LEBESGUE POINTS OF MEASURES AND NON-TANGENTIAL CONVERGENCE OF POISSON-HERMITE INTEGRALS

G. FLORES, G. GARRIGÓS, B. VIVIANI

Abstract. We study differentiability conditions on a complex measure ν at a point $x_0 \in \mathbb{R}^d$, in relation with the boundary convergence at that point of the Poisson-type integral $\mathbb{P}_t\nu = e^{-t\sqrt{L}}\nu$, where $L = -\Delta + |x|^2$ is the Hermite operator. In particular, we show that x_0 is a Lebesgue point for ν iff a slightly stronger notion than non-tangential convergence holds for $\mathbb{P}_t\nu$ at x_0. We also show non-tangential convergence when x_0 is a σ-point of ν, a weaker notion than Lebesgue point, which for $d = 1$ coincides with the classical Fatou condition.

1. Introduction

Let ν be a locally finite complex measure in $\mathbb{R}^d$, meaning that $\nu = \nu_1 + i\nu_2$ and ν_1, ν_2 are signed measures which are finite in compact sets.

We shall use the notion of Lebesgue point for ν as defined by Saeki in [11]. Namely, we say that $x_0 \in \mathbb{R}^d$ is a Lebesgue point of ν, denoted $x_0 \in \mathcal{L}_\nu$, if there exists $\ell \in \mathbb{C}$ such that

$$\lim_{r\to 0} \frac{|\nu - \ell\, dy|\big(B_r(x_0)\big)}{|B_r(x_0)|} = 0. \tag{1.1}$$

Here $dy(E) = |E|$ denotes the standard Lebesgue measure of a set $E \subset \mathbb{R}^d$ and $B_r(x_0) = \{x \in \mathbb{R}^d \ : \ |x - x_0| < r\}$. If we write ν in terms of the Lebesgue-Radon-Nikodym decomposition, that is

$$\nu = f\, dy + \lambda, \quad \text{with } \lambda \perp dy \tag{1.2}$$

(see e.g. [5, Thm 3.12]), then using the property that $|\mu + \lambda| = |\mu| + |\lambda|$ when $\mu \perp \lambda$, we see that $x_0 \in \mathcal{L}_\nu$ if and only if for some $\ell \in \mathbb{C}$ it holds

$$\lim_{r\to 0} \fint_{B_r(x_0)} |f(y) - \ell|\, dy = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{|\lambda|\big(B_r(x_0)\big)}{|B_r(x_0)|} = 0. \tag{1.3}$$

Date: April 16, 2024.

2010 *Mathematics Subject Classification.* 42C10, 35C15, 33C45, 40A10, 31B25, 28A15.

Key words and phrases. Hermite operator, Poisson integral, Fatou theorem, Lebesgue point.

G.G. was supported in part by grants PID2022-142202NB-I00 from Agencia Estatal de Investigación (Spain), and grant 21955-PI22 from Fundación Séneca (Región de Murcia, Spain).

F.G. was partially supported by grants from CONICET and SECYT-UNC. .

In particular, if $\nu = f dx$, this is the usual notion of Lebesgue point for the function f, while for a general ν as in (1.2) we have

$$\mathcal{L}_\nu = \mathcal{L}_f \cap \mathcal{L}_\lambda.$$

Observe also that, if $x_0 \in \mathcal{L}_\nu$, we can express the value of ℓ in (1.1) without appealing to the Lebesgue decomposition, by letting

$$\ell = D\nu(x_0) := \lim_{r\to 0} \frac{\nu\big(B_r(x_0)\big)}{|B_r(x_0)|}. \tag{1.4}$$

Indeed, this identity follows from the elementary bound

$$\left|\frac{\nu\big(B_r(x_0)\big)}{|B_r|} - \ell\right| \;=\; \left|\frac{(\nu - \ell\, dy)\big(B_r(x_0)\big)}{|B_r|}\right| \le \frac{|\nu - \ell\, dy|\big(B_r(x_0)\big)}{|B_r|} \longrightarrow 0.$$

The quantity $D\nu(x_0)$ in (1.4) is sometimes called the *symmetric derivative* of the measure ν at x_0; see e.g. [10, Chapter 7].

Consider now the Hermite operator

$$L = -\Delta + |x|^2, \quad \text{in } \mathbb{R}^d,$$

and its associated Poisson semigroup

$$\mathbb{P}_t = e^{-t\sqrt{L}} = \tfrac{t}{\sqrt{4\pi}} \int_0^\infty e^{-\frac{t^2}{4\tau}}\, e^{-\tau L}\, \frac{d\tau}{\tau^{3/2}};$$

see e.g. [14, Ch 2.2]. This semigroup (and its close relative involving the Ornstein-Uhlenbeck operator $-\Delta + 2x\cdot\nabla$) have been widely studied in Harmonic Analysis; see e.g. [9, 16, 15, 8, 17]. We wish to find mild differentiability conditions on a measure ν at an individual point $x_0 \in \mathbb{R}^d$ so that the Poisson integrals

$$\mathbb{P}_t\nu(x) = \int_{\mathbb{R}^d} \mathbb{P}_t(x,y)\, d\nu(y), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \tag{1.5}$$

have a non-tangential limit when $(t,x) \to (0,x_0)$.

When L is the Hermite operator, the kernel $\mathbb{P}_t(x,y)$ is partially explicit (see (2.1) below for a precise expression), and its growth (for fixed t and x) is well determined by the function

$$\Phi(y) := \frac{e^{-|y|^2/2}}{(1+|y|)^{\frac{d}{2}}\,[\log(e+|y|)]^{\frac{3}{2}}}, \quad y \in \mathbb{R}^d. \tag{1.6}$$

Namely, it was shown in [7, Lemma 4.1] that for each $t > 0$ and $x \in \mathbb{R}^d$ there exist $c_j(t,x) > 0$, $j = 1,2$, such that

$$c_1(t,x)\,\Phi(y) \le \mathbb{P}_t(x,y) \le c_2(t,x)\,\Phi(y), \quad y \in \mathbb{R}^d. \tag{1.7}$$

In particular, if $\mathcal{M}(\Phi)$ denotes the set of (locally finite) complex measures ν in $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \Phi(y)\, d|\nu|(y) < \infty,$$

then it follows from (1.7) that the function $\mathbb{P}_t\nu(x)$ in (1.5) is well-defined for all $t>0$ and $x\in\mathbb{R}^d$. Moreover, $u(t,x)=\mathbb{P}_t\nu(x)$ is smooth in $\mathbb{R}^{d+1}_+=(0,\infty)\times\mathbb{R}^d$ and satisfies the PDE

$$u_{tt}=-\Delta_x u+|x|^2u,$$

see e.g. [4, Theorem 1.3].

Our first result in this paper investigates the relation between the Lebesgue point condition, $x_0\in\mathcal{L}_\nu$, and the existence of the non-tangential limit

$$\lim_{|x-x_0|<\alpha t\to 0}\mathbb{P}_t\nu(x),$$

for every $\alpha>0$. We shall prove the following

Theorem 1.1. *Let $\nu\in\mathcal{M}(\Phi)$ and $x_0\in\mathbb{R}^d$. Then, the following assertions are equivalent*

(i) $x_0\in\mathcal{L}_\nu$;

(ii) $\lim_{t\to 0}\mathbb{P}_t\big(|\nu-\ell\,dy|\big)(x_0)=0$, *for some* $\ell\in\mathbb{C}$;

(iii) $\lim_{|x-x_0|<\alpha t\to 0}\mathbb{P}_t\big(|\nu-\ell\,dy|\big)(x)=0$, *for some* $\ell\in\mathbb{C}$ *and some (all)* $\alpha>0$.

Morever, if these assertions hold we can take $\ell=D\nu(x_0)$, and for every $\alpha>0$ it also holds

$$\lim_{|x-x_0|<\alpha t\to 0}\mathbb{P}_t\nu(x)=D\nu(x_0). \tag{1.8}$$

As a second result, we give a weaker condition than Lebesgue point which still ensures non-tangential convergence. This is the notion of σ-point introduced by V. Shapiro in [13]; see also [12]. We say that $x_0\in\mathbb{R}^d$ is a σ-point of a (locally finite) complex measure ν, denoted $x_0\in\mathfrak{S}_\nu$, if there exists $\ell\in\mathbb{C}$ such that

$$\lim_{|x-x_0|+r\to 0}\frac{\big|(\nu-\ell\,dy)(B_r(x))\big|}{(|x-x_0|+r)^d}=0. \tag{1.9}$$

Note that we can take $\ell=D\nu(x_0)$ in (1.9), by just restricting the above limit to $x=x_0$. Also, since $B_r(x)\subset B_{r+|x-x_0|}(x_0)$, we have

$$\mathcal{L}_\nu\subset\mathfrak{S}_\nu,$$

and the inclusion can be strict in view of the examples in [13, §3]. Finally, when $d=1$ there is a simple characterization: write ν as a Lebesgue-Stieltjes measure $\nu=dm_F$, with

$$\nu\big((a,b]\big)=F(b)-F(a);$$

see e.g. [5, Thm 1.16]. Then $x_0\in\mathfrak{S}_\nu$ iff F is differentiable at x_0, in which case

$$F'(x_0)=\ell=D\nu(x_0).$$

The proof is elementary; see also [12, Proposition 3.4].

Our second result has the following statement.

Theorem 1.2. *Let $\nu \in \mathcal{M}(\Phi)$ and let $x_0 \in \mathbb{R}^d$ be a σ-point of ν with value ℓ.*

(i) If $d \in \{1, 2, 3\}$ then

$$\lim_{|x-x_0|<\alpha t \to 0} \mathbb{P}_t \nu(x) = \ell, \quad \forall\, \alpha > 0. \tag{1.10}$$

(ii) If $d \geq 4$ then (1.10) *holds if it is additionally assumed that*

$$\lim_{r\to 0} \frac{|\nu - \ell\, dy|(B_r(x_0))}{r^{d-3}} = 0. \tag{1.11}$$

As a corollary, when $d = 1$ we obtain the non-tangential convergence under the classical Fatou condition, see [1] or [18, Thm III.7.9.ii].

Corollary 1.3. *Let $d = 1$ and $\nu = m_F$ a Lebesgue-Stieljes measure in $\mathcal{M}(\Phi)$. If F is differentiable in x_0 then*

$$\lim_{|x-x_0|<\alpha t \to 0} \mathbb{P}_t \nu(x) = F'(x_0), \quad \forall\, \alpha > 0.$$

We comment on previous results on these matters for the Poisson-Hermite semigroup. When the initial datum is a function f, generic results on a.e. convergence of $\mathbb{P}_t f$ go back to [9], see also [15, 7]. These references contain the main kernel estimates, but do not consider convergence at individual Lebesgue points of f. A statement for these it seems first given in [6] (for the related Laguerre operator), and requires an argument that we elaborate further here in the new Lemma 2.4. Concerning measures ν as initial data, we are only aware of the a.e. results for $\mathbb{P}_t \nu$ in [4, Theorem 1.3], which again do not consider the behavior at individual points.

Regarding Theorem 1.2, no results concerning weaker notions than Lebesgue points seem to appear in the literature for Poisson-Hermite integrals, even when ν equals f or $d = 1$. In fact, such weaker notions seem to have only been considered before associated with convolution kernels with a dilation structure, that is,

$$K_t(x, y) = t^{-d} \phi((x - y)/t), \quad t > 0,$$

and most often with a radially decreasing ϕ; see [11, 13, 12] and references therein. So one novelty here is to consider kernels $\mathbb{P}_t(x, y)$ without this convolution structure, and perform fine estimates on them in order to carry out the proofs. Concerning the additional condition (1.11) for dimensions $d \geq 4$, we remark that it is quite weak (certainly weaker than the Lebesgue point condition). We do not know whether Theorem 1.2 may hold without this hypothesis, although it seems unlikely in view of examples in [3], where a similar condition in relation with normal convergence of $\mathbb{P}_t \nu(x_0)$ appears for dimensions $d \geq 4$, and cannot be removed in that setting.

2. Proof of Theorem 1.1

It is clear that (iii) implies (ii). We shall show in separate subsections the other implications. Below we use the notation

$$p_t(z) = \frac{t}{(t^2 + |z|^2)^{\frac{d+1}{2}}},$$

which except for a multiplicative constant is the standard Poisson kernel. Recall also that the Poisson kernel $\mathbb{P}_t(x,y)$ associated with the Hermite operator can be explicitly given by

$$\mathbb{P}_t(x,y) = c_d\, t \int_0^1 \frac{e^{-\frac{t^2}{\ell(s)}}\,(1-s^2)^{\frac{d}{2}-1}\, e^{-\frac{1}{4}(\frac{|x-y|^2}{s}+s|x+y|^2)}}{s^{\frac{d}{2}}\,\big(\ell(s)\big)^{3/2}}\, ds, \tag{2.1}$$

where $\ell(s) = 2\ln\frac{1+s}{1-s}$ and $c_d > 0$; see e.g. [7, (4.1)]. Note that $\ell(s) \approx s$ when $s \in (0, 1/2)$, a fact that we shall use often below.

2.1. **Proof of** $(ii) \implies (i)$**.** We quote the following estimate from below for the kernel $\mathbb{P}_t(x,y)$.

Lemma 2.1. *Given $R \geq 1$, there exists a constant $\delta_R > 0$ such that*

$$\mathbb{P}_t(x,y) \geq \delta_R\, p_t(x-y), \quad \text{when } |x|, |y|, t \leq R. \tag{2.2}$$

Proof. If we restrict the range of integration in (2.1) to $s \in (0, 1/2)$, so that $\ell(s) \approx s$, after disregarding inessential terms we have

$$\begin{aligned}
\mathbb{P}_t(x,y) &\gtrsim t \int_0^{1/2} \frac{e^{-\frac{c\,t^2}{4s}}\, e^{-\frac{1}{4}(\frac{|x-y|^2}{s}+s|x+y|^2)}}{s^{\frac{d+3}{2}}}\, ds \\
&\geq e^{-\frac{R^2}{2}}\, t \int_0^{1/4} \frac{e^{-\frac{c\,t^2+|x-y|^2}{4s}}}{s^{\frac{d+1}{2}}}\, \frac{ds}{s} \\
&= \frac{c_R\, t}{(ct^2+|x-y|^2)^{\frac{d+1}{2}}} \int_{ct^2+|x-y|^2}^{\infty} e^{-u} u^{\frac{d+1}{2}}\, \frac{du}{u},
\end{aligned}$$

performing in the last step the change of variables $u = (c\,t^2 + |x-y|^2)/(4s)$. Using again the assumption $|x|, |y|, t \leq R$, the last integral is bounded below by

$$\int_{(4+c)R^2}^{\infty} e^{-u} u^{\frac{d+1}{2}}\, \frac{du}{u} = c'_R.$$

Combining these expressions one obtains (2.2). □

Going back to the proof of Theorem 1.1, assume that (ii) holds, and denote $|\nu - \ell dx| = \mu$. Pick $R \geq |x_0| + 1$ and $t \leq 1$. Then the previous lemma gives

$$\mathbb{P}_t\big(\mu\big)(x_0) \gtrsim \delta_R \int_{|y-x_0|\leq t} p_t(x_0 - y)\, d\mu(y).$$

Since $p_t(x_0-y)\approx t^{-d}$ when $|x_0-y|\le t$, this implies

$$\mathbb{P}_t\big(\mu\big)(x_0)\;\gtrsim\;\frac{\mu\big(B_t(x_0)\big)}{|B_t(x_0)|}.$$

By (ii), the left-hand side goes to 0 as $t\to 0$, hence so does the right hand side. Since $\mu=|\nu-\ell\,dx|$, this implies that x_0 is a Lebesgue point of ν (and therefore, also that $\ell=D\nu(x_0)$).

2.2. **Proof of** *(i)* $\Longrightarrow$ *(iii)***.** We shall use the following upper bound for $\mathbb{P}_t(x,y)$, which can be found in [7, Lemma 4.2].

Lemma 2.2. *There exists a constant $\gamma\ge 2$ and a continuous function $C(x)$ such that*

$$\mathbb{P}_t(x,y)\le C(x)\Big[p_t(x-y)\mathbb{1}_{|y|\le\gamma\max\{|x|,1\}}\;+\;t\,\Phi(y)\Big],$$

for all $t>0$ and $x,y\in\mathbb{R}^d$.

We also quote the following elementary lemma.

Lemma 2.3. *Let $\alpha>0$. Then, there exists $C_\alpha>0$ such that*

$$\frac{1}{C_\alpha}\big(|x_0-y|+t\big)\le|x-y|+t\le C_\alpha\big(|x_0-y|+t\big), \tag{2.3}$$

whenever $|x-x_0|\le\alpha t$ and $y\in\mathbb{R}^d$.

Proof. The proof is elementary using the condition $|x-x_0|\le\alpha t$ and the triangle inequality. Indeed, on the one hand

$$|x-y|+t\le|x-x_0|+|x_0-y|+t\le|x_0-y|+(\alpha+1)t,$$

which implies the upper bound in (2.3) with $C_\alpha=\alpha+1$. The lower bound follows from the upper one after interchanging the roles of x and x_0. $\square$

We turn to the proof of (iii) in Theorem 1.1. Since we are interested in non-tangential limits at x_0, we fix $\alpha>0$ and consider only points (t,x) such that $|x-x_0|\le\alpha t\le 1$. For such points, Lemmas 2.2 and 2.3 imply that

$$\mathbb{P}_t(x,y)\lesssim C_{x_0}\Big[p_t(x_0-y)\mathbb{1}_{|y|\le\gamma(|x_0|+2)}\;+\;t\,\Phi(y)\Big],\quad\forall\,y\in\mathbb{R}^d,$$

for some constant $C_{x_0}>0$ (which we could take $\max_{|x|\le|x_0|+1}C(x)$). For simplicity, we write

$$K=\big\{y\in\mathbb{R}^d\;:\;|y|\le\gamma(|x_0|+2)\big\},$$

and as before we denote $\mu=|\nu-\ell dx|$. The previous inequalities then imply

$$\mathbb{P}_t\mu(x)\;\lesssim\;C_{x_0}\Big[\int_K p_t(x_0-y)\,d\mu(y)\;+\;t\int_{\mathbb{R}^d}\Phi(y)\,d\mu(y)\Big]. \tag{2.4}$$

The assumption that $\nu\in\mathcal{M}(\Phi)$ implies that the last summand is $O(t)$ as $t\to 0$, so we must only estimate the first term

$$\int_K p_t(x_0-y)\,d\mu(y).$$

Here one could use the known results about the standard Poisson kernel to conclude that

$$\lim_{t\to 0}\int_K p_t(x_0-y)\,d\mu(y)=0, \tag{2.5}$$

under the assumption $x_0\in\mathcal{L}_\nu$. We briefly sketch the argument for completeness. Given $\varepsilon>0$, we choose $\delta>0$ such that

$$\frac{\mu\big(B_r(x_0)\big)}{r^d}<\varepsilon,\quad r\in(0,2\delta]. \tag{2.6}$$

When $|y-x_0|\ge\delta$ we have $p_t(x_0-y)\le t/\delta^{d+1}$, and hence

$$\int_{y\in K\setminus B_\delta(x_0)} p_t(x_0-y)\,d\mu(y)\le\frac{t\,\mu(K)}{\delta^{d+1}}<\varepsilon, \tag{2.7}$$

if $t<t_0:=\min\{\varepsilon\,\delta^{d+1}/\mu(K),\delta\}$. On the other hand, given any such t, we can find an integer $J\in\mathbb{N}$ such that

$$\delta<2^Jt\le 2\delta.$$

So letting $S_j:=B_{2^jt}(x_0)\setminus B_{2^{j-1}t}(x_0)$ and using that $p_t(x_0-y)\lesssim t^{-d}2^{-j(d+1)}$ in S_j we have

$$\begin{aligned}\int_{B_\delta(x_0)} p_t(x_0-y)\,d\mu(y) &\lesssim \int_{B_t(x_0)} t^{-d}\,d\mu(y)+\sum_{j=1}^J\int_{S_j}(2^jt)^{-d}2^{-j}\,d\mu(y)\\ &\le \frac{\mu(B_t(x_0))}{t^d}+\sum_{j=1}^J 2^{-j}\frac{\mu(B_{2^jt}(x_0))}{(2^jt)^d}\\ &\le 2\varepsilon,\end{aligned}$$

using (2.6) in the last step. This shows (2.5).

2.3. **Last assertion in Theorem 1.1.** It remains to prove that, under the assumption (iii), it also holds

$$\lim_{|x-x_0|<\alpha t\to 0}\mathbb{P}_t\nu(x)=\ell.$$

This will be a consequence of the following lemma.

Lemma 2.4. *Let $\nu\in\mathcal{M}(\Phi)$ and $\ell\in\mathbb{C}$. Then the function*

$$E(t,x):=\big(\mathbb{P}_t\nu(x)-\ell\big)-\mathbb{P}_t(\nu-\ell\,dy)(x)$$

satisfies

$$\lim_{(t,x)\to(0,x_0)}E(t,x)=0,\quad\forall\,x_0\in\mathbb{R}^d.$$

Indeed, assuming the lemma and using (iii) we have

$$\big|\mathbb{P}_t\nu(x)-\ell\big|\le\mathbb{P}_t|\nu-\ell\,dy|(x)+|E(t,x)|\longrightarrow 0,$$

when $|x-x_0|<\alpha t\to 0$.

We now prove Lemma 2.4. We remark that this is not completely straightforward since for the Hermite semigroup we do not have $\mathbb{P}_t(\mathbb{1})=\mathbb{1}$. We shall use instead an argument, borrowed from [2, §3.2] (see also [6, §6]), which can be adapted to more general operators L. The only tool is the existence of a *regular positive eigenvector* of L, that is $\psi \in \mathrm{Dom}(L)$ such that

(a) $\psi \in C^\infty(\mathbb{R}^d)$
(b) $\psi(x) > 0$, $\forall\, x \in \mathbb{R}^d$
(c) $L(\psi) = \lambda\psi$, for some $\lambda \geq 0$.

When $L = -\Delta + |x|^2$, it is elementary to check that

$$\psi(x) = e^{-|x|^2/2} \tag{2.8}$$

satisfies these properties with $\lambda = d$.

Proof of Lemma 2.4: Let $\psi(x)$ be as in (2.8). Since $\mathbb{P}_t = e^{-t\sqrt{L}}$, we also have

$$\mathbb{P}_t\psi = e^{-t\sqrt{\lambda}}\psi.$$

Then

$$\mathbb{P}_t\nu(x) - \ell \;=\; \big(\mathbb{P}_t\nu(x) - \ell\, e^{-t\sqrt{\lambda}}\big) \;+\; \big(e^{-t\sqrt{\lambda}} - 1\big)\,\ell.$$

The last summand goes to 0 as $t \to 0$, so we look at the first summand. Since $\psi(x_0) > 0$, we may as well consider

$$A(t,x) := \psi(x_0)\,\big(\mathbb{P}_t\nu(x) - \ell\, e^{-t\sqrt{\lambda}}\big),$$

which using that $\mathbb{P}_t\psi = e^{-t\sqrt{\lambda}}\psi$ we can write as

$$\begin{aligned} A(t,x) \;&=\; \big[\psi(x_0)\,\mathbb{P}_t\nu(x) - \ell\,\mathbb{P}_t\psi(x)\big] \;+\; \ell\,\big[\mathbb{P}_t\psi(x) - \mathbb{P}_t\psi(x_0)\big] \\ &=\; A_1(t,x) + A_2(t,x). \end{aligned}$$

Now,

$$|A_2(t,x)| = |\ell|\, e^{-t\sqrt{\lambda}}|\psi(x) - \psi(x_0)| \longrightarrow 0, \quad \text{if } |x - x_0| \to 0,$$

by the continuity of ψ. Finally,

$$\begin{aligned} A_1(t,x) \;&=\; \psi(x_0)\,\mathbb{P}_t(\nu - \ell\, dy)(x) \;-\; \ell\,\mathbb{P}_t\big(\psi - \psi(x_0)\big)(x) \\ &=\; A_{1,1}(t,x) + A_{1,2}(t,x). \end{aligned}$$

Since the function $g = \psi - \psi(x_0)$ is continuous everywhere (and vanishes at x_0), using Lemma 2.2 and standard results on approximations of the identity one can show that

$$|A_{1,2}(t,x)| \leq |\ell|\,\mathbb{P}_t|g|(x) \;\lesssim\; \big(p_t * |g|\big)(x) + t\int |g|\Phi \longrightarrow 0,$$

as $|x - x_0| + t \to 0$. Thus,

$$A(t,x) = A_{1,1}(t,x) + O(1),$$

and using that $\psi(x_0) > 0$ we derive

$$\mathbb{P}_t\nu(x) - \ell \,= \mathbb{P}_t(\nu - \ell\, dy)(x) + O(1).$$

This proves Lemma 2.4, and hence it concludes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

It will suffice to consider the case when $\ell = 0$; otherwise, one would apply the reasoning to the measure $\mu = \nu - \ell\, dy$ together with Lemma 2.4. We fix $\alpha > 0$ and must show that

$$\lim_{|x-x_0|<\alpha t \to 0} \mathbb{P}_t \nu(x) = 0. \tag{3.1}$$

By the condition $x_0 \in \mathfrak{S}_\nu$ (with $\ell = 0$), for any fixed $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\big|\nu\big(B_r(x)\big)\big| \leq \varepsilon\,(|x-x_0|+r)^d, \quad \text{whenever } |x-x_0|+r<2\delta. \tag{3.2}$$

If the dimension $d \geq 4$, we also assume that δ is chosen such that

$$|\nu|\big(B_r(x_0)\big) < \varepsilon\, r^{d-3}, \quad \text{when } r < 2\delta, \tag{3.3}$$

in view of the hypothesis in (1.11).

The first part of the proof is similar to Theorem 1.1: we consider only points (t,x) such that $|x-x_0| \leq \alpha t \leq 1$; in particular $|x| \leq |x_0|+1$. By Lemma 2.2 we have

$$\mathbb{P}_t(x,y) \lesssim C_{x_0}\Big[p_t(x-y)\mathbb{1}_{|y|\leq \gamma(|x_0|+2)} + t\,\Phi(y)\Big], \quad \forall\, y \in \mathbb{R}^d,$$

for some constant $C_{x_0} > 0$. In the region $|x-y| \geq \delta$, we have $p_t(x-y) \lesssim t/\delta^{d+1}$, so the above bound gives

$$\begin{aligned}\int_{|x-y|\geq\delta} \mathbb{P}_t(x,y)\,d\,|\nu|(y) \;&\lesssim\; C'_{x_0}\left(\delta^{-d-1} + \int \Phi\,d|\nu|\right)\cdot t \\ &=\; c(x_0,\delta)\cdot t \;<\; \varepsilon, \end{aligned} \tag{3.4}$$

provided we assume $t < t_0 := \min\{\varepsilon/c(x_0,\delta), \delta\}$. So, in the remainder of the proof we shall aim to show

$$\Big|\int_{|x-y|<\delta} \mathbb{P}_t(x,y)\,d\,\nu(y)\Big| = \mathcal{O}(\varepsilon),$$

when $|x-x_0| \leq \alpha t$ and t is sufficiently small. Since $|x| \leq |x_0|+1$ we have

$$B_\delta(x) \subset \{y \in \mathbb{R}^d \;:\; |y| \leq |x_0|+2\} =: K,$$

so in the sequel we may assume ν to be supported in the compact set K.

We next remove another inessential term: we define the "local" part of the Poisson kernel by

$$\mathbb{P}^0_t(x,y) := c_d\, t \int_0^{1/2} \frac{e^{-\frac{t^2}{\ell(s)}}\,(1-s^2)^{\frac{d}{2}-1}\, e^{-\frac14\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)}}{s^{\frac{d}{2}}\,\big(\ell(s)\big)^{3/2}}\,ds. \tag{3.5}$$

Then we have

$$\mathbb{P}^1_t(x,y) := \mathbb{P}_t(x,y) - \mathbb{P}^0_t(x,y) \;\leq\; C\,t\int_{1/2}^1 (1-s^2)^{\frac{d}{2}-1}ds = C'\,t,$$

for some $C' > 0$ (depending only on d), and therefore

$$\int_K \mathbb{P}^1_t(x,y)\,d|\nu|(y) \;\leq\; C'\,|\nu|(K)\cdot t = C''\,t < \varepsilon, \tag{3.6}$$

if $t < t_1 = \min\{t_0, \varepsilon/C''\}$. So it suffices to show that

$$A(t,x) := \int_{|x-y|<\delta} \mathbb{P}^0_t(x,y)\, d\nu(y) = \mathcal{O}(\varepsilon)$$

when $|x - x_0| \le \alpha t$ and t is sufficiently small. We fix x in what follows, and changing variables $y = x + h$, we rewrite the above expression as

$$A(t,x) = \int_{|h|<\delta} \mathbb{P}^0_{t,x}(h)\, d\nu_x(h),$$

where for simplicity we denote

$$\mathbb{P}^0_{t,x}(h) := \mathbb{P}^0_t(x, x+h) \quad \text{and} \quad \nu_x(E) := \nu(x+E), \;\; E \subset \mathbb{R}^d.$$

The kernel now takes the form

$$\mathbb{P}^0_{t,x}(h) = c_d\, t \int_0^{1/2} \frac{e^{-\frac{t^2}{\ell(s)}}\,(1-s^2)^{\frac{d}{2}-1}}{s^{\frac{d}{2}}\,\big(\ell(s)\big)^{3/2}}\, e^{-\frac{|h|^2}{4s}}\, e^{-\frac{s|2x+h|^2}{4}}\, ds.$$

This is not a radial function in h, so we shall split it into a radial part and a remainder. To do so, we write

$$\begin{aligned} e^{-\frac{s|2x+h|^2}{4}} &= e^{-s|x|^2}\, e^{-\frac{s|h|^2}{4}}\, e^{-s\,x\cdot h} \\ &= e^{-s|x|^2}\, e^{-\frac{s|h|^2}{4}}\, \big(1 + R(s\,x\cdot h)\big), \end{aligned}$$

where letting $u = s\,x\cdot h$, we have

$$|R(u)| = |e^{-u} - 1| = \Big|\int_0^u e^{-v}\, dv\Big| \le |u|\, e^{|u|} \le c_{x_0}\, s\,|x|\,|h|, \tag{3.7}$$

since $s \in (0, 1/2)$, $|h| \le 1$ and $|x| \le |x_0| + 1$. We now split

$$\mathbb{P}^0_{t,x}(h) = \mathcal{K}_{t,x}(h) + \mathcal{R}_{t,x}(h), \tag{3.8}$$

with

$$\mathcal{K}_{t,x}(h) = c_d\, t \int_0^{1/2} \frac{e^{-\frac{t^2}{\ell(s)}}\,(1-s^2)^{\frac{d}{2}-1}}{s^{\frac{d}{2}}\,\big(\ell(s)\big)^{3/2}}\, e^{-\frac{|h|^2}{4}(s+\frac{1}{s})}\, e^{-s|x|^2}\, ds. \tag{3.9}$$

The kernel $\mathcal{R}_{t,x}(h)$ has a similar expression with an additional factor $R(s\,x\cdot h)$ inside the integral. Disregarding inessential terms, we can estimate it by

$$\begin{aligned} \mathcal{R}_{t,x}(h) &\lesssim t \int_0^{1/2} \frac{e^{-\frac{c\,t^2}{4s}}\, e^{-\frac{|h|^2}{4s}}\, s\,|x|\,|h|}{s^{\frac{d+3}{2}}}\, ds \qquad (3.10) \\ &= \frac{t\,|x|\,|h|}{(ct^2 + |h|^2)^{\frac{d-1}{2}}} \int_{\frac{ct^2+|h|^2}{2}}^{\infty} e^{-v}\, v^{\frac{d-1}{2}}\, \frac{dv}{v}, \end{aligned}$$

where in the last step we have changed variables $v = (ct^2 + |h|^2)/(4s)$. At this point we distinguish two cases, if $d > 1$, the above integral is bounded by a finite constant, so we have

$$\big|\mathcal{R}_{t,x}(h)\big| \lesssim \frac{t\,|x|\,|h|}{(ct^2 + |h|^2)^{\frac{d-1}{2}}}. \tag{3.11}$$

If $d = 1$, using that $t, |h| \ll 1$ we have

$$\big|\mathcal{R}_{t,x}(h)\big| \lesssim t\,|x|\,|h|\,\log\tfrac{1}{ct^2+|h|^2}. \tag{3.12}$$

Note that in both cases the involved constants depend only on x_0 (and the dimension d), but not on x or δ. Note further that

$$|x| \le |x - x_0| + |x_0| \le \alpha t + |x_0| \le 1 + |x_0|,$$

so below we shall absorb the factor $|x|$ in (3.11) and (3.12) into the constants; however, in the special case that $x_0 = 0$ this factor is $\mathcal{O}(t)$ and will lead to a slightly better result; see Remark 3.2 below.

We will now show that

$$A_1(t,x) := \int_{|h|<\delta} \mathcal{R}_{t,x}(h)\,d\,\nu_x(h) = \mathcal{O}(1), \quad \text{as } |x - x_0| < \alpha t \searrow 0. \tag{3.13}$$

The argument is different depending on the dimension.

Case $d = 1$. In this case we simply have

$$\begin{aligned} |A_1(t,x)| &\le \int_{|h|<\delta} |\mathcal{R}_{t,x}(h)|\;d|\nu_x|(h) \\ &\lesssim t\,\log(1/t) \int_{|h|<\delta} |h|\;d|\nu_x|(h) \\ &\le t\,\log(1/t)\;|\nu|(K), \end{aligned}$$

which vanishes as $t \searrow 0$.

For higher dimensions $d > 1$, the bound in (3.11) gives

$$\begin{aligned} |A_1(t,x)| &\le \int_{|h|<\delta} |\mathcal{R}_{t,x}(h)|\;d|\nu_x|(h) \\ &\lesssim t \int_{|h|<\delta} \frac{|h|}{(t+|h|)^{d-1}}\;d|\nu_x|(h) \\ &\le t \int_{|h|<\delta} \frac{d|\nu_x|(h)}{(t+|h|)^{d-2}}. \end{aligned} \tag{3.14}$$

We again distinguish cases.

Case $d = 2$. In this case, (3.14) clearly becomes $\mathcal{O}(t)$ as $t \searrow 0$ (with a bound independent of x since $|\nu|(K) < \infty$).

Case $d = 3$. We now have

$$\begin{aligned} |A_1(t,x)| &\lesssim \int_{|h|<\delta} \frac{t}{t+|h|}\;d|\nu_x|(h) = \int_{B_\delta(x)} \frac{t}{t+|y-x|}\;d|\nu|(y) \\ \text{(by Lemma 2.3)} &\lesssim \int_K \frac{t}{t+|y-x_0|}\;d|\nu|(y) \end{aligned}$$

and the right hand side vanishes as $t \searrow 0$ by the dominated convergence theorem (independently of x).

Case $d \geq 4$. In this case, rather than (3.13), we show that

$$\limsup_{|x-x_0|\leq \alpha t\to 0^+} |A_1(t,x)| \leq C\,\varepsilon.$$

To do so, we shall need the additional hypothesis in (1.11), in the form (3.3). We break the integral in (3.14) into pieces. Assuming $t \ll \delta/2$, we can find $J \in \mathbb{N}$ such that $2^J t < \delta \leq 2^{J+1}t$. We then write

$$\int_{|h|<\delta} \frac{t\, d|\nu_x|(h)}{(t+|h|)^{d-2}}\, dh \leq \int_{|h|<t} \cdots \;+\; \sum_{j=0}^{J} \int_{2^j t\leq |h|<2^{j+1}t} \cdots =: I_0 + I_1.$$

Then, using (3.3) one finds that

$$I_0 \leq \frac{1}{t^{d-3}} \int_{|h|<t} d|\nu_x| = \frac{|\nu|\big(B_t(x)\big)}{t^{d-3}} < \varepsilon,$$

and

$$I_1 \leq \sum_{j=0}^{J} \int_{|h|<2^{j+1}t} \frac{t\;d|\nu_x|}{(2^j t)^{d-2}} \;=\sum_{j=0}^{J} \frac{2^{-j}}{(2^j t)^{d-3}} |\nu|\big(B_{2^{j+1}t}(x)\big) \leq C\,\varepsilon,$$

with $C = 2^{d-2}$. Combining these estimates with (3.14) we obtain

$$\limsup_{|x-x_0|\leq \alpha t\to 0} |A_1(t,x)| \;\leq\; c_{x_0}\,\varepsilon. \tag{3.15}$$

This finishes the estimate involving the remainder part $\mathcal{R}_{t,x}(h)$ of the kernel $\mathbb{P}^0_{t,x}(h)$; see (3.8). We remark that the hypothesis (1.10) has not been used in this part.

We now turn to estimate the piece involving the radially decreasing part $\mathcal{K}_{t,x}(h)$, which makes a crucial use of this hypothesis (in the form (3.2)). We shall need a lemma from measure theory, whose proof is similar to [5, Proposition 6.23], but that we skecth for completeness. This gives a formula of polar coordinates (3.18) for a general measure μ.

Lemma 3.1. *Given a (locally finite) complex measure μ in $\mathbb{R}^d$, let*

$$\varrho(r) := \mu\big(\overline{B}_r(0)\big) = \mu\Big(\big\{|x| \leq r\big\}\Big), \quad r \geq 0, \tag{3.16}$$

and denote by m_ϱ the associated Lebesgue-Stieltjes measure in $[0,\infty)$, that is

$$m_\varrho\big((a,b]\big) = \varrho(b) - \varrho(a), \quad \textit{and} \quad m_\rho\big(\{0\}\big) = \varrho(0). \tag{3.17}$$

Then, for every radial $f(x) = f_0(|x|) \in L^1(\mathbb{R}^d; |\mu|)$, it holds

$$\int_{\mathbb{R}^d} f(x)\, d\mu(x) = \int_{[0,\infty)} f_0(r)\, dm_\varrho(r). \tag{3.18}$$

Proof. We may assume that the measure μ is positive. Consider the new measure μ_0 defined on Borel sets $A \subset [0,\infty)$ by

$$\mu_0(A) := \mu\Big(\big\{x \;:\; |x| \in A\big\}\Big) = \int_{\mathbb{R}^d} \mathbb{1}_A(|x|) d\mu(x).$$

When A is an interval $(a,b] \subset (0,\infty)$ (or $A=\{0\}$), it is clear by definition that

$$\mu_0(A) = m_\varrho(A).$$

Since these elementary sets generate the σ-algebra of all Borel sets in $[0,\infty)$, by uniqueness, the two measures μ_0 and m_ϱ will coincide over all such sets; see e.g. [5, Theorem 1.14]. Thus, for every function of the form $f(x) = \mathbb{1}_A(|x|)$ with A a Borel set, it will hold

$$\int_{\mathbb{R}^d} f(x)\, d\mu(x) = \int_{[0,\infty)} f_0(r)\, d\mu_0(r) = \int_{[0,\infty)} f_0(r)\, dm_\varrho(r).$$

By linearity, this identity extends to all simple functions, and by monotone convergence to all non-negative $f(x) = f_0(|x|)$. A further extension by linearity to complex functions in $L^1(d|\mu|)$ establishes the result. $\square$

We shall apply the previous lemma to the measure $\mu = \nu_x$, so that

$$A_2(t,x) := \int_{|h|<\delta} \mathcal{K}_{t,x}(h)\, d\,\nu_x(h) = \int_{[0,\delta)} \mathcal{K}^0_{t,x}(r)\, dm_{\varrho_x}(r),$$

with $\mathcal{K}_{t,x}(h) = \mathcal{K}^0_{t,x}(|h|)$ and

$$\varrho_x(r) := \nu_x\big(\overline{B}_r(0)\big) = \nu\big(\overline{B}_r(x)\big). \tag{3.19}$$

Notice that condition (3.2) implies

$$\big|\varrho_x(r)\big| \leq \varepsilon\,(|x-x_0|+r)^d, \quad \text{if } |x-x_0|+r<2\delta. \tag{3.20}$$

Then, integrating by parts (see e.g. [5, Theorem 3.36]) we have

$$\begin{aligned} A_2(t,x) &= \mathcal{K}^0_{t,x}(0)\varrho_x(0) + \int_{(0,\delta)} \mathcal{K}^0_{t,x}(r)\, dm_{\varrho_x}(r) \\ &= \mathcal{K}^0_{t,x}(0)\varrho_x(0) + \Big[\mathcal{K}^0_{t,x}(r)\varrho_x(r)\Big]_{r=0}^{r=\delta^-} - \int_0^\delta \varrho_x(r)\, \frac{d\mathcal{K}^0_{t,x}}{dr}(r)\, dr \\ &= \mathcal{K}^0_{t,x}(\delta)\varrho_x(\delta^-) + \int_0^\delta \varrho_x(r) \left|\frac{d\mathcal{K}^0_{t,x}}{dr}(r)\right| dr, \end{aligned} \tag{3.21}$$

using in the last line that (for each fixed t and x) the function

$$r \longmapsto \mathcal{K}^0_{t,x}(r)$$

is decreasing in $[0,\infty)$; see (3.9). Note also that

$$\mathcal{K}_{t,x}(h) \lesssim \frac{t}{(t+|h|)^{d+1}}, \tag{3.22}$$

which can be proved with a similar argument as we did in (3.10) for the kernel $\mathcal{R}_{t,x}(h)$ (removing the factor $s\,|x|\,|h|$ that appears in that kernel due to (3.7)). So, using this and (3.20), the boundary term in (3.21) satisfies

$$\mathcal{K}^0_{t,x}(\delta)\,\big|\varrho_x(\delta^-)\big| = \mathcal{K}^0_{t,x}(\delta)\,\big|\nu\big(B_\delta(x)\big)\big| \lesssim \frac{t}{\delta^{d+1}}\,(|x-x_0|+\delta)^d \cdot \varepsilon \lesssim \varepsilon,$$

since $|x - x_0| \le \alpha t < \delta$ (and also $t < \delta$). To estimate the integral in (3.21) we split it as $\int_0^t + \int_t^\delta$. In the first range we use that

$$\big|\varrho_x(r)\big| \le \, (|x - x_0| + r)^d \, \varepsilon \lesssim \varepsilon \, t^d,$$

since $|x - x_0| \le \alpha t$ and $r \le t$. So,

$$\begin{aligned} \int_0^t |\varrho_x(r)| \Big| \frac{d\mathcal{K}^0_{t,x}}{dr}(r) \Big| \, dr &\lesssim \varepsilon \, t^d \int_0^t \Big| \frac{d\mathcal{K}^0_{t,x}}{dr}(r) \Big| \, dr \\ &= \varepsilon \, t^d \Big[\mathcal{K}^0_{t,x}(0) - \mathcal{K}^0_{t,x}(t) \Big] \\ &\le \varepsilon \, t^d \, \mathcal{K}^0_{t,x}(0) \lesssim \varepsilon, \end{aligned}$$

with the last bound due to (3.22). Finally, if $r \in (t, \delta)$ we have

$$\big|\varrho_x(r)\big| \le \, (|x - x_0| + r)^d \, \varepsilon \le (\alpha t + r)^d \, \varepsilon \lesssim \varepsilon \, r^d.$$

Hence

$$\begin{aligned} \int_t^\delta |\varrho_x(r)| \Big| \frac{d\mathcal{K}^0_{t,x}}{dr}(r) \Big| \, dr &\lesssim \varepsilon \int_0^\delta \Big| \frac{d\mathcal{K}^0_{t,x}}{dr}(r) \Big| \, r^d \, dr \\ \text{(parts)} \quad &= \varepsilon \Big(- \Big[r^d \, \mathcal{K}^0_{t,x}(r) \Big]_0^\delta + d \int_0^\delta \mathcal{K}_{t,x}(r) \, r^{d-1} \, dr \Big) \\ \text{(by (3.22))} \quad &\lesssim \varepsilon \int_0^\infty \frac{t \, r^{d-1} \, dr}{(t+r)^{d+1}} = c \, \varepsilon. \end{aligned}$$

This shows that, if $|x - x_0| \le \alpha t$ and $t < t_2 = \min\{\delta, \delta/\alpha\}$ then

$$|A_2(t, x)| \le C \, \varepsilon.$$

Combining this with the previous estimates (3.4), (3.6) and (3.15), we conclude that, given $\varepsilon > 0$ there exists $\tau_0 = \tau_0(\varepsilon) > 0$ such that, if $|x - x_0| < \alpha t$ and $t \in (0, \tau_0)$ then

$$\big|\mathbb{P}_t \nu(x)\big| < \varepsilon.$$

Thus, (3.1) holds and we have completed the proof of Theorem 1.2. □

Remark 3.2. As observed above, in the special case that $x_0 = 0$, the estimate of the remainder term $\mathcal{R}_{t,x}$ can be slightly improved, due to the factor $|x| = \mathcal{O}(t)$ appearing in (3.11). Indeed, this will give a better factor t^2 in the estimate (3.14), which in turn implies that

$$A_1(t, x) = \mathcal{O}(1), \quad \text{as } t \searrow 0$$

for all dimensions $d \le 4$. Moreover, in order to have, for dimensions $d \ge 5$, the estimate

$$\limsup_{|x - x_0| < \alpha t \to 0} |A_1(t, x)| \le C \, \varepsilon,$$

one only needs the assumption

$$\lim_{r \to 0} \frac{|\nu - \ell \, dy|(B_r(0))}{r^{d-4}} = 0, \tag{3.23}$$

which at the point $x_0 = 0$ is weaker than (1.11). So, overall, we can formulate this special case as a separate theorem.

Theorem 3.3. *Let $\nu \in \mathcal{M}(\Phi)$ and assume that* 0 *is a σ-point of ν with value ℓ.*
(i) If $d \in \{1, 2, 3, 4\}$ then

$$\lim_{|x|<\alpha t \to 0} \mathbb{P}_t \nu(x) = \ell, \quad \forall\, \alpha > 0. \tag{3.24}$$

(ii) If $d \geq 5$ then (3.24) *holds under the additional assumption* (3.23).

References

[1] P. Fatou, Series trigonometriques et series de Taylor. Acta Math **30** (1906), 335-400.
[2] G. Flores, G. Garrigós, T. Signes, B. Viviani, Pointwise convergence of fractional powers of Hermite type operators. Rev Unión Matem Arg. **66** (1) (2023), 187–205.
[3] G. Flores, G. Garrigós, B. Viviani, Symmetric derivatives and normal convergence of Poisson-Hermite integrals. Work in progress, 2024.
[4] G. Flores, B. Viviani, A Calderón theorem for the Poisson semigroups associated with the Ornstein–Uhlenbeck and Hermite operators. Math. Ann. **386** (2023), 329–342.
[5] G. Folland, *Real Analysis*, 2nd Ed. John Wiley & Sons Inc, 1989.
[6] G. Garrigós, S. Hartzstein, T. Signes, B. Viviani, A.e. convergence and 2-weight inequalities for Poisson-Laguerre semigroups. Ann. Matem. Pura Appl. **196** (2017), 1927–1960.
[7] G. Garrigós, S. Hartzstein, T. Signes, J.L. Torrea, B. Viviani, Pointwise convergence to initial data of heat and Laplace equations. Trans. Amer. Math. Soc. **368** (9) (2016), 6575–6600.
[8] L. Liu, P. Sjögren, A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein-Uhlenbeck Poisson kernel. Rev. Mat. Iberoam. **32** (4) (2016), 1189–1210.
[9] B. Muckenhoupt, Poisson integrals for Hermite and Laguerre expansions. Trans. Amer. Math. Soc. **139** (1969), 231–242.
[10] W. Rudin, *Real and Complex Analysis*, 3rd Ed. McGraw-Hill, 1987.
[11] S. Saeki. On Fatou-type theorems for non-radial kernels. Math. Scand. **78** (1996), 133-160.
[12] J. Sarkar. A note on σ-point and nontangential convergence. Preprint, 2021. Available at `https://arxiv.org/abs/2101.05660`.
[13] V. Shapiro. Poisson integrals and nontangential limits. Proc. Amer. Math. Soc. **134** (2006), 3181-3189.
[14] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Ann. Math. Stud. **63**, Princeton Univ Press and Univ Tokio Press, Princeton NJ 1970.
[15] K. Stempak and J.L. Torrea, Poisson integrals and Riesz transforms for Hermite function expansions with weights. Jour. Funct. Anal. **202** (2) (2003), 443–472.
[16] S. Thangavelu, On conjugate Poisson integrals and Riesz transforms for the Hermite expansions. Colloq. Math. **64** (1) (1993), 103–113.
[17] W. Urbina-Romero, *Gaussian harmonic analysis*. Springer Monogr. Math. Springer, 2019.
[18] A. Zygmund, *Trigonometric series*, 3rd Ed, Vol I, Cambridge Univ Press, NY, 1959.

G. Flores, CIEM-CONICET & FaMAF, Universidad Nacional de Córdoba, Av. Medina Allende s/n, Ciudad Universitaria, CP:X5000HUA Córdoba, Argentina
Email address: `guilleflores@unc.edu.ar`

G. Garrigós, Departamento de Matemáticas, Universidad de Murcia, 30100, Espinardo, Murcia, Spain
Email address: `gustavo.garrigos@um.es`

Beatriz Viviani, IMAL (UNL-CONICET) y FIQ (Universidad Nacional del Litoral), Colectora Ruta Nac. N 168, Paraje El Pozo - 3000 Santa Fe - Argentina
Email address: `viviani@santafe-conicet.gov.ar`