d+2\sigma-3)_+}}, \quad \textit{when} \quad |y|\le\delta. \tag{2.15}$$

Then, for every locally integrable f it holds

$$f\in\mathcal{D}^{2\sigma}_{\rm st}(x_0) \quad \Longrightarrow \quad x_0\in\mathcal{A}^{\rm loc}_f(L^\sigma). \tag{2.16}$$

Moreover, (2.13) *holds with the same A and δ as in* (2.14) *and* (2.15).

Before proving the result, recall the standard notation

$$f_{\rm even}(x)=\frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_{\rm odd}(x)=\frac{f(x)-f(-x)}{2}.$$

We shall use the following elementary lemma.

Lemma 2.7. *Let F and G be locally integrable in $\mathbb{R}^d$, and B a ball centered at the origin. Then*

$$\int_B F(x)G(x)\,dx=\int_B F_{\rm even}(x)G_{\rm even}(x)\,dx+\int_B F_{\rm odd}(x)G_{\rm odd}(x)\,dx. \tag{2.17}$$

We also introduce the notation

$$\triangle^1_z f(x)=f(x+z)-f(x-z), \quad \text{and} \quad \triangle^2_z f(x)=f(x+z)+f(x-z)-2f(x).$$

Observe that, after dividing by 2, these expressions are respectively the odd and even parts of the function $z\mapsto f(x+z)-f(x)$.

Proof of Proposition 2.6: Changing variables $y=x_0+z$ in (2.12), we can write

$$\mathcal{I}_\delta f(x_0,s)=\int_{|z|<\delta} h_s(x_0,x_0+z)\,\big[f(x_0+z)-f(x_0)\big]dz.$$

Then, using the identity in (2.17) and simple manipulations, we can rewrite this expression as

$$\begin{aligned}\mathcal{I}_\delta f(x_0,s) \;&=\; \tfrac12\int_{|z|<\delta} h_s(x_0,x_0+z)\triangle^2_z f(x_0)dz\\ &+\tfrac14\int_{|z|<\delta}\big(h_s(x_0,x_0+z)-h_s(x_0,x_0-z)\big)\,\triangle^1_z f(x_0)dz\end{aligned} \tag{2.18}$$

Thus, using the kernel assumptions in (2.14) and (2.15), we clearly have

$$\begin{aligned}\int_0^A |\mathcal{I}_\delta f(x_0,s)|\,\frac{ds}{s^{1+\sigma}} \;&\le\; \tfrac12\int_{|z|<\delta}|\triangle^2_z f(x_0)|\Big(\int_0^A h_s(x_0,x_0+z)\,\frac{ds}{s^{1+\sigma}}\Big)dz\\ &+\tfrac14\int_{|z|<\delta}|\triangle^1_z f(x_0)|\int_0^A |h_s(x_0,x_0+z)-h_s(x_0,x_0-z)|\,\frac{ds}{s^{1+\sigma}}dz\\ &\lesssim \int_{|z|<\delta}\frac{|\triangle^2_z f(x_0)|}{|z|^{d+2\sigma}}\,dz+\int_{|z|<\delta}\frac{|\triangle^1_z f(x_0)|}{|z|^{(d+2\sigma-3)_+}}\,dz\end{aligned}$$

which is a finite quantity when $f\in\mathcal{D}^{2\sigma}_{\rm st}(x_0)$. □

Remark 2.8. When the heat kernel at a given x_0 satisfies

$$h_s(x_0, y) = \rho_{s,x_0}(|x_0 - y|),$$

for some function ρ_{s,x_0} (for instance, if h_s is of convolution type and radial), then the condition (2.15) is automatically satisfied (since the integrand is 0). Moreover, in the proof of the proposition the integral in (2.18) vanishes, so no bound is needed involving $\triangle_z^1 f(x_0)$. Thus, in that setting, the conclusion (2.16) of the proposition holds with the weaker smoothness assumption $f \in \mathcal{D}^{2\sigma}(x_0)$.

3. The Hermite operator $L = -\Delta + |x|^2 + m$, with $m \geq -d$

In this section we specialize to the case when

$$L = -\Delta + |x|^2 + m, \quad \text{with } m \geq -d.$$

We recall the kernel expressions in this setting. For the heat kernel $h_t(x, y)$, associated with e^{-tL}, we have the Mehler formula

$$h_t(x, y) = e^{-tm} \, \frac{e^{-\frac{|x-y|^2}{2\text{th}\, 2t} - \text{th}\, t\, x\cdot y}}{[2\pi \text{sh}\, 2t]^{\frac{d}{2}}}, \quad t > 0, \ x, y \in \mathbb{R}^d;$$

see e.g. [16, (4.3.14)]. Changing variables to $t = t(s) = \frac{1}{2} \ln(\frac{1+s}{1-s})$ (or equivalently, $s = \text{th}\,(t)$), the kernel takes the form

$$h_{t(s)}(x, y) = \frac{(1 - s)^{\frac{m+d}{2}}}{(1 + s)^{\frac{m-d}{2}}} \, \frac{e^{-\frac{1}{4}(\frac{|x-y|^2}{s} + s|x+y|^2)}}{(4\pi s)^{\frac{d}{2}}}. \tag{3.1}$$

In the next subsection we shall collect the decay and smoothness estimates of this kernel that will be needed in the proof of Theorem 1.1.

3.1. **Kernel estimates.**

Throughout this section we denote

$$\mathcal{K}(x, y) := \int_0^A h_t(x, y) \, \frac{dt}{t^{1+\sigma}}, \quad x, y \in \mathbb{R}^d, \tag{3.2}$$

where we select $A > 0$ so that $\text{th}\, A = 1/2$ (any other $A > 0$ would also be fine). Performing the change of variables in (3.1) (so that $dt = \frac{ds}{1-s^2}$) we obtain

$$\mathcal{K}(x, y) = \int_0^{1/2} \frac{(1 - s)^{\frac{m+d}{2} - 1} \, e^{-\frac{1}{4}(\frac{|x-y|^2}{s} + s|x+y|^2)}}{(1 + s)^{\frac{m-d}{2} + 1} \, (4\pi s)^{\frac{d}{2}} \, \left(\frac{1}{2} \ln \frac{1+s}{1-s}\right)^{1+\sigma}} \, ds. \tag{3.3}$$

Observe that in this range of integration we have $\ln \frac{1+s}{1-s} \approx s$, and $1 \pm s \approx 1$, so $\mathcal{K}(x, y)$ becomes comparable to

$$\mathcal{K}_1(x, y) = \int_0^{1/2} \frac{e^{-\frac{1}{4}(\frac{|x-y|^2}{s} + s|x+y|^2)}}{s^{1+\sigma+\frac{d}{2}}} \, ds. \tag{3.4}$$

The first lemma shows the decay condition in (2.14). The argument in the proof is similar to the one used in [8, (4.13)] (where a better estimate is obtained).

Lemma 3.1. *With the notation in* (3.3)*, for every* $\sigma > 0$ *there exists* $c = c(\sigma) > 0$ *such that*

$$\mathcal{K}(x,y) \le \frac{c}{|x-y|^{d+2\sigma}}, \quad \forall\, x, y \in \mathbb{R}^d.$$

Proof. Changing variables $u = \frac{|x-y|^2}{4s}$ in (3.4) we see that

$$\begin{aligned} \mathcal{K}(x,y) \;&\approx\; \mathcal{K}_1(x,y) = \Big(\frac{4}{|x-y|^2}\Big)^{\sigma+\frac{d}{2}} \int_{\frac{|x-y|^2}{2}}^{\infty} e^{-u}\, e^{-\frac{|x+y|^2\,|x-y|^2}{16u}}\, u^{\sigma+\frac{d}{2}}\,\frac{du}{u} \\ &\le\; \frac{4^{\sigma+\frac{d}{2}}}{|x-y|^{d+2\sigma}} \int_0^\infty e^{-u}\, u^{\sigma+\frac{d}{2}}\,\frac{du}{u} = \frac{c}{|x-y|^{d+2\sigma}}. \end{aligned}$$

□

Remark 3.2. This lemma may also be proved directly from (3.2) using the property

$$h_t(x,y) \lesssim t^{-d/2} e^{-c\frac{|x-y|^2}{t}}, \quad 0 < t \lesssim 1.$$

This property is known to hold for many other operators L.

We now show the smoothness condition in (2.15).

Lemma 3.3. *For every* $\sigma > 0$*, there exists* $c = c(\sigma) > 0$ *such that*

$$\int_0^A |h_t(x,x+y) - h_t(x,x-y)|\,\frac{dt}{t^{1+\sigma}} \le \frac{c\,|x|}{|y|^{(d+2\sigma-3)_+}}, \quad \forall\, x \in \mathbb{R}^d,\ |y| \le 1. \tag{3.5}$$

In the proof of (3.5) we shall use the following elementary inequality.

Lemma 3.4. *If* $x, y \in \mathbb{R}^d$ *then*

$$\Big|e^{-|x+y|^2} - e^{-|x-y|^2}\Big| \le 4|x|\,|y|\, e^{-\min|x\pm y|^2}.$$

Proof. Using $|x \pm y|^2 = |x|^2 + |y|^2 \pm 2x\cdot y$ we can write

$$\Big|e^{-|x+y|^2} - e^{-|x-y|^2}\Big| = e^{-|x|^2-|y|^2}\,\Big|e^{2x\cdot y} - e^{-2x\cdot y}\Big|.$$

Now, letting $t = 2|x\cdot y|$, and using the inequality

$$e^t - e^{-t} = \int_{-t}^t e^s\,ds \le 2te^t,$$

we obtain

$$\Big|e^{-|x+y|^2} - e^{-|x-y|^2}\Big| \le 4|x|\,|y|\, e^{2|x\cdot y|}\, e^{-|x|^2-|y|^2} = 4|x|\,|y|\, e^{-\min|x\pm y|^2}.$$

□

Proof of Lemma 3.3: Denote by $\mathcal{K}'(x,y)$ the left hand side of (3.5). Then, performing the change of variables in (3.1), and disregarding the inessential terms (as discussed before (3.4)) we obtain

$$\begin{aligned} \mathcal{K}'(x,y) &\approx \int_0^{1/2} \frac{e^{-\frac{|y|^2}{4s}}\,\left|e^{-\frac{s}{4}|2x+y|^2}-e^{-\frac{s}{4}|2x-y|^2}\right|}{s^{1+\sigma+\frac{d}{2}}}\,ds \\ \text{(by Lemma 3.4)} &\lesssim \int_0^{1/2} \frac{e^{-\frac{|y|^2}{4s}}\,s\,|2x|\,|y|}{s^{1+\sigma+\frac{d}{2}}}\,ds \;\lesssim\; |x|\,|y| \int_0^{1/2} \frac{e^{-\frac{|y|^2}{4s}}}{s^{\sigma+\frac{d}{2}}}\,ds \\ (u=|y|^2/(4s)) &= \frac{c\,|x|\,|y|}{|y|^{d+2\sigma-2}} \int_{|y|^2/2}^{\infty} e^{-u}\,u^{\sigma+\frac{d}{2}-1}\,\frac{du}{u} \;\le\; \frac{c'\,|x|}{|y|^{d+2\sigma-3}}, \end{aligned}$$

the last bound being valid when $\sigma+d/2-1>0$. This is always the case when $d\ge 2$ and $\sigma>0$, and also if $d=1$ and $\sigma>1/2$.

In the special case that $d=1$ and $\sigma\in(0,1/2]$, one observes that the integral

$$I(y) := \int_{|y|^2/2}^{\infty} e^{-u}\,u^{\sigma-\frac12}\,\frac{du}{u} \approx \begin{cases} \log(e/|y|), & \text{if } \sigma=1/2 \\ |y|^{2\sigma-1}, & \text{if } \sigma\in(0,1/2), \end{cases}$$

when $|y|\le 1$. Inserting this into the above estimates, it leads to

$$\mathcal{K}'(x,y) \lesssim \begin{cases} |x|\,|y|\,\log(e/|y|), & \text{if } \sigma=1/2 \\ |x|\,|y|, & \text{if } \sigma\in(0,1/2), \end{cases}$$

which implies

$$\mathcal{K}'(x,y)\lesssim |x|, \quad \forall\,|y|\le 1.$$

Note that this matches (3.5) in the special case $d=1$ and $\sigma\le 1/2$. □

Our last result is a strengthening of the decay estimate in Lemma 3.1 when $|y|\gg|x|$. The proof follows a similar reasoning as in [8, Lemma 4.2].

Lemma 3.5. *Let $\sigma>0$. Then there exist $c=c(\sigma)>0$ and $\gamma>0$ such that*

$$\mathcal{K}(x,y)\le c\,e^{-(\frac12+\gamma)|y|^2}, \quad \text{when } |y|\ge 10\max\{|x|,1\}.$$

Proof. For simplicity denote $a=|x+y|$ and $b=|x-y|$. Note that, the condition $|y|\ge 10|x|$ implies

$$a^2,b^2\ge \left(\tfrac{9}{10}\right)^2|y|^2.$$

Given a small $\eta\in(0,1)$ (to be determined) we have, for all $s\in(0,1/2)$

$$\begin{aligned} e^{-\frac14(sa^2+\frac{b^2}{s})} &= e^{-\frac{\eta b^2}{4s}}\,e^{-\frac14(sa^2+(1-\eta)\frac{b^2}{s})} \\ &\le e^{-\frac{\eta b^2}{4s}}\,e^{-\frac14\left(\frac{9}{10}\right)^2|y|^2\left(s+(1-\eta)\frac1s\right)} \\ &\le e^{-\frac{\eta b^2}{4s}}\,e^{-\frac14\left(\frac{9}{10}\right)^2|y|^2\,(1-\eta)\,\frac52}, \end{aligned}$$

using that $s+\frac1s\ge 5/2$ when $s\in(0,1/2)$. Note that, if $\eta>0$ is chosen sufficiently small, we can find some $\gamma>0$ such that

$$\tfrac14\left(\tfrac{9}{10}\right)^2(1-\eta)\,\tfrac52 \;>\; \tfrac12+\gamma.$$

So we have

$$e^{-\frac14(sa^2+\frac{b^2}{s})}\le e^{-\frac{\eta b^2}{4s}}\,e^{-(\frac12+\gamma)\,|y|^2}, \quad s\in(0,1/2).$$

Thus, inserting these estimates into (3.4), we obtain

$$\mathcal{K}(x,y) \approx \mathcal{K}_1(x,y) \le e^{-(\frac12+\gamma)\,|y|^2} \int_0^{1/2} e^{-\frac{\eta|x-y|^2}{4s}} \frac{ds}{s^{1+\sigma+\frac{d}{2}}}.$$

Finally, in the last integral we perform the change of variables $u = \frac{\eta|x-y|^2}{4s}$ and obtain

$$\mathcal{K}(x,y) \lesssim \frac{e^{-(\frac12+\gamma)\,|y|^2}}{|x-y|^{2\sigma+d}} \int_0^\infty e^{-u}\, u^{\sigma+\frac{d}{2}}\, \frac{du}{u} \lesssim e^{-(\frac12+\gamma)\,|y|^2},$$

using in the last step that $|x-y|\approx|y|\ge 1$, under the conditions in the statement. □

3.2. **Regular positive eigenvectors.**

Definition 3.6. We say that $\psi(x)\in \mathrm{Dom}(L)$ is a *regular positive eigenvector* of L if

(a) $\psi\in C^\infty(\mathbb{R}^d)$
(b) $\psi(x)>0$, $\forall\, x\in\mathbb{R}^d$
(c) $L(\psi)=\lambda\psi$, for some $\lambda\ge 0$.

When $L=-\Delta+|x|^2+m$, it is elementary to find an explicit regular positive eigenvector, namely

$$\psi(x) = e^{-|x|^2/2}.$$

Indeed, it is easily verified that $L(\psi)=\lambda\psi$ with $\lambda = m+d\ge 0$.

We have the following simple lemma, which is valid for general operators L.

Lemma 3.7. *Let ψ be a regular positive eigenvector of L. Then, for all $\sigma\in(0,1)$ and all $x\in\mathbb{R}^d$ it holds*

$$\int_0^\infty \Big| e^{-tL}\psi(x) - \psi(x)\Big| \frac{dt}{t^{1+\sigma}} \;<\; \infty.$$

That is, $x\in\mathcal{A}_\psi(L^\sigma)$, for all $x\in\mathbb{R}^d$.

Proof. Since $e^{-tL}\psi = e^{-t\lambda}\psi$, the result is clear if $\lambda=0$. If $\lambda>0$, then we have

$$\int_0^\infty \Big| e^{-tL}\psi(x) - \psi(x)\Big| \frac{dt}{t^{1+\sigma}} = \psi(x)\int_0^\infty |e^{-t\lambda}-1|\,\frac{dt}{t^{1+\sigma}}. \tag{3.6}$$

Now, from the elementary estimate

$$|e^{-t\lambda}-1| = \Big|\int_0^t \lambda\, e^{-s\lambda}\, ds\Big| \le \min\{\lambda\, t, 2\}.$$

one deduces that (3.6) is a finite expression when $\sigma\in(0,1)$. □

Remark 3.8. In this paper we shall not pursue this notion with other operators L, but it is well-known that such eigenvectors exist when $L=-\Delta+V(x)$, under very general conditions on $V(x)$; see e.g. [11, Theorem 11.8].

3.3. **Proof of Theorem 1.1.** In this section we prove Theorem 1.1 for the Hermite operator

$$L = -\Delta + |x|^2 + m.$$

That is, if $\sigma \in (0,1)$ and Φ_σ is given as in (1.9), we must show that, for $x = x_0$,

$$\int_0^\infty \left| e^{-tL} f(x) - f(x) \right| \frac{dt}{t^{1+\sigma}} < \infty,$$

under the conditions $f \in L_1(\Phi_\sigma)$ and $f \in \mathcal{D}^{2\sigma}_{\rm st}(x)$. In that case, the assertions in the theorem will follow directly from Proposition 2.3. In view of Lemma 2.4, it suffices to show that

$$\int_0^A \left| e^{-tL} f(x) - f(x) \right| \frac{dt}{t^{1+\sigma}} < \infty, \tag{3.7}$$

where $A > 0$ can be chosen as in §3.1.

Let ψ be a regular positive eigenvector for L, as described in §3.2. Since $\psi(x) > 0$, a multiplication by this number does not affect the finiteness of (3.7). Now we have

$$\begin{aligned} J \;&:=\; \psi(x) \int_0^A \left| e^{-tL} f(x) - f(x) \right| \frac{dt}{t^{1+\sigma}} \qquad (3.8)\\ &= \int_0^A \left| \big(e^{-tL} f\big)(x)\psi(x) - f(x)\psi(x) \right| \frac{dt}{t^{1+\sigma}} \\ &\leq \int_0^A \left| \big(e^{-tL} f\big)(x)\psi(x) - \big(e^{-tL}\psi\big)(x) f(x) \right| \frac{dt}{t^{1+\sigma}} \\ &\quad + |f(x)| \int_0^A \left| \big(e^{-tL}\psi\big)(x) - \psi(x) \right| \frac{dt}{t^{1+\sigma}} \;=\; J_1 + J_2. \end{aligned}$$

Note that $J_2 < \infty$ by Lemma 3.7, so we must only prove the finiteness of J_1. For that term, we have the following inequalities

$$\begin{aligned} J_1 \;&=\; \int_0^A \Big| \int_{\mathbb{R}^d} h_t(x,y) \big[f(y)\psi(x) - f(x)\psi(y) \big] \, dy \Big| \frac{dt}{t^{1+\sigma}} \\ &\leq\; \psi(x) \int_0^A \Big| \int_{\mathbb{R}^d} h_t(x,y) \, \big[f(y) - f(x) \big] \, dy \Big| \frac{dt}{t^{1+\sigma}} \\ &\quad + |f(x)| \int_0^A \Big| \int_{\mathbb{R}^d} h_t(x,y) \, \big[\psi(x) - \psi(y) \big] \, dy \Big| \frac{dt}{t^{1+\sigma}} \;=\; J_{11} + J_{12}. \end{aligned}$$

The two summands, J_{11} and J_{12}, can be treated similarly, since both functions f and ψ belong to $L_1(\Phi_\sigma) \cap \mathcal{D}^{2\sigma}_{\rm st}(x)$, by assumption*. So in the sequel we will just prove that $J_{11} < \infty$ and this will be enough to conclude the theorem. In fact, since $\psi(x) > 0$, it will suffice to show that

$$\mathcal{J}_{11} := \int_0^A \Big| \int_{\mathbb{R}^d} h_t(x,y) \, \big[f(y) - f(x) \big] \, dy \Big| \frac{dt}{t^{1+\sigma}} < \infty.$$

*Actually, ψ is much smoother than just $\mathcal{D}^{2\sigma}_{\rm st}(x)$, so J_{12} is formally easier.

At this point we let $\delta = 11\max\{|x|,1\}$, and split the inner integral into the two regions $\{|y-x|<\delta\}$ and $\{|y-x|\geq\delta\}\subset\big\{|y|\geq 10\max\{|x|,1\}\big\}$. So recalling the notation for $\mathcal{I}_\delta f(x,t)$ in (2.12) we have

$$\mathcal{J}_{11} \leq \int_0^A \big|\mathcal{I}_\delta f(x,t)\big|\,\frac{dt}{t^{1+\sigma}} + \mathcal{J}_{11}^*, \tag{3.9}$$

where

$$\begin{aligned}\mathcal{J}_{11}^* &= \int_0^A \int_{|y|\geq 10\max\{|x|,1\}} h_t(x,y)\,\big|f(y)-f(x)\big|\,dy\,\frac{dt}{t^{1+\sigma}}\\ &= \int_{|y|\geq 10\max\{|x|,1\}} \big|f(y)-f(x)\big|\,\mathcal{K}(x,y)\,dy,\end{aligned}$$

using this time the notation for $\mathcal{K}(x,y)$ in (3.2). By Lemma 3.5, this last kernel has a gaussian decay, which leads to

$$\mathcal{J}_{11}^* \lesssim \int_{\mathbb{R}^d} \big(|f(x)|+|f(y)|\big)\,e^{-(\frac12+\gamma)|y|^2}\,dy \lesssim |f(x)| + \int_{\mathbb{R}^d}|f(y)|\,\Phi_\sigma(y)\,dy,$$

since, from the definition in (1.9), one has $e^{-(\frac12+\gamma)|y|^2}\lesssim\Phi_\sigma(y)$ (actually, for all $\sigma>0$). Thus, the assumption $f\in L_1(\Phi_\sigma)$ gives $\mathcal{J}_{11}^*<\infty$, and hence, in view of (3.9), we have reduced matters to verify that

$$\int_0^A \big|\mathcal{I}_\delta f(x,t)\big|\,\frac{dt}{t^{1+\sigma}} < \infty.$$

But this is precisely the condition $x\in\mathcal{A}_f^{\rm loc}(L^\sigma)$ in Definition 2.5. Now, in view of Proposition 2.6, this property is a consequence of the smoothness assumption $f\in\mathcal{D}^{2\sigma}_{\rm st}(x)$, since the heat kernel $h_t(x,y)$ satisfies the two hypotheses in the proposition, (2.14) and (2.15), due to Lemmas 3.1 and 3.3. This completes the proof of Theorem 1.1. □

Remark 3.9. When $x_0=0$, Theorem 1.1 holds with the weaker smoothness condition $f\in\mathcal{D}^{2\sigma}(x_0)$. This is because of the observation in Remark 2.8, since the heat kernel for the Hermite operator, $h_t(0,y)$, only depends on $|y|$; see (3.1).

4. The Ornstein-Uhlenbeck operator $\mathcal{O} = -\Delta + 2x\cdot\nabla$

4.1. **Proof of Theorem 1.2.** We now turn to the proof of Theorem 1.2 for the operator

$$\mathcal{O} = -\Delta + 2x\cdot\nabla,$$

which is positive and self-adjoint in $L_2(e^{-|y|^2}dy)$. In this case, there is a well-known transference principle, see e.g. [1, Prop 3.3], that reduces matters to the Hermite operator with $m=-d$, that is

$$L = -\Delta + |x|^2 - d, \quad \text{in } L_2(\mathbb{R}^d). \tag{4.1}$$

Indeed, if we set $\tilde f(x) = e^{-\frac{|x|^2}{2}}f(x)$ then it is easily seen that $\mathcal{O}f(x) = e^{\frac{|x|^2}{2}}[L\tilde f](x)$. Thus,

$$e^{-t\mathcal{O}}f(x) = e^{\frac{|x|^2}{2}}\,e^{-tL}\tilde f(x) \quad\text{and}\quad P_t^{\sigma,\mathcal{O}}f(x) = e^{\frac{|x|^2}{2}}\,P_t^{\sigma,L}\tilde f(x),$$

so that the convergence properties of $\mathcal{O}$ and L in (4.1) are linked by the mapping $f \mapsto \tilde{f}$. Indeed, just observe that, if

$$\Phi_\sigma^{\mathcal{O}}(y) = \frac{e^{-|y|^2}}{[\ln(e+|y|)]^\sigma},$$

then

(i) $f \in L_1(\Phi_\sigma^{\mathcal{O}})$ iff $\tilde{f} \in L_1(\Phi_\sigma^L)$
(ii) $\mathcal{O}^\sigma(f)(x) = e^{|x|^2/2}\, L^\sigma(\tilde{f})(x)$, as defined in (2.9)
(iii) $\lim_{t\to 0^+} t^{1-2\sigma}\,\partial_t\big[P_t^{\sigma,\mathcal{O}} f(x)\big] = e^{\frac{|x|^2}{2}} \lim_{t\to 0^+} t^{1-2\sigma}\,\partial_t\big[P_t^{\sigma,L}\tilde{f}(x)\big]$.

Since we also have $f \in \mathcal{D}_{\rm st}^{2\sigma}(x)$ iff $\tilde{f} \in \mathcal{D}_{\rm st}^{2\sigma}(x)$, then Theorem 1.2 is an immediate consequence of Theorem 1.1. □

5. Results for other operators L

One can ask whether Theorem 1.1 continues to hold for other positive self-adjoint operators L. If one aims at *optimal* integrability conditions on f, the first step would be to find a suitable function $\Phi_\sigma^L(y)$ such that

$$c_1(t,x)\,\Phi_\sigma^L(y) \;\le\; p_t^{\sigma,L}(x,y) \;\le\; c_2(t,x)\,\Phi_\sigma^L(y)\;, \quad \forall\, y \in \mathbb{R}^d, \tag{5.1}$$

as stated in Lemma 2.1. Such optimal estimates, for certain families of operators L, have already been investigated by the authors (and their collaborators) in earlier papers. For instance, besides the already mentioned reference [8] for Hermite type operators, we have also considered a large class of *Laguerre type operators* L in [7], while the Bessel operators (in the case $\sigma = 1/2$) were treated by I. Cardoso in [3].

In this paper we have tried to state our results in sufficient generality, so that one part of the arguments can be applied to general operators L (such as in §2), while the other parts concern with specific estimates of the kernels $h_t(x,y)$ associated with L (such as in §3.1). We remark that, following this line of reasoning, one can derive versions of Theorem 1.1 when L is any the aforementioned Laguerre or Bessel operators; we expect to take up these matters in [4].

In this section we illustrate this fact in just one specific but particularly simple case. For a fixed† parameter $R > 0$, we consider the perturbed Laplacian

$$\mathbb{L} = -\Delta + R.$$

In this case, $e^{-t\mathbb{L}}$ has a well-known convolution kernel

$$h_t(x,y) = e^{-tR}\,W_t(x-y), \quad \text{where } W_t(x) = (4\pi t)^{-d/2}\,e^{-\frac{|x|^2}{4t}}. \tag{5.2}$$

It was also proved in [8, §5.2] that (5.1) does hold with

$$\Phi_\sigma^{\mathbb{L}}(y) := \frac{e^{-\sqrt{R(1+|y|^2)}}}{(1+|y|)^{\frac{d+1}{2}+\sigma}}. \tag{5.3}$$

†In the sequel we shall not track the dependence of the constants on $R > 0$.

We now define a similar kernel as in (3.2) (this time letting $A=\infty$)

$$\mathcal{K}(x,y) := \int_0^\infty h_t(x,y)\,\frac{dt}{t^{1+\sigma}}. \tag{5.4}$$

Lemma 5.1. *With the notation in* (5.2), (5.3) *and* (5.4), *if* $x\in\mathbb{R}^d$, *there exists* $c(x)>0$ *such that*

$$\mathcal{K}(x,y) \le c(x)\,\Phi^{\mathbb{L}}_\sigma(y), \quad \textit{for all } |y|\ge 2\max\{|x|,1\}.$$

Proof. Note from (5.2) and (5.4) that

$$\mathcal{K}(x,y) = (4\pi)^{-d/2}\int_0^\infty e^{-tR}e^{-\frac{|x-y|^2}{4t}}\,\frac{dt}{t^{1+\sigma+\frac{d}{2}}}. \tag{5.5}$$

For $\nu>0$, consider the special function

$$F_\nu(z) := \int_0^\infty e^{-s}e^{-\frac{z^2}{4s}}\,s^{\nu-1}\,ds \lesssim (1+z)^{\nu-\frac12}\,e^{-z}, \quad z>0,$$

where the inequality follows from the asymptotics of the integral; see e.g. [10, p. 136]. If we change variables $s=|x-y|^2/(4t)$ in (5.5) we obtain that

$$\mathcal{K}(x,y) = c\,\frac{F_{\sigma+d/2}\big(\sqrt{R}\,|x-y|\big)}{|x-y|^{2\sigma+d}} \lesssim \frac{\big(1+\sqrt{R}\,|x-y|\big)^{\sigma+\frac{d-1}{2}}}{|x-y|^{2\sigma+d}}\,e^{-\sqrt{R}\,|x-y|}.$$

If we now assume that $|y|\ge 2\max\{|x|,1\}$, the right side is easily seen to be controlled by $c(x)\,\Phi^{\mathbb{L}}_\sigma(y)$; see e.g. [8, (5.6)] and subsequent lines for a detailed argument. □

We can now state the corresponding theorem for the operator $\mathbb{L}=-\Delta+R$.

Theorem 5.2. *Let* $\mathbb{L}=-\Delta+R$ *with* $R>0$ *fixed. Let* $\sigma\in(0,1)$ *and* $\Phi^{\mathbb{L}}_\sigma(y)$ *be as in* (5.3). *Suppose that*

$$f\in L_1(\Phi^{\mathbb{L}}_\sigma) \quad \textit{and} \quad f\in\mathcal{D}^{2\sigma}(x_0) \quad \textit{at some } x_0\in\mathbb{R}^d.$$

Then $(-\Delta+R)^\sigma(x_0)$ *is well defined in the limiting sense of* (2.10), *and as the absolutely convergent integral in* (2.9), *and both definitions agree.*

Proof. By Proposition 2.3 we must show that $x=x_0\in\mathcal{A}_f(\mathbb{L}^\sigma)$. Observe that $\psi(y)\equiv1$ is a regular positive eigenvector for $\mathbb{L}$, according to Definition 3.6. So, by Lemma 3.7 and the inequalities following (3.8) (applied with $A=\infty$), if suffices to show that

$$\begin{aligned} J_1 &= \int_0^\infty \Big|\big(e^{-t\mathbb{L}}f\big)(x) - f(x)\big(e^{-t\mathbb{L}}\psi\big)(x)\Big|\,\frac{dt}{t^{1+\sigma}} \\ &= \int_0^\infty \Big|\int_{\mathbb{R}^d} h_t(x,y)\big[f(y)-f(x)\big]\,dy\Big|\,\frac{dt}{t^{1+\sigma}} < \infty. \end{aligned}$$

We let $\delta=3\max\{|x|,1\}$, and as before, split the inner integral into the regions $\{|y-x|<\delta\}$ and $\{|y-x|\ge\delta\}$. So, using the notation for $\mathcal{I}_\delta f(x,t)$ in (2.12) we have

$$J_1 \le \int_0^\infty \big|\mathcal{I}_\delta f(x,t)\big|\,\frac{dt}{t^{1+\sigma}} + J_1^*$$

where

$$J_1^* = \int_0^\infty \int_{|y-x|\geq\delta} h_t(x,y)\big|f(y)-f(x)\big|\,dy\,\tfrac{dt}{t^{1+\sigma}}$$
$$\leq \int_{|y|\geq\max\{|x|,1\}} \big|f(y)-f(x)\big|\,\mathcal{K}(x,y)\,dy,$$

with $\mathcal{K}(x,y)$ as in (5.4). By Lemma 5.1,

$$J_1^* \leq c(x)\int_{\mathbb{R}^d}\big(|f(x)|+|f(y)|\big)\,\Phi_\sigma^{\mathbb{L}}(y)\,dy,$$

which is a finite expression. So we have reduced matters to show that

$$\int_0^\infty \big|\mathcal{I}_\delta f(x,t)\big|\,\frac{dt}{t^{1+\sigma}} < \infty.$$

But under the smoothness assumption that $f\in\mathcal{D}^{2\sigma}(x)$, this is a consequence of Proposition 2.6 (setting $A=\infty$), since the kernel $h_t(x,y)$ trivially satisfies (2.14) (by Remark 3.2) and (2.15) (whose left hand side is identically 0; see Remark 2.8). Finally observe, also by Remark 2.8, that only the weaker smoothness condition $f\in\mathcal{D}^{2\sigma}(x)$ is used, due to the convolution structure of the kernel $h_t(x,y)$. □

6. Smoothness conditions

In this section we give some examples to illustrate the smoothness conditions from §1. Recall that, for $\alpha\in(0,2)$, a locally integrable function $f\in\mathcal{D}^\alpha(x_0)$ if

$$\int_{|h|\leq\delta}\frac{|f(x_0+h)+f(x_0-h)-2f(x_0)|}{|h|^{d+\alpha}}\,dh < \infty,$$

for some $\delta>0$ (hence for all $\delta>0$). Also, $f\in\mathcal{D}^\alpha_{\rm st}(x_0)$ if

$$f\in\mathcal{D}^\alpha(x_0)\quad\text{and}\quad\int_{|h|\leq\delta}\frac{|f(x_0+h)-f(x_0-h)|}{|h|^{d+\alpha-3}}\,dh < \infty.$$

Observe that if f is bounded near x_0, this last condition is redundant, so α-smooth and strictly α-smooth agree in this case. The two classes also coincide if $d+\alpha-3\leq 0$, that is

$$\mathcal{D}^\alpha(x_0)=\mathcal{D}^\alpha_{\rm st}(x_0),\quad\text{if } d=1,\text{ or } d=2\text{ and }\alpha\in(0,1]. \tag{6.1}$$

In other cases, the classes are different, as shown by the example in (6.4) below. Finally, note that strict α-smoothness can also be characterized as follows.

Lemma 6.1. *Let $\alpha\in(0,2)$. Then $f\in\mathcal{D}^\alpha_{\rm st}(x_0)$ if and only if*

$$f\in\mathcal{D}^\alpha(x_0)\quad\textit{and}\quad\int_{|h|\leq\delta}\frac{|f(x_0+h)-f(x_0)|}{|h|^{d+\alpha-3}}\,dh < \infty. \tag{6.2}$$

Proof. The implication "⇐" is obvious since

$$|f(x_0+h)-f(x_0-h)|\leq|f(x_0+h)-f(x_0)|+|f(x_0-h)-f(x_0)|.$$

For the converse implication "⇒" note that

$$2\big(f(x_0+h)-f(x_0)\big)=\Big[f(x_0+h)-f(x_0-h)\Big]+\Big[f(x_0+h)+f(x_0-h)-2f(x_0)\Big].$$

□

We next collect a few further elementary observations.

(1) If f is odd about x_0 and $f(x_0) = 0$, then $f \in \mathcal{D}^\alpha(x_0)$, for all $\alpha \in (0,2)$. For instance, if $\gamma \in [0,d)$ then

$$f(x) = \operatorname{sign}(x \cdot \mathbf{e}_1)/|x|^\gamma \quad \text{if } x \neq 0, \quad f(0) = 0, \tag{6.3}$$

belongs to $\mathcal{D}^\alpha(0)$ for all $0 < \alpha < 2$, even though it is discontinuous there. However, by (6.2), $f \in \mathcal{D}^\alpha_{\rm st}(0)$ only if additionally $\alpha \in (0, 3-\gamma)$. In particular, the function

$$g(x) = \frac{\operatorname{sign}((x-x_0)\cdot \mathbf{e}_1)}{|x-x_0|^{3-\alpha}} \quad \text{if } x \neq x_0, \quad g(x_0) = 0, \tag{6.4}$$

which is locally integrable if $d + \alpha - 3 > 0$, shows that $\mathcal{D}^\alpha_{\rm st}(x_0) \subsetneq \mathcal{D}^\alpha(x_0)$ in the complementary range of (6.1).

(2) There exists a function $f \in \mathcal{D}^\alpha_{\rm st}(x_0)$, for all $\alpha \in (0,2)$, but which is discontinuous and unbounded at x_0. Indeed, if $d \geq 2$ (and $x_0 = 0$), the function f defined in (6.3) with parameter $\gamma = 1$ has this property. If $d = 1$, one may take any (locally integrable) odd function unbounded near x_0.

(3) If $f \in \mathrm{Lip}_\beta(x_0)$ for some $\beta \in (0,2]$ then $f \in \mathcal{D}^\alpha_{\rm st}(x_0)$ for all $\alpha < \beta$. Here, $f \in \mathrm{Lip}_\beta(x_0)$, if $\beta \in (0,1]$, means that

$$|f(x_0+h) - f(x_0)| \leq c\,|h|^\beta, \quad \forall\, |h| \leq \delta,$$

for some $c, \delta > 0$. If $\beta \in (1,2]$, it means that f is differentiable at x_0 and

$$\big|f(x_0+h) - f(x_0) - \nabla f(x_0)\cdot h\big| \leq c\,|h|^\beta, \quad \forall\, |h| \leq \delta.$$

Indeed, in either case it is clear that $f \in \mathrm{Lip}_\beta(x_0)$ implies

$$|\triangle^2_h f(x_0)| = \big|f(x_0+h) - 2f(x_0) + f(x_0-h)\big| \;\leq\; 2c\,|h|^\beta,$$

and

$$|\triangle^1_h f(x_0)| = \big|f(x_0+h) - f(x_0-h)\big| \;\leq\; c'\,|h|^{\min\{\beta,1\}},$$

which in turn implies $f \in \mathcal{D}^\alpha_{\rm st}(x_0)$, for all $\alpha < \beta$.

(4) The following examples relate $\mathrm{Lip}_\beta(x_0)$ and $\mathcal{D}^\alpha(x_0)$ when $\beta = \alpha$:

$$f(x) = |x-x_0|^\alpha \in \mathrm{Lip}_\alpha(x_0) \setminus \mathcal{D}^\alpha(x_0), \quad \forall\, \alpha \in (0,2)$$
$$g(x) = \operatorname{sign}\big((x-x_0)\cdot \mathbf{e}_1\big)\,|x-x_0|^\alpha \in \mathrm{Lip}_\alpha(x_0) \cap \mathcal{D}^\alpha_{\rm st}(x_0), \quad \forall\, \alpha \in (0,2).$$

(5) We mention an example relating the above smoothness conditions at a point x_0 and the existence of $(-\Delta)^{\frac{\alpha}{2}} f(x_0)$, as defined in (1.3). Consider the two functions f, g, defined as in point (4) above but additionally multiplied by a smooth cut-off $\varphi(|x-x_0|)$, where $\varphi \in C^\infty_c(\mathbb{R})$ with $\varphi \equiv 1$ in $[-1,1]$. Then, it is easily seen that

$$(-\Delta)^{\frac{\alpha}{2}} f(x_0) = -\infty \quad \text{but} \quad (-\Delta)^{\frac{\alpha}{2}} g(x_0) = 0.$$

So in general, $f \in \mathrm{Lip}_\alpha(x_0)$ is not enough to define pointwise fractional powers, $L^{\frac{\alpha}{2}} f(x_0)$, justifying the search for a different condition such as $f \in \mathcal{D}^\alpha(x_0)$ or $f \in \mathcal{D}^\alpha_{\rm st}(x_0)$.

(6) When L is the Hermite operator in (1.7), one can show that the function $g(x)$ defined in (6.4), when additionally multiplied by a smooth cut-off $\varphi(|x-x_0|)$ supported near the point $x_0 = \mathbf{e}_1$, has the property

$$L^{\frac{\alpha}{2}} g(x_0) = \infty.$$

Thus, when $x_0 \neq 0$, in Theorem 1.1 one cannot replace the assumption $f \in \mathcal{D}^{2\sigma}_{\rm st}(x_0)$ by the weaker condition $f \in \mathcal{D}^{2\sigma}(x_0)$ (except of course when the two classes coincide; see (6.1)). This example also shows that there are compactly supported g such that

$$(-\Delta)^{\frac{\alpha}{2}} g(x_0) = 0 \quad \text{but} \quad L^{\frac{\alpha}{2}} g(x_0) = \infty$$

(the latter in the sense of Definition 2.2).

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